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ON ACYCLICITY OF CIRCUITS OF A DIGRAPH AND THE DUAL CONCEPT

Yoji KAJITANI (梶谷洋司)
Tokyo Institute of Technology

I. INTRODUCTION

For a circuit C of a given digraph G, consider the reference direction. With respect to this reference direction, C (represented by the edge set) can be partitioned as $C=C^+ \cup C^-$, $C^+ \cap C^- = \phi$ where C^+ is the set of all the edges whose directions follow the reference. Define the acyclicity of C by

$$a(C)=\min\ (|C^+|,|C^-|).$$

The acyclicity of the whole graph G is

$$a(G)=\min\ (a(C))$$

where min ranges over all the circuits. Trivially, $a(C)\leq\lfloor\frac{|C|}{2}\rfloor$.

Usually, C is called cyclic or acyclic corresponding to $a(C)=0$ or $a(C)>0$, respectively. G also is called acyclic if $a(G)>0$.

The acyclicities of circuits of a certain set cannot be independent. For example, consider three circuits C_1, C_2 and C_2' of G shown in Fig.1. It is easy to see that

$$a(C_1)+a(C_2)-a(C_2')\leq 3.$$

Theorem 1 presents this kind of dependence relations.

The dependencies of circuits with respect to acyclicity lead to the concept of k-th acyclicity dominating set D of circuits, which is defined as a set such that

$$\min_{C\in D}\ a(C)\geq k \quad \text{implies} \quad a(G)\geq k.$$

Theorem 2 determins the minimum first acyclicity dominating set.

It follows the complete dual discussion which treats the co-circuits (cuts) and their co-acyclicity (strongness of connectivity).

All these results are a version of our earlier works[1,2].

2. DEPENDENCY OF CIRCUITS WITH RESPECT TO ACYCLICITY

In this paper, terms "circuit" and "cut" are used to denote the simple ones. Circuits C_1 and C_2 are said to be confluent if $C_1\cap C_2$ forms a nonempty simple path. If C_1 and C_2 are confluent, $C_2'=C_1\oplus C_2=(C_1\cup C_2)-(C_1\cap C_2)$ is again a circuit.

Lemma 1: Suppose C_1, C_2 are confluent and let $C_2'=C_1\oplus C_2$. Then

$$|C_2'|-a(C_2')\leq|C_1\cup C_2|-(a(C_1)+a(C_2)).$$

Proof: Let p_i, q_i, r_i denote the numbers of edges whose directions are coincide or not with the reference direction of those circuits contained in C_1-C_2, $C_1\cap C_2$, C_2-C_1, respectively, as indicated in Fig.1. Then,

$$\begin{aligned} a(C_1)+a(C_2)&=\min(p_1+q_2,\ P_2+q_1)+\min(r_1+q_2,\ r_2+q_1)\\ &\leq\min(p_1+r_2+q_1+q_2,\ p_2+r_1+q_1+q_2)\\ &=\min(p_1+r_2,\ p_2+r_1)+q_1+q_2\\ &=a(C_2')+|C_1\cup C_2|-|C_1\cup C_2|. \end{aligned}$$

Q.E.D.

Theorem 1: Let $(C_1,C_2,\dots,C_n)$ be a sequence of circuits such that

$$C_k'=C_1\oplus C_2\dots\oplus C_k$$

is a circuit and C_k' and C_{k+1} are confluent for $k=1,2,\dots,n-1$. Then

$$|C_n'|-a(C_n)\leq|\bigcup_{i=1}^{n}C_i|-\sum_{i=1}^{n}a(C_i).$$

Proof: For n=2, the proposition is Lemma 2. Suppose it is true for $n\leq m-1$. By the lemma, since $C_m'=C_{m-1}'\oplus C_m$,

$$\begin{aligned} |C_m'|-a(C_m')&\leq|C_{m-1}'\cup C_m|-a(C_{m-1}')-a(C_m)\\ &=|C_{m-1}'\cup C_m|-|C_{m-1}'|+\{|C_{m-1}'|-a(C_{m-1}')\}-a(C_m)\\ &\leq|C_{m-1}'\cup C_m|-|C_{m-1}'|+\{|\bigcup^{m-1}C_i|-\sum^{m-1}a(C_i)\}-a(C_m)\\ &=|\bigcup^{m}C_i|-\sum^{m}a(C_i). \end{aligned}$$

Q.E.D.

A typical example is when $\cup C_i=E$ forms a planar subgraph G in which $C_1,C_2,\dots,C_n$ are the inner meshes properly ordered and C_n' the outer mesh of G which is properly drawn on a plane. If we put $E_I=E-C_n'$, the set of inner edges, the theorem is

$$a(C_n')\geq\Sigma a(C_i)-|E_I|.$$

Thus, the theorem has a meaning only when $\Sigma a(C_i)-|E_I|>0$.

III. THE k-TH ACYCLICITY DOMINATING SET

Let S be the set of all the circuits of G. A subset $D\subset S$ is called the k-th acyclicity dominating set if

(P) $a(C)\geq k$ (for all $C\in D$) implies $a(G)\geq k$.

A circuit C is called to associate the i-chord if the contraction of C produces a new circuit of length i. Let S(i) be the set of all circuits which are associated with i-chords.

Suppose C_1 and C_2 are confluent and $|C_1\cap C_2|\leq k$. Then Lemma 1 insists that $a(C_1)\geq k$ and $a(C_2)\geq k$ lead to $a(C_1\oplus C_2)\geq k$. Hence the lemma.

Lemma 2: For any $C\in\overset{k}{\cup}S(i)$, $S-\{C\}$ is a k-th acyclicity dominating set.

If G contains a circuit of length 2k-1 or less, it is trivial that $a(G)\ngeq k$. Hence the determination of the k-th acyclicity dominating set is meaningful only when G contains no circuit of length 2k-1 or less. That is, the girth of G is 2k or more.

Theorem 2: Suppose the girth of a digraph G is 2k+1 or more. Then, $D=S-\overset{k}{\cup}S(i)$ is a k-th acyclicity dominating set.

Proof: Consider a circuit $C\notin D$. There exist two other circuits C_1 and C_2 such that C_1 and C_2 are confluent, $C_1\oplus C_2=C$, and $|C_1\cap C_2|\leq k$. Furthermore, both $|C_1|$ and $|C_2|$ are strictly less than $|C|$ because, for i,j=1,2

$$\begin{aligned}|C_i|&<|C_i|+(|C_j|-2k)\\&\leq|C_i|+|C_j|-2|C_i\cap C_j|=|C|.\end{aligned}$$

If $C_i\notin D$, there is another pair (C_{i_1},C_{i_2}) the length of each strictly less than that of C_i. Continuing the discussion, it is true that acyclicity being k or more or less of any circuit not in D can be checked by the members of D. Thus D is a k-th dominating set.

Q.E.D.

Most interesting is the consideration when k=1 for it distinct G between being acyclic or not. For this case, we can give the minimum dominating set.

Theorem 3: If the girth of G is 3 or more (i.e. G contains no parallel edges or self-loops), D=S-S(1) is the unique and minimum first acyclicity dominating set.

Proof: It suffices to show that for any $C\varepsilon D$ we cannot prove whether $a(C)\geq 1$ from the information of other circuits being acyclic.

Consider the graph G' which is obtained from G by contracting the edges of C to a vertex v. Suppose G is so oriented as: G' is acyclic and C cyclic.

Every graph has this possibility since G' contains no self-loops by assumption.

Then, in G, all circuits except C is acyclic. Hence S-{C} is not a dominating set.

Q.E.D.

IV. THE DUAL CONCEPT

We do not follow the dual discussion faithfully but, for reference, only the fact corresponding to Theorem 3 will be cited.

A 1-chorded cut is a cut which, after deletion of its edges, produce a bridge. A cut is co-acyclic if all of its edge do not follow the same direction.

Theorem 3': Suppose that G contains no cut of cardinality 2 or less. Then, the set D' of all cuts that are not 1-chorded is the minimum and unique set such that every cut of D' being co-acyclic implies G being strongly connected.

V. CONCLUSION

This paper studied the dependency of acyclisity of circuits. Theorem 3 and 3' can be extended to the general cases in which no restriction is imposed[1].

REFERENCES

[1] Y. Kajitani and F. Hirose,"On acycle basis and co-acycle basis," Proc. Tech. Group on Circuits and Systems, IECE of Japan, CST76-15, 1976.

[2] E. Okamoto, H. Satoh and Y. Kajitani,"The acyclicity of digraphs," Proc. 1977 Convention Record, 17, 1977.

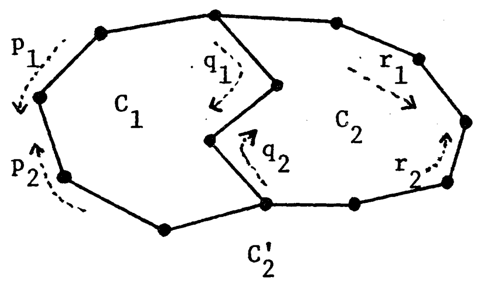

Fig.1 Confluent pair of circuits C_1 and C_2 and $C_2'=C_1\oplus C_2$