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Recent development of QST

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Quantum spectral transform (QST) method was introduced as a natural quantization of the famous inverse scattering transform (IST) introduced in [1] . The following considerations stimulated its derivation :

1. Hamiltonian formulation of IST, derived in [2].
2. Group theoretical interpretation, found in [3], [4].
3. Certain similarity between trace formulae, introduced to IST in [2] and trasfer -matrix method in the statistical mechanics lattice models given in [5].

Now the QST seems to be more natural than IST which can be considered as a suitable quasi-classical limit. The literature on QST is rapidly growing. The surveys [6], [7] contain main ideas and history of the method. More recent developements are described in [8], [9], where many references can be found.

The main results of QST can be formulated as follows:

1. Algebraization of the Bethe Ansats method for finding the eigenvalues and eigenvectors of the Hamiltonian in question.
2. Selecting the ground state as a Dirac sea of the negative energy states and finding the particle-like excitations. For relativistic models it leads to describing the mass spectrum which can be quite different from that of the perturbation theory.

3. Calculation of the phase-shifts for scattering of these excitations.

Unfortunately until now QST can be applied only to field-theoretic models in 1+1 dimensional space-time.

In these lectures I am going to relate some results in QST obtained in Leningrad last year. They belong to the point 1 above. But first I shall briefly formulate the basic ideas of the method.

The quantum mechanical dynamical system on a one dimensional periodic lattice of finite length N is described in a Hilbert space $\mathfrak{h}$ of the form

$$\mathfrak{h} = \prod_n \otimes \mathfrak{h}_n$$

Here $\mathfrak{h}_n$ is a one site Hilbert space where the field operators are represented. These operators for different sites are supposed to commute.

Examples :

1. Heisenberg model for spin 1/2 (XXX - model).

The space $\mathfrak{h}_n$ is $\mathbb{C}^2$ and field operators are three spin operators S_n^α , $\alpha = 1, 2, 3$

$$S_n^\alpha = I \otimes I \otimes \cdots \underbrace{\tfrac{1}{2}\sigma_n^\alpha}_{n} \cdots \otimes I \tag{1}$$

where σ^α are ordinary Pauli matrices and nontrivial factor stands on the n-th place. The operators S_n^α satisfy the commutation relations

$$[S_m^\alpha , S_n^\beta] = i \varepsilon^{\alpha\beta\gamma} S_m^\gamma \delta_{mn} .$$

The Hamiltonian is given by

$$H = \sum_n (S_n^\alpha S_{n+1}^\alpha - \frac{1}{4})$$

where it is supposed that $S_{N+1}^\alpha = S_1^\alpha$.

2. Nonlinear Schroedinger (NS) model.

The space $\mathfrak{h}_n$ is isomorphic to $\mathcal{L}_2(\mathbb{R}^1)$ and the field operators ψ_n^*, ψ_n represent the Heisenberg algebra

$$[\psi_m, \psi_n^*] = \delta_{mn} .$$

The exact Hamiltonian is not defined explicitly and will be discussed later. It must turn to the classical continuous Hamiltonian

$$H_{NS} = \int_0^L (\partial\psi^*\partial\psi + \kappa(\psi^*\psi)^2) \, dx \qquad (2)$$

in the limit $\Delta \to 0$, $\psi(x) = \Delta^{-1/2} \psi_n$, $\kappa = n\Delta$, $L = N\Delta$ when order of factors is not paid attention to. There Δ is lattice distance.

3. Sine - Gordon (SG) model.

The space $\mathfrak{h}_n$ is the same as in the previous example but the field operators u_n, v_n represent Weyl algebra with parametre γ

$$u_n \, v_n = v_n \, u_n \, e^{i\gamma}$$

$$u_m \, v_n = v_n \, u_m \, , \qquad n \neq m \, .$$

The Hamiltonian will be commented upon later.

The corresponding classical continuous Hamiltonian is given by

$$H_{SG} = \int [\frac{1}{2} (\pi^2 + (\partial\phi)^2) + \frac{m^2}{\beta^2}(1 - \cos\beta\phi)] \, dx \quad (3)$$

where $\beta^2 = 8\gamma$ and m is a mass parameter. The relation between the Weyl operators and the fields $\pi(x)$, $\phi(x)$ is approximately as follows ; let

$$\pi_n = \int_{\Delta_n} \pi(x) \, dx \quad ; \qquad \phi_n = \frac{1}{\Delta} \int_{\Delta_n} \phi(x) dx$$

where Δ_n is interval of the length Δ adjacent to the lattice point n, then

$$u_n = \exp \{i\beta\pi_{n/4}\} \, , \quad v_n = \exp \{i\beta\phi_{n/2}\} \, .$$

4. XYZ model

The algebra is the same as in example 1, but the Hamiltonian is nonisotropic

$$H = \sum_n J_\alpha \, S_n^\alpha \, S_{n+1}^\alpha \, .$$

It reduces to XXX model when the parametres J_α are equal. In case when two of them (say J_1 and J_2) are equal it is called XXZ model.

The list of examples can be continued and references can be found in the literature cited above.

The main object of QST is a so called $L_n(\lambda)$ operator which is a matrix, acting in an auxiliary space V with matrix elements being operators in $\mathfrak{h}_n$ and depending on a complex parametre λ. The main property of $L_n(\lambda)$ consists in the family of commutation relations which can be written in the form

$$R(\lambda-\mu)(L_n(\lambda) \otimes L_n(\mu)) = ((L_n(\mu) \otimes L_n(\lambda)) \ R(\lambda-\mu) . \quad (4)$$

Here the tensor product is defined in V in ordinary algebaric way but the order of noncommuting matrix elements is to be taken into account. The matrix $R(\lambda)$ acting in $V \otimes V$ does not depend on field operators and is the same for all n.

For the examples mentioned above the space V is two-dimensional, $V = \mathbb{C}^2$. Corresponding L_n are given as follows

$$L_n^{XXX} = \begin{pmatrix} \lambda + iS_n^3 , & iS_n^+ \\ iS_n^- & \lambda - iS_n^3 \end{pmatrix} \quad (5)$$

$$S_n^+ = S_n^1 - iS_n^2 \ ; \quad S_n^- = S_n^1 + iS_n^2$$

$$L_n^{NS} = \begin{pmatrix} 1 + \frac{i\lambda\Delta}{2} - \frac{\alpha^2}{2} \psi_n^* \psi_n & \alpha \psi_n^* (1-\frac{\alpha^2}{4} \psi_n^* \psi_n) \\ \alpha \psi_n & 1 - \frac{i\lambda\Delta}{2} - \frac{\alpha^2}{2} \psi_n^* \psi_n \end{pmatrix}$$

where $\alpha^2 = \kappa\Delta$;

$$L_n^{SG} = \begin{pmatrix} u_n^{-1/2} f(v_n) u_n^{-1/2} & \frac{m\Delta}{4} (e^{\lambda} v_n^{-1} - e^{-\lambda} v_n) \\ \frac{m\Delta}{4} (e^{-\lambda} v_n^{-1} - e^{\lambda} v_n) & u_n^{1/2} f(v_n) u_n^{1/2} \end{pmatrix}$$

where $f(v) = (1 + \frac{m^2\Delta^2}{32}(v^2 + v^{-2}))^{1/2}$;

$$L_n^{XYZ} = \begin{pmatrix} w_0(\lambda) + w_3(\lambda)\, S_n^3 \;;\; w_1(\lambda)\, S_n^1 - iw_2(\lambda)\, S_n^2 \\ w_1(\lambda)\, S_n^1 + iw_2(\lambda)\, S_n^2 \;;\; w_0(\lambda) - w_3(\lambda)\, S_n^3 \end{pmatrix}$$

where the coefficiants $w_\alpha(\lambda)$ are given in terms of elliptic functions of modulus k

$$w_0(\lambda) = \frac{sn\,(\lambda+\eta\,,\,k)}{sn(\eta,k)} \;;\quad w_1(\lambda) = 1$$
$$w_2(\lambda) = \frac{dn\,(\lambda+\eta\,,\,k)}{dn(\eta,k)} \;;\quad w_3(\lambda) = \frac{cn(\lambda+\eta\,,\,k)}{cn(\eta,k)} \;. \tag{6}$$

The matrices $L_n(\lambda)$ above contain correspondingly 0, 1, 2, and 3 parametres besides λ.

The R-matricies can be written in the form

$$R = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix} \tag{7}$$

in a suitable basis with the following coefficients

XXX case

$d = 0$; $a = 1$; $b = \frac{i}{\lambda+i}$; $c = \frac{\lambda}{\lambda+i}$

N S case

$d = 0$; $a = 1$; $b = \frac{i\kappa}{\lambda+i\kappa}$; $c = \frac{\lambda}{\lambda+i\kappa}$

S G case

$d = 0$; $a = 1$; $b = \frac{i\sin\gamma}{sh(\lambda+i\gamma)}$; $c = \frac{sh\,\lambda}{sh(\lambda+i\gamma)}$, XYZ case

$$a = w_0 + w_3 \qquad b = w_0 - w_3$$
$$c = w_0 - w_2 \qquad d = w_0 + w_2 \tag{8}$$

Several comments are in order.

1. The fundamental relation (4) in generic case is consistent if R-matrix satisfies some condition. To write it down let us introduce the matrices $R \otimes I$ and $I \otimes R$ acting in $V \otimes V \otimes V$, where I is a unit matrix in V. The condition called Yang - Baxter relation looks as follows

$$(R(\lambda-\mu) \otimes I)\ (I \otimes R(\lambda-\sigma))(R(\mu-\sigma) \otimes I) =$$
$$= (I \otimes R(\mu-\sigma))(R(\lambda-\sigma) \otimes I)\ (I \otimes R(\lambda-\mu)). \tag{9}$$

2. In XYZ case the auxiliary space and local quantum space $\mathfrak{h}_n$ coincide and R matrix practically coincides with L_n, so that the relations (4) and (9) are identical. Such models and corresponding L-operators are called fundamental.

3. Operator L_n^{XXX} is a degenerate case of L_n^{XYZ} in an appropriate limit.

4. The R-matrices of XXX and NS models coincide after trivial renormalization $\lambda \to \lambda/\kappa$.

5. Consider NS example and let us introduce a new L_n-operator

$$\tilde{L}_n = -\frac{2i}{\alpha}\, \sigma_3\, L_n \quad . \tag{10}$$

It satisfies the fundamental relation (4) because corresponding R - matrix commutes with $\sigma_3 \otimes \sigma_3$. Now $\tilde{L}_n$ can be written in the form

$$\tilde{L}_n = \begin{pmatrix} \lambda/\kappa + i\, S_n^3 & i\, S_n^{(+)} \\ i\, S_n^{(-)} & \lambda/\kappa - i\, S_n^3 \end{pmatrix}$$

analogous to L_n^{XXX} with operator S_n^α given by

$$S_n^{(+)} = 2i\ \psi_n^* \left(1 - \frac{\alpha^2}{4}\ \psi_n^*\ \psi_n\right) ; \quad S_n^{(-)} = 2i\ \psi_n$$
$$S_n^3 = \frac{2i}{\alpha} \left(1 - \frac{\alpha^2}{2}\ \psi_n^*\ \psi_n\right) \tag{11}$$

which satisfy the commutation relations of $O(3)$ or $O(2,1)$ group depending on the sign of κ. Moreover the corresponding representation is irreducible, the Casimir operator $K = \Sigma\, S_n^\alpha\, S_n^\alpha$ being equal to $\ell(\ell+1)$, $\ell = -2/\alpha$. We see that after the change $L_n \to \tilde{L}_n$ the NS model is nothing but XXX model for the noninteger value of spin. One must comment that S_n^α realize the representation of the Lie algebra which is not integrable to that of the group.

6. In the XXX and NS examples there exists a local vacuum, namely a state $\omega_n \in \mathfrak{h}_n$ such that $L_n(\lambda)$ becomes triangular when applied to it. In XXX case ω_n is given by vector

$$\omega_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and for NS case ω_n is a state annihilated by the operator ψ_n

$$\psi_n\, \omega_n = 0 \ .$$

For SG model local vacuum exists for the product of two consecutive L_n ; for matrix $L_{n+1}L_n$ it lies in $\mathfrak{h}_{n+1} \otimes \mathfrak{h}_n$ and is given by function

$$\left(1 - \frac{m^2\Delta^2}{16} \cos\left(\frac{1}{i} \ln v_n v_{n+1}\right)\right)^{-1/2} \quad \delta\left(\frac{1}{i} \ln v_n/v_{n+1} + \gamma - \pi \right)$$

in the representation where operators v_n are diagonal.

In all these cases we have the property

$$L_n \omega_n = \begin{pmatrix} \alpha(\lambda) & * \\ 0 & \delta(\lambda) \end{pmatrix} \omega_n$$

where $\alpha(\lambda)$ and $\delta(\lambda)$ are functions independent of the field operators. For SG case this formula is true for the product $L_{n+1}L_n$. The values of $\alpha(\lambda)$ and $\delta(\lambda)$ (local vacuum eigenvalues) are as follows

XXX case

$$\alpha(\lambda) = \lambda + i/2 \quad , \qquad \delta(\lambda) = \lambda - i/2$$

NS case

$$\alpha(\lambda) = 1 + \frac{i\lambda\Delta}{2} \quad , \qquad \delta(\lambda) = 1 - \frac{i\lambda\Delta}{2}$$

SG case

$$\alpha(\lambda) = 1 + \frac{m^2\Delta^2}{8} \operatorname{ch}(2\lambda - i\gamma) \quad ; \qquad \delta(\lambda) = \alpha(-\lambda)$$

Once more we see that α_{NS} and α_{XXX} are practically the same.

We are ready now to explain the formalism of QST. First of all the monodromy matrix $T(\lambda)$ is defined as as ordered product of the L_n - metrices over the lattice

$$T(\lambda) = \overset{\curvearrowleft}{\Pi} L_n(\lambda) .$$

Its matrix elements will be denoted as A, B, C, D

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} .$$

We do not introduce explicitly the index N on T because N is fixed throughout all our lectures.

The main formula (4) leads to the commutation relation

$$R(\lambda-\mu)\ (T(\lambda) \otimes T(\mu)) = (T(\mu) \otimes T(\lambda))R(\lambda-\mu) \tag{12}$$

and in case when the local vacuum exists we have

$$T\Omega = \begin{pmatrix} \alpha(\lambda)^N & * \\ 0 & \delta(\lambda)^N \end{pmatrix} \Omega \tag{13}$$

where $\Omega = \Pi \otimes \omega_n$ and N is to be changed to N/2 in SG case. In other words Ω is an eigenvector both for $A(\lambda)$ and $D(\lambda)$

$$A(\lambda)\Omega = \alpha(\lambda)^N \Omega \quad ; \quad D(\lambda)\Omega = \delta(\lambda)^N \Omega \tag{14}$$

and is annihilated by $C(\lambda)$

$$C(\lambda)\Omega = 0 \quad .$$

We shall write explicitly some of the commutation relations contained in (12)

$$[A(\lambda) + D(\lambda) \quad , \quad A(\mu) + D(\mu)] = 0 \quad ;$$

$$[B(\lambda) \ , \ B(\mu)] = 0 \ ;$$

$$A(\lambda)\ B(\mu) = B(\mu)\ A(\lambda)\ \frac{1}{c(\mu-\lambda)} - B(\lambda)\ A(\mu)\ \frac{b(\mu-\lambda)}{c(\mu-\lambda)} \quad ;$$

$$D(\lambda)\ B(\mu) = B(\mu)\ D(\lambda)\ \frac{1}{c(\lambda-\mu)} - B(\lambda)\ D(\mu)\ \frac{b(\lambda-\mu)}{c(\lambda-\mu)} \quad ;$$

$$B(\lambda)\ C(\mu) - C(\mu)\ B(\lambda) = \frac{b(\mu-\lambda)}{c(\mu-\lambda)}\ (A(\mu)\ D(\lambda) - A(\lambda)\ D(\mu)\ ) \quad .$$

From these relations and (14) it follows that for a given set $\{\lambda\} = (\lambda_1,\cdots\lambda_n)$ the vector

$$\psi(\{\lambda\}) = B(\lambda_1)\ \cdots\ B(\lambda_n)\Omega \qquad (15)$$

is an eigenvector of $A(\lambda) + D(\lambda) = \mathrm{tr}T(\lambda)$ with the eigenvalue

$$\Lambda(\lambda\ ,\ \{\lambda\}) = \alpha(\lambda)^N \prod_\ell \frac{1}{c(\lambda_\ell,\lambda)} + \delta(\lambda)^N \prod_\ell \frac{1}{c(\lambda,\lambda_\ell)}$$

if $(\lambda_1,\cdots,\lambda_n)$ satisfy the set of equations

$$\left(\frac{\alpha(\lambda_j)}{\delta(\lambda_j)}\right)^N = \prod_{\ell\neq j} \frac{c(\lambda_j,\lambda_\ell)}{c(\lambda_\ell,\lambda_j)} \quad . \qquad (16)$$

($N \to N/2$ in SG case)

The last equation is equivalent to the natural requirement that $\Lambda(\lambda,\{\lambda\})$ is an entire function of λ.

Thus the described formalism gives the system of eigenvectors of an infinite set of commuting operators for which $\mathrm{tr}T(\lambda)$ is a generating function. The hamiltonians of the model is contained in this set. Indeed in the example of XXX model the hamiltonian is given by the formula

$$H = \frac{d}{d\lambda}\ \ln(A(\lambda) + D(\lambda)\ )\Big|_{\lambda = i/2}$$

which reminds the trace formulae mentioned in the introduction. For other models such simple formula is not true. Recent ideas on this subject will be discussed below.

The limit $N \to \infty$ is most interesting in physical applications as it reveals the structure of ground state and excitations. Many results on this limit can be found in the original papers, see for example [10], [11]. We shall not treat this subject in these lectures. Instead we shall concentrate on several formal aspects of QST which have got a new insight recently, namely

1. Normalization of the state $\psi(\{\lambda\})$.
2. The local hamiltonian for the nonfundamental models.
3. The general structure of the operator matrix $L_n(\lambda)$.

Normalization problem was treated on the example of XXZ model by Gaudin et al in [12]. Recently Korepin has got a complete and general solution [13]. We shall make a comment on his solution.

Korepin considers the matrix element

$$\langle \Omega \mid C(\lambda_1) \cdots C(\lambda_n)\, B(\lambda_1) \cdots B(\lambda_n) \mid \Omega \rangle$$

which in most cases is nothing but normalization factor due to the condition of the type $C(\lambda) \sim B(\lambda)^*$. Apart from a trivial factor this matrix element is shown to be proportional to the determinant of a matrix M, where

$$M_{ik} = \frac{\partial \psi_i}{\partial \lambda_k} \quad .$$

There $\psi_i(\{\lambda\})$ are essentially the logarithms of the equations (16)

$$\psi_i(\{\lambda\}) = N \ln \frac{\alpha(\lambda_i)}{\delta(\lambda_i)} - \sum_{i \neq j} \ln \frac{c(\lambda_i - \lambda_j)}{c(\lambda_j - \lambda_i)} \quad .$$

It is easy to see that matrix M is symmetric, so that there exists a function $\Phi(\{\lambda\})$ such that

$$\psi_i = \frac{\partial \Phi}{\partial \lambda_i} \quad , \qquad M_{ik} = \frac{\partial^2 \Phi}{\partial \lambda_i \partial \lambda_k} \quad .$$

Moreover this function is invariant with respect to the permutation of $\lambda_1, \cdots \lambda_n$.

I beleive that the function $\Phi(\{\lambda\})$ will be very useful in the future understanding of the mathematical structure of QST. Indeed the equations

$$\frac{\partial \Phi}{\partial \lambda_i} = 0 \qquad (\mathrm{mod}\ 2\pi i) \tag{17}$$

equivalent to (16) and the fact that the normalization factor is proportional to the Hessian of Φ look very much like quasiclassical formulae giving exact quantum result. Alternatively one can imagine that (17) is some kind of the integervaluedness condition inherent in the geometric quantization a là Kirillov-Kostant. All this must be considered as an indication that QST is connected with the representation theory of an infinite dimensional group.

Now I turn to the problem of the local hamiltonians. One natural requirement for such a hamiltonian is the additivity of its eigenvalues ; more exactly the eigenvalues of the vector $\psi(\{\lambda\})$(15) is to have the form

$$E_0 + \sum_i h(\lambda_i)$$

where E_0 and $h(\lambda)$ are vacuum and quasiparticle energies, correspondingly. This property is satisfied by the operators given as follows.

$$H_k = \left(\frac{d}{d\lambda}\right)^k \ln \operatorname{tr} T(\lambda)|_{\lambda=\eta} \qquad (18)$$

where η is a zero of the local vacuum eigenvalue $\delta(\lambda)$

$$\delta(\eta) = 0 \quad ,$$

E_0 and $h(\lambda)$ being given by

$$E_0 = N\left(\frac{d}{d\lambda}\right)^k \ln \alpha(\lambda)|_{\lambda=\eta} \quad ; \quad h_k(\lambda) = \left(\frac{d}{d\mu}\right)^k \ln \frac{1}{c(\lambda,\mu)}\Big|_{\mu=\eta} \quad :$$

Indeed in the vicinity of η the eigenvalue $\Lambda(\lambda,\{\lambda\})$ is multiplicative thereby leading to the given formulae.

The hamiltonians H_k are local in the case of fundamental models. However for nonfundamental models such as NS and SG this is not true. With some effort Korepin and Izergin have modified the approach given above and obtained reasonably quasilocal hamiltonians approaching the expressions (2) and (3) in the continuous limit. (see [14]). However their results are not completely satisfactory.

The alternative way to solve the problem is based on the proper interpretation of the results of Kulish and Sklyanin [9] on the XXX model for higher spin. The L_n operator of the form (5) satisfy the fundamental relation (4) whenever the spin operators, S^+ , S^- and S^3 give the representation of the rotation group. The R-matrix is the same as in spin 1/2 case ; only the local vacuum eigenvalues change

$$\alpha = \lambda + iS \quad , \quad \delta = \lambda - iS$$

where S is full spin. However this L_n-operator which we denote L_n^S is not fundamental and formula (18) does not lead to the local hamiltonians.

Kulish and Sklyanin have found a remedy for this. They constructed the L_n operator with auxiliary space having the dimension $2S + 1$. This operator will be denoted $L_n^{S,S}$. It is fundamental so that (18) can be used to produce the local hamiltonians. But to diagonalize them we can use old operator L_n^S . Indeed, there exists an R-matrix such that the following relation holds

$$R(L_n^{S,S}(\lambda) \otimes L_n^S(\mu))\, R^{-1} = (I \otimes L_n^S(\mu))(L_n^{S,S}(\lambda) \otimes I)$$

(observe slight change of notations due to the difference of two auxiliary spaces). The last formula shows in particular that the traces of $T^{S,S}(\lambda)$ and $T^S(\lambda)$ commute so that they have mutual eigenvectors.

In this way the family of local hamiltonians for any spin appears. For $S = 1$ the simplest of them looks as follows

$$H = \sum_n X_n^2 - X_n \quad , \quad X_n = S_n^\alpha S_{n+1}^\alpha \quad .$$

For any spin analogous H has the form

$$H = \sum_n P_S(X_n) \tag{19}$$

where P_S is some polynom of order $2S$.

The group - theoretical interpretation of these results and explicit formula for $P_S(X)$ is given in [15]. The $N \to \infty$ limit was recently investigated by Takhtajan [16].

This obervation together with the comment above on the connection of NS and XXX modle makes us beleive that to get the local hamiltonian for NS case one is to continue formula (19) to the $S = -1/\alpha$. To do it we must introduce the infinite dimensional auxiliary space V and construct the corresponding fundamental operator L_n . The σ_3 factor in (10) can be dropped out after the change of ψ operators

$$\psi_n \to (-1)^n \psi_n$$

which is easily seen from the relation

$$\sigma_3 L_n(\psi) \sigma_3 = L_n(-\psi) \quad .$$

It is instructive to see the form of the operator X_n for spin operators given by (11). The simple calculation shows that X_n up to a factor and a constant is given by the expression

$$(\psi^*_{n+1} + \psi_{n+1})(\psi^*_n + \psi_n) + \frac{\kappa}{4}(2\psi^*_{n+1}\psi^*_n\psi_{n+1}\psi_n + \psi^*_n\psi^*_n\psi_{n+1}\psi_n + \psi^*_{n+1}\psi^*_{n+1}\psi_{n+1}\psi_n)$$

which is a simple finite difference opproximation for the energy density in (2) .

These quite satisfactory results make me suspect that the model for which QST can be applied are associated with some algebraic structure represented in the local quantum space $\mathfrak{h}_n$. For XXX and NS models this is a representation of the Lie algebra of the O(3) group. More general algebraic structure was uncovered recently by Sklyanin, who made the following observation.

Let us consider operator L_n of the form

$$L_n = w_0(\lambda)\ S_n^0 \otimes I \quad + \sum_{\alpha=1}^{3} w_\alpha(\lambda)\ S_n^\alpha \otimes \sigma^\alpha$$

when I and σ^α are matricies in $\mathbb{C}^2$, coefficients $w_0(\lambda)$, $w_\alpha(\lambda)$ are given in (6) and S_n^0, S_n^α some operators represented in $\mathcal{H}_n$. This L_n reduces to L_n^{XYZ} in case $\mathcal{H}_n = \mathbb{C}^2$, $S_n^0 = I$ and S_n^α given in (1). Let us further suppose that this L_n satisfies the fundamental relation (4) with the Baxter R-matrix (7), (8). Then the following commutation relations are to be satisfied by S_n^0, S_n^α :

$$[S_n^\alpha, S_n^0] = -\, i\, J_{\beta\gamma}\ (S_\beta\, S_\gamma + S_\gamma\, S_\beta)\ \delta_{mn}$$

$$[S_m^\alpha, S_n^\beta] = i\ (S_0\, S_\gamma + S_\gamma\, S_0)\ \delta_{mn} \quad .$$

There $\alpha\beta\gamma$ is an even permutation of 123 and $J_{\beta\gamma}$ are given by

$$J_{\beta\gamma} = \frac{w_\alpha^2 - w_\beta^2}{w_\gamma^2 - w_0^2}$$

and are independent of λ due to the special form of $w_0(\lambda)$, $w_\alpha(\lambda)$.

We see that the problem of finding the general L_n is reduced to the description of possible representations of these commutation relations. In XYZ case they are satisfied due to the anticommutativity of σ-matrices. It is clear that to generalize XYZ model to "higher spin" we must consider nontrivial operator S_n^0. In particular the SG model is such a generalization of the XXZ model.

To get the local hamiltonians the theory of tensor multiplications of the representations of operators S_n^α is to be developed

This problem is under investigation now.

At this point my lectures are finished. The outstanding unsolved problems are the following :

1. Expression for the Green functions.

2. Finding the L_n operator for the non-linear σ-model.

The last results give us optimism with respect to the second problem ; we must think about the E(3) structure analogous to the O(3) structure of the XXX model. For the first problem the interpretation of QST in terms of representation of infinite dimensional group will be very useful.

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