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LU-Decomposition of a Matrix with Entries of Different Kinds

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Abstract

Let $\underline{F} \supset \underline{K}$ be fields and consider a matrix A over $\underline{F}$ whose entries not belonging to $\underline{K}$ are algebraically independent transcendentals over $\underline{K}$. It is shown that if det $A \in \underline{K}^*$ $(=\underline{K}-\{0\})$, the matrix A, with suitable permutations of its rows and columns, is decomposed into LU-factors with the entries of the U-factor belonging to $\underline{K}$.

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1. Introduction

Let $\underline{K}$ be a field and $\underline{F}$ ($\supset \underline{K}$) an extension field. For $S \subset \underline{F}$ we denote by $M(S)$ the set of matrices with entries belonging to S. Suppose an n by n matrix $A=(A_{ij}) \in M(\underline{F})$ is expressed as

$$A = Q + T, \qquad (1)$$

where

i) $Q \in M(\underline{K})$,

ii) non-zero entries of T are algebraically independent transcendentals over $\underline{K}$.

In the following we shall denote by T^* the set of non-zero entries of T.

As is well known, A is invertible in the ring $\underline{K}[T^*]$ of polynomials in T^* over $\underline{K}$, i.e., $A^{-1} \in M(\underline{K}[T^*])$, iff $\det A \in \underline{K}^*(=\underline{K}-\{0\})$. Here we are interested in whether we can compute A^{-1} by means of pivot operations in $\underline{K}[T^*]$; moreover, how simple we can make the LU-factors of A by applying suitable permutations to its rows and columns.

By way of illustration, we will start with an example. Let $\underline{K}=\underline{Q}$ (the field of rational numbers) and set $\underline{F}=\underline{Q}(x,y,z)$, where $\{x,y,z\}$, as a collection, is assumed to be algebraically independent over $\underline{Q}$. Matrix

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left(\begin{array}{ccccc} -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & x & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ y & -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & z & 0 \end{array}\right) \end{array},$$

is expressed as the sum of the following Q and T according to (1):

$$Q = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 \end{pmatrix}.$$

Note that det A = 2 and hence A is invertible in $\underline{Q}[x,y,z]$. The matrix A is decomposed into LU-factors in $\underline{F}$ as

$$A = L\,U,$$

with

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -y & y-1 & y-1-2/x & 1 & 0 \\ -1 & 2 & 2+1/x & -(xz+1)/2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & x+1 & 1 & 1 \\ 0 & 0 & -x & -1 & 0 \\ 0 & 0 & 0 & -2/x & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is observed that some of the entries of L and U, especially some of the diagonals of U, do not belong to $\underline{K}[T^*]$.

However, after rearranging the rows and the columns of A as

$$\tilde{A} = \begin{array}{c} \\ 1 \\ 3 \\ 4 \\ 2 \\ 5 \end{array} \begin{array}{c} \begin{matrix} 1 & 5 & 3 & 4 & 2 \end{matrix} \\ \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ y & -1 & 1 & 0 & -1 \\ 1 & 0 & x & 1 & 0 \\ 1 & 0 & 0 & z & 1 \end{pmatrix} \end{array},$$

we obtain the LU-decomposition

$$\tilde{A} = \tilde{L}\,\tilde{U}$$

with

$$\tilde{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -y & y-1 & 1 & 0 & 0 \\ -1 & 1 & x/2 & 1 & 0 \\ -1 & 1 & 0 & z & 1 \end{pmatrix}, \quad \tilde{U} = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The LU-factors are much simpler in the sense that all the entries of $\tilde{U}$ are numbers in $\underline{K}=\underline{Q}$, i.e., $\tilde{U} \in M(\underline{K})$ and, consequently, the entries of $\tilde{L}$ are polynomials in x, y and z over $\underline{K}$ of degree at most 1.

In this paper, we establish a theorem stating to the effect that this is always the case for any matrix A which admits the expression of (1) with det $A \in \underline{K}^*$, i.e., that it is always possible to find a permutation of rows and that of columns, through which the matrix A can be brought to the form decomposable into LU-factors with a U-factor in $M(\underline{K})$. Furthermore, it is shown how to find suitable permutations. Some implications of the theorem are also discussed.

2. The Theorem

In this section we prove the following theorem.

<u>Theorem</u>. Let A be a matrix of form (1). If det $A \in \underline{K}^*$, then there exist permutation matrices P_r, P_c and LU-factors $\tilde{L} = (\tilde{L}_{ij})$, $\tilde{U} = (\tilde{U}_{ij})$:

$$P_r' \, A \, P_c = \tilde{L} \, \tilde{U}$$

such that

(i) $\tilde{L}_{ij}$ is a polynomial of degree at most 1 in non-zero entries T^* of T over $\underline{K}$ ($\tilde{L}_{ii}=1$; $\tilde{L}_{ij}=0$ for $i<j$)

and

(ii) $\tilde{U} \in M(\underline{K})$; $\tilde{U}_{ii} \in \underline{K}^*$ ($\tilde{U}_{ij}=0$ for $i>j$). ▯

To prove the theorem, the following lemma is crucial, giving a necessary and sufficient condition for a matrix of form (1) to be invertible in $\underline{K}[T^*]$. We will say that a matrix is strictly lower triangular if it is a lower triangular matrix with zero diagonals.

Lemma 1. Let A be a matrix in (1). Then $\det A \in \underline{K}^*$ iff $\det Q \neq 0$ and $P_r'(TQ^{-1})P_r$ is strictly lower triangular for some permutation matrix P_r. ▯

Proof: ["if" part] Suppose $P_r'(TQ^{-1})P_r$ is strictly lower triangular for some permutation matrix P_r. Then, since $\det Q \neq 0$ and $A = Q + T$, we have

$$\begin{aligned}\det A &= \det[(I+TQ^{-1})\,Q] \\ &= \det[I + P_r'(TQ^{-1})P_r]\,\det Q \\ &= \det Q \in \underline{K}^*.\end{aligned}$$

["only if" part] If $\det A \in \underline{K}^*$, then $\det Q = \det A \neq 0$, so that we may put $S = Q^{-1}$. Suppose, to the contrary, that $P_r'(TS)P_r$ is not strictly lower triangular for any permutation matrix P_r. Then TS has a cycle of non-zero entries, that is, there exist an integer $M \geq 1$ and a sequence of indices $i(m)$ and $j(m)$ $(m=1,\dots,M)$ such that

$$T_{i(m-1),j(m)} \neq 0 \text{ and } S_{j(m),i(m)} \neq 0 \text{ for } m=1,\dots,M,$$

where $i(0)=i(M)$. Choose M to be the minimal of such integers. For notational simplicity, we write $T_{i(m-1),j(m)}=t_m$ and $S_{j(m),i(m)}=s_m$.

For $k=0,1,\dots,$ consider the expression of the $(j(1),i(1))$ entry of $S(TS)^{kM}$ in the form of the sum of products of T_{ij}'s and S_{ji}'s. Corresponding to the above cycle, it contains a term

$$s_1(s_1s_2\dots s_M)^k(t_1\dots t_M)^k,$$

since no other similar terms of $(t_1\dots t_M)^k$ exist due to the minimality of M and since it cannot be cancelled out by non-similar terms by virtue of the algebraic independence of elements of T^*.

Next we formally expand A^{-1} as

$$A^{-1} = [(I+TQ^{-1})Q]^{-1}$$
$$= S - STS + STSTS - \dots .$$

Each entry of A^{-1} on the left-hand side is a polynomial in T* over $\underline{K}$ since det $A \in \underline{K}^*$. On the right-hand side, we first observe that each entry of the m-th term is a homogeneous polynomial in T* of degree m-1. Hence, by algebraic independence of T*, no cancellation occurs among distinct terms in this expansion.

It follows in particular that the (j(1),i(1)) entry of the right-hand side contains a term of arbitrarily high degree, since the non-zero term $(t_1 \dots t_M)^k$ of degree kM, stemming from $S(TS)^{kM}$ as above, cannot be cancelled out for k=0,1,.... This is a contradiction. □

We make use of the following well-known lemma, the proof of which is omitted.

<u>Lemma</u> <u>2</u>. If det Q ≠ 0, then for any permutation matrix P_r, there exists a permutation matrix P_c and LU-factors M, $\tilde{U}$ such that

$$P_r' \, Q \, P_c = M \, \tilde{U},$$

where M is a lower triangular matrix with unit diagonals in $M(\underline{K})$ and $\tilde{U}$ a nonsingular upper triangular matrix in $M(\underline{K})$. □

With Lemmas 1 and 2, the Theorem is easy to establish as shown below.

Proof of Theorem: Let P_r and P_c be permutation matrices as in Lemmas 1 and 2, respectively. Then from Lemma 2 we obtain

$$\begin{aligned}\tilde{A} &= P_r' \, A \, P_c \\ &= P_r' \, (Q+T) \, P_c \\ &= (I+P_r'(TQ^{-1})P_r) \, (P_r'QP_c) \\ &= (I+P_r'(TQ^{-1})P_r) \, M \, \tilde{U}\end{aligned}$$

$= \tilde{L}\,\tilde{U},$

where

$$\tilde{L} = (I+P_r'(TQ^{-1})P_r)\ M.$$

Since both factors of $\tilde{L}$ are lower triangular matrices with unit diagonals, $\tilde{L}$ is also a lower triangular matrix with unit diagonals and therefore $\tilde{A} = \tilde{L}\,\tilde{U}$ is actually the LU-decomposition of $\tilde{A}$. Obviously $\tilde{U}$ belongs to $M(\underline{K})$ and, consequently, the entries of $\tilde{L} = \tilde{A}\,\tilde{U}^{-1}$ are polynomials in T^* of degree at most 1. □

Remark 1. In parallel with the Theorem, it is likewise possible to find permutations through which A can be brought to a form decomposable into LU-factors in such a way that the L-, instead of U-, factor belongs to $M(\underline{K})$.

Remark 2. Consider a matrix A in $M(\underline{F})$. Then it can be written as

$$A = Q + T,$$

where $Q \in M(\underline{K})$ and $T \in M(\underline{F}\backslash\underline{K})$. In general, the non-zero entries T^* of T are not algebraically independent over $\underline{K}$ and the LU-decomposition of the above-mentioned kind does not necessarily exist even if $\det A \in \underline{K}^*$, as is the case with

$$A = \begin{pmatrix} x & 1+x \\ 1-x & -x \end{pmatrix},$$

where $\underline{K}=\underline{Q}$ and $\underline{F}=\underline{Q}(x)$.

However, it may happen that the matrix $A_0=Q+T_0$, where T_0 is obtained from T by replacing its non-zero entries by algebraically independent transcendentals, satisfies the condition $\det A_0 \in \underline{K}^*$. Then the Theorem can be applied to A_0, which, in turn, implies that A itself can be decomposed, with suitable permutations, into the LU-factors with a U-factor belonging to M(K).

3. Discussions

When given a matrix A of form (1) satisfying the condition det $A \in K^*$, we can find the suitable permutations P_r and P_c on the basis of Lemmas 1 and 2. P_r can be determined by the zero/non-zero pattern of TQ^{-1} and P_c by pivoting operations on the matrix Q. Thus both permutations can be found with $O(n^3)$ arithmetic operations in K.

Lemma 1 gives an efficient way, with $O(n^3)$ arithmetic operations in K, for testing whether a matrix A of form (1) satisfies the condition det $A \in K^*$.

The problem dealt with in the present paper has arisen when the author was investigating the following problem of large-scale system analysis.

Let R and C be the set of row and column numbers, respectively, and A(I,J) denote the submatrix of A corresponding to $I(\subset R)$ and $J(\subset C)$. For a matrix A of form (1), it is known [1] (cf. also the concept of 2-block rank in [2]) that we can find, by an efficient algorithm, two subsets $I \subset R$ and $J \subset C$ such that

$$\text{rank } A = \text{rank } A(I,J) + \text{rank } A(R\backslash I, C\backslash J),$$

$$\text{rank } A(I,J) = \text{rank } Q(I,J)$$

and

$$\text{rank } A(R\backslash I, C\backslash J) = \text{rank } T(R\backslash I, C\backslash J),$$

where the rank is considered over F. If we take I and J to be the minimal of such subsets, we have $|I| = |J|$ and

$$\det A(I,J) \in K^*,$$

The submatrix A(I,J) above meets the condition of the Theorem. This implies that a matrix A of form (1) with det $A \neq 0$ can be decomposed, after suitable permutations P_r and P_c, into LU-factors as

$$P_r' \, A \, P_c = \tilde{L} \, \tilde{U}$$

with a lower triangular matrix

$$\tilde{L} = \begin{pmatrix} \tilde{L}_{11} & 0 \\ \tilde{L}_{21} & \tilde{L}_{22} \end{pmatrix}$$

with unit diagonals and a nonsingular upper triangular matrix

$$\tilde{U} = \begin{pmatrix} \tilde{U}_{11} & \tilde{U}_{12} \\ 0 & \tilde{U}_{22} \end{pmatrix}$$

such that

i) the entries of $\tilde{L}_{11}$ and $\tilde{L}_{21}$ are polynomials in T* over $\underline{K}$ of degree at most 1.

ii) $\tilde{U}_{11} \in M(\underline{K})$ and the diagonal entries of $\tilde{U}_{22}$ are algebraically independent over $\underline{K}$.

This procedure is applied to the iterative solution of a system of linear/non-linear equations f(x)=0 in real unknown variables x, as follows. Let us suppose that a sequence of approximate solutions are computed by means of the Newton method, which would involve the solution of $J(x) \Delta x = f(x)$ for Δx through the LU-decomposition of J(x), where J(x) is the Jacobian matrix.

Since the non-constant derivatives of f(x) may vary in value at each iteration, we regard them as being algebraically independent, or in other words, denoting the non-linear part of J(x) by T(x), we express J(x) in the form (1):

$$J(x) = Q + T(x)$$

with $\underline{K}=\underline{Q}$ or $\underline{K}=\underline{R}$ (the field of real numbers). Furthermore we assume that $\det J(x) \in \underline{Q}^*$ or $\underline{R}^*$.

As the Theorem guarantees, we can obtain the LU-decomposition of J(x):

$$J(x) = L(x)\ U$$

with

$$L(x) = (I + T(x)\ Q^{-1})\ M$$
$$= M + T(x)\ U^{-1},$$

where $Q = M\ U$, as above, and the permutation matrices are suppressed for simplicity. Since M and U do not depend on x, they can be computed before the iteration process starts. At each iteration step, only the L-factor $L(x)$ of $J(x)$ is to be computed. Note that U^{-1} on the right-hand side of $L(x)$ does not cost much since U is triangular. As pointed out in Remark 1 in the previous section, we may alternatively adopt the LU-decomposition $J(x) = L\ U(x)$ with the L-factor being independent of x.

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