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Pseudo-Particle (Instantons, Solitons, Vortices,...) Solutions of SU(2) Gauge Field Equations

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§1. Introduction

Recently, much attention has been focussed to the self-dual Yang-Mills field equations. It has been shown that the equations share the characteristics such as the inverse scattering formulation,[1] Bäcklund transformations and conservation laws[2] with many two-dimensional completely ingegrable systems. Of various attempts, Yang's formulation of SU(2) gauge fields has been recognized to be particularly useful, since by introduction of suitable gauge, the so-called R-gauge, the condition of self duality can be reduced to a system of Lapalce-type nonlinear differential equations of two variables, one of which is real and the other is complex. Here a study is made of the Yang equations[3] in connection with nonlinear σ models along the line somewhat different from that of Pohlmeyer[4] with particular attention paid to psudo-particle solutions such as instantons, solitons and vortices.

§2. The Yang equations

Let us define SU(2) gauge potentials A^a_μ (μ=1,2,3,4; a=1,2,3) in four-dimensional Euclidean space (x_1,x_2,x_3,x_4). Yang considered an analytic continuation of A^a_μ into complex space where $x_1,x_2,x_3,$ x_4 are complex. The condition of self-duality is valid also in complex space, in a region containing real space where the x's are real. Then, by introducing complex coordinates

$$y = 2^{-1/2}(x_1 + i x_2), \quad \bar{y} = 2^{-1/2}(x_1 - i x_2), \quad z = 2^{-1/2}(x_3 - i x_4), \quad \bar{z} = 2^{-1/2}(x_1 + i x_4) \tag{2.1}$$

and

$$A_y = 2^{-1/2}(A_1 - i A_2), \quad A_{\bar{y}} = 2^{-1/2}(A_1 + i A_2), \quad A_z = 2^{-1/2}(A_3 + i A_4), \quad A_{\bar{z}} = 2^{-1/2}(A_3 - i A_4) \tag{2.2}$$

with

$$A_\mu = (g/2i)\,\sigma^a A_\mu^a \qquad (g\text{;const.; } \sigma^a\text{: Pauli matrix}) \tag{2.3}$$

and by chhosing the R-gauge, any solution of the self-duality equation can be brought to the folowing form

$$A_u = \begin{pmatrix} -f_u/2f & 0 \\ \sigma_u/f & f_u/2f \end{pmatrix}, \qquad A_{\bar{u}} = \begin{pmatrix} f_{\bar{u}}/2f & -\bar{\sigma}_{\bar{u}}/f \\ 0 & -f_{\bar{u}}/2f \end{pmatrix} \tag{2.4}$$

with

$$f \doteq \text{real}, \quad \bar{\sigma} \doteq \sigma^*, \quad u = y, z, \quad f_u = \partial f/\partial u, \text{ etc.} \tag{2.5}$$

Here the quantities f, σ , and $\bar{\sigma}$ are governed by the Yang equations[3)]

$$f(f_{y\bar{y}} + f_{z\bar{z}}) - f_y f_{\bar{y}} - f_z f_{\bar{z}} + \sigma_y \bar{\sigma}_{\bar{y}} + \sigma_z \bar{\sigma}_{\bar{z}} = 0, \tag{2.6a}$$

$$(f^{-2}\sigma_y)_{\bar{y}} + (f^{-2}\sigma_z)_{\bar{z}} = 0, \qquad (f^{-2}\bar{\sigma}_{\bar{y}})_y + (f^{-2}\bar{\sigma}_{\bar{z}})_z = 0. \tag{2.6b}$$

In the above equations the simbols $\doteq$ and $=$ denote that the equations are valid for real and all complex values of the x's, respectively. By putting

$$f = \exp(-g), \tag{2.7}$$

Eqs.(2.6) are rewritten as

$$g_{y\bar{y}} + g_{z\bar{z}} = \exp(2g)(\sigma_y \bar{\sigma}_{\bar{y}} + \sigma_z \bar{\sigma}_{\bar{z}}), \tag{2.8a}$$

$$(\exp(2g)\,\sigma_y)_{\bar{y}} + (\exp(2g)\,\sigma_z)_{\bar{z}} = 0, \qquad (\exp(2g)\,\bar{\sigma}_{\bar{y}})_y + (\exp(2g)\,\bar{\sigma}_{\bar{z}})_z = 0. \tag{2.8b}$$

Equations (2.8) imply that the Yang equations can be reduced to Liouville-type equations.

We introduce a transformation $(f, \sigma, \bar{\sigma}) \longrightarrow (f', \sigma', \bar{\sigma}')$:

$$f' = 1/f, \quad \sigma_y' = \mp \bar{\sigma}_{\bar{z}}/f^2; \quad \bar{\sigma}_{\bar{y}}' = \mp \sigma_z/f^2; \quad \sigma_z' = \pm \bar{\sigma}_{\bar{y}}/f^2; \quad \bar{\sigma}_{\bar{z}} = \pm \sigma_y/f^2 \tag{2.9}$$

to reduce Eqs.(2.6) to

$$f'(f'_{y\bar{y}} + f'_{z\bar{z}}) - f_y' f_{\bar{y}}' - f_z' f_{\bar{z}}' - \sigma_y' \bar{\sigma}_{\bar{y}}' - \sigma_z \bar{\sigma}_{\bar{z}}' = 0, \tag{2.10a}$$

$$(f'^{-2}\sigma_y')_{\bar{y}} + (f'^{-2} f'_z)_{\bar{z}} = 0, \quad (f'^{-2}\bar{\sigma}_{\bar{y}}')_y + (f'^{-2}\bar{\sigma}_{\bar{z}}')_z = 0. \tag{2.10b}$$

Equations (2.10) are interpreted as the self-duality equations for an SU(1,1) gauge theory. The above transformation, though similar to the well-known Atiyah-Ward anzatz,[5] are different in an important respect from theirs in that Eqs.(2.9) preserve the requirement that f is real and that σ and $\bar{\sigma}$ are complex conjugate with each other for real values of the x's. By putting f'=exp(g'), Eqs.(2.10) can be reduced to equations similar to Eqs.(2.8). Equations (2.6) and ((2.10) are invariant for the following transformations

$$f \to \frac{f}{f^2 + \sigma\bar{\sigma}}, \quad \sigma \longrightarrow \frac{\bar{\sigma}}{f^2 + \sigma\bar{\sigma}}, \quad \bar{\sigma} = \frac{\sigma}{f^2 + \sigma\bar{\sigma}} \tag{2.11}$$

and

$$f' \longrightarrow \frac{f'}{\sigma'\bar{\sigma}' - f'^2}, \quad \sigma' \longrightarrow \frac{\bar{\sigma}'}{\sigma'\bar{\sigma}' - f'^2}, \quad \bar{\sigma}' = \frac{\sigma'}{\sigma'\bar{\sigma}' - f'^2}, \tag{2.12}$$

respectively. To study the solutions to Eqs.(2.6) it may sometimes be convenient to treat Eqs.(2.10). This is illustrated by noting that Eq.(2.10) possess a particular solutions

$$\sigma' = \pm f' \tag{2.13}$$

with f' satisfying the equation

$$f'(f'_{y\bar{y}} + f'_{z\bar{z}}) - 2(f'_y f'_{\bar{y}} + f'_z f'_{\bar{z}}) = 0. \tag{2.14}$$

Inserting this into Eqs.(2.9) give the well-known Corrigan-Fairlie-t'Hooft-Wilczek ansatz,[2] for which Eqs.(2.6) reduce to the one and the same Laplace-type equation

$$f_{y\bar{y}} + f_{z\bar{z}} = 0. \tag{2.15}$$

A particular solution to this equation gives the t'Hooft multi-instanton solutions:

$$f = f_0 + \sum_j \frac{\lambda_j^2}{(y - y_j)(\bar{y} - \bar{y}_j) + (z - z_j)(\bar{z} - \bar{z}_j)}, \tag{2.16}$$

where f_o, y_j, y_j, z_j, z_j are constant, the latter four being identifined as the position of the jth instanton.

Let us denote the transformations (2.11) and (2.12) by the superscripts + and −, respectively, namely $(f, \sigma, \bar{\sigma}) \longrightarrow (f^+, \sigma^+, \bar{\sigma}^+)$ and $(f', \sigma', \bar{\sigma}') \longrightarrow (f'^-, \sigma'^-, \bar{\sigma}'^-)$. Then, we can introduce the following two types of the Bäcklund transformations for each of Eqs. (2.6) and (2.10)

$$(f, \sigma, \bar{\sigma}) \to (f', \sigma', \bar{\sigma}') \to (f'^-, \sigma'^-, \bar{\sigma}'^-) \to (f'^-, \sigma'^-, \bar{\sigma}'^-)^- \equiv (f^I, \sigma^I, \bar{\sigma}^I)$$
$$\to \cdots\cdots, \tag{2.17a}$$

$$(f, \sigma, \bar{\sigma}) \to (f^+, \sigma^+, \bar{\sigma}^+) \to (f^{+\prime}, \sigma^{+\prime}, \bar{\sigma}^{+\prime}) \to (f^{+\prime-}, \sigma^{+\prime-}, \bar{\sigma}^{+\prime-})$$
$$\to (f^{+\prime-}, \sigma^{+\prime-}, \bar{\sigma}^{+\prime-})' \equiv (f^I, \sigma^I, \bar{\sigma}^I) \to \cdots\cdot, \tag{2.17b}$$

$$(f', \sigma', \bar{\sigma}') \to (f'', \sigma'', \bar{\sigma}'') \to (f''^+, \sigma''^+, \bar{\sigma}''^+) \to (f''^+, \sigma''^+, \bar{\sigma}''^+)' \equiv (f'^I, \sigma'^I, \bar{\sigma}'^I)$$
$$\to \cdots\cdots, \tag{2.18a}$$

$$(f', \sigma', \bar{\sigma}') \to (f'^-, \sigma'^-, \bar{\sigma}'^-) \to (f'^{-\prime}, \sigma'^{-\prime}, \bar{\sigma}'^{-\prime}) \to (f'^{-\prime+}, \sigma'^{-\prime+}, \bar{\sigma}'^{-\prime+})$$
$$\to (f'^{-\prime+}, \sigma'^{-\prime+}, \bar{\sigma}'^{-\prime+})' \equiv (f'^I, \sigma'^I, \bar{\sigma}'^I) \cdots\cdot, \tag{2.18b}$$

Here the cost of the preservation of the reality of f and of σ and $\bar{\sigma}$ being complex conjugate with each other for real values of the x_j's by the transformation (2.9) is paid by the fact that at least three successive transformations are required to return to $(f, \sigma, \bar{\sigma}) \equiv (f^I, \sigma^I, \bar{\sigma}^I)$ or $(f', \sigma', \bar{\sigma}') = (f'^I, \sigma'^I, \bar{\sigma}'^I)$ for the first time.

§3. Nonlinear σ model

In this section we limit our discussion to the real space where all the x's are real. By introduction of the coordinate transformation

$$x_1' = a_{11}x_1 + a_{12}x_2, \quad x_2' = a_{21}x_1 + a_{22}x_2, \quad a_{11}a_{22} - a_{12}a_{21} = 0,$$

$$x_3' = a_{33}x_3 + a_{34}x_4, \quad x_4' = a_{43}x_3 + a_{44}x_4, \quad a_{33}a_{44} - a_{34}a_{43} = 0, \qquad (3.1)$$

Eqs.(2.6) can then be rewritten as

$$f\Delta f - (\nabla f)^2 + \nabla\sigma\cdot\nabla\bar{\sigma} = 0, \qquad (3.2a)$$

$$\nabla\cdot(f^{-2}\nabla\sigma) = 0 \quad \text{and c.c.} \qquad (3.2b)$$

By using the same procedure, Eqs.(2.10) reduce to

$$f'\Delta f' - (\nabla f')^2 = \nabla\sigma'\cdot\nabla\bar{\sigma}' = 0, \qquad (3.3a)$$

$$\nabla\cdot(f'^{-2}\nabla\sigma') = 0. \qquad (3.3b)$$

It is shown in the following that Eqs.(3.2) and (3.3) can be reduced to equations for the O(3,1) and O(1,3) nonlinear σ models. We consider these two cases separately.

(i) O(3,1) nonlinear σ model

It is a straightforward matter to show that by taking

$$f = 1/(\xi^0 + \xi^3), \quad \sigma = (\xi^1 + i\xi^2)/(\xi^0 + \xi^3) \qquad (3.4)$$

with

$$\langle\xi,\xi\rangle \equiv (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 - (\xi^0)^2 = -1, \qquad (3.5)$$

we can rewrite Eqs.(3.2) as

$$\Delta\xi^\mu - \langle\nabla\xi,\nabla\xi\rangle\,\xi^\mu = 0, \qquad \mu = 0, 1, 2, 3, \qquad (3.6)$$

where

$$\langle\nabla\xi,\nabla\xi\rangle \equiv (\nabla\xi^1)^2 + (\nabla\xi^2)^2 + (\nabla\xi^3)^2 - (\nabla\xi^0)^2. \qquad (3.5')$$

We can also introduce three angles of rotation θ, φ, α:

$$\xi^0 = \cosh\theta, \quad \xi^1 + i\xi^2 = \sinh\theta\,\sin\varphi\,\exp(i\alpha), \quad \xi^3 = \sinh\theta\,\cos\varphi \qquad (3.7)$$

to rewrite Eqs.(3.6) as

$$\nabla\cdot(\sinh^2\theta\,\sin^2\varphi\,\nabla\alpha)=0, \tag{3.8a}$$

$$\nabla\cdot(\sinh^2\theta\,\nabla\varphi)-\sinh^2\theta\,\sin\varphi\,\cos\varphi\,(\nabla\alpha)^2=0, \tag{3.8b}$$

$$\Delta\theta-\sinh\theta\cosh\theta\,(\nabla\varphi)^2-\sinh\theta\cosh\theta\,\sin^2\varphi\,(\nabla\alpha)^2=0. \tag{3.8b}$$

As an integrable two-dimensional system associated with the O(3.1) nonlinear σ model, we can consider the following equations

$$u_{xt}-\frac{u_x u_t}{uu^*+1}-u(uu^*+1)=0, \tag{3.9}$$

or

$$(\varphi_t\tanh^2\theta)_x+(\varphi_x\tanh^2\theta)_t=0, \tag{3.10a}$$

$$\theta_{xt}-\sinh\theta\cosh\theta-\frac{\sinh\theta}{\cosh^3\theta}\varphi_x\varphi_t=0, \tag{3.10b}$$

with $\varphi=\sinh\theta\exp(i\varphi)$. Equations (3.9) and (3.10) are, respectively, a hyperbolic-version of the Getmanov equation and the Pohlmeyer-Lund-Regge equations.[6]

Several specific cases are considered below associated with the O(3.1) nonlinear σ model.

(i.1) O(2,1) nonlinear σ model

It is well known that if α is taken to be constant, Eqs.(3.8) are reduced to the form identical to the Ernst equation[7] for the axisymmetric gravitational field problem

$$f\Delta f-(\nabla f)^2+(\nabla\sigma)^2=0, \qquad \nabla\cdot(f^{-2}\nabla\sigma)=0, \tag{3.11a}$$

$$(\xi\xi^*-1)\Delta\xi=2\xi^*(\nabla\xi)^2, \qquad f+i\sigma=(\xi-1)/(\xi+1). \tag{3.11b}$$

This can also be regarded as equations for O(2.1) nonlinear σ model field equations

$$\Delta\phi^\mu+\langle\nabla\phi,\nabla\phi\rangle\phi^\mu=0, \qquad \mu=1,2,3, \tag{3.12}$$

with

$$\langle\phi,\phi\rangle\equiv(\phi^1)^2+(\phi^2)^2-(\phi^3)^2=-1,\quad \langle\nabla\phi,\nabla\phi\rangle=(\nabla\phi^1)^2+(\nabla\phi^2)^2-(\nabla\phi^3)^2, \tag{3.13}$$

or

$$\nabla\cdot(\sinh^2\theta\,\nabla\varphi)=0, \qquad \Delta\theta-\sinh\theta\cosh\theta\,(\nabla\varphi)=0. \tag{3.14}$$

The relationship which exists among the ϕ's, ζ, θ, ψ is

$$\phi^1 = (\zeta+\zeta^*)/(\zeta\zeta^*-1) = \sinh\theta\cos\varphi, \qquad (3.15a)$$

$$\phi^2 = (1/i)(\zeta-\zeta^*)/(\zeta\zeta^*-1) = \sinh\theta\sin\varphi, \qquad (3.15b)$$

$$\phi^3 = (\zeta\zeta^*+1)/(\zeta\zeta^*-1) = \cosh\theta. \qquad (3.15c)$$

Another case exists in which Eqs.(3.8) reduce to Eqs.(3.14). This is the case $\varphi = \pi/2$, which implies that $f^2 + \sigma\bar{\sigma} = 1$. This amounts to parametrizing f, σ, $\bar{\sigma}$ as

$$f = \operatorname{sech}\theta, \quad \sigma = \tanh\theta\exp(i\alpha), \quad \bar{\sigma} = \tanh\theta\exp(-i\alpha), \qquad (3.16)$$

for which Eqs.(3.2) take the form

$$(1-\sigma\bar{\sigma})\Delta\sigma + \bar{\sigma}(\nabla\sigma)^2 + \sigma\nabla\sigma\cdot\nabla\bar{\sigma} = 0. \qquad (3.17)$$

Equation (3.17) is somewhat similar to the Ernst equation (3.11b).

(i.2) O(3) nonlinear σ model

For the specific case

$$q^0 = \text{const} > 1 \quad \text{or} \quad \theta = \text{const}, \qquad (3.18)$$

for which Eqs.(3.5) and (3.7) become

$$\langle\xi,\xi\rangle \equiv (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = 1, \qquad (3.19)$$

$$\langle\nabla\xi,\nabla\xi\rangle \equiv (\nabla\xi^1)^2 + (\nabla\xi^2)^2 + (\nabla\xi)^3. \qquad (3.20)$$

Eqs.(3.6) are equations for the O(3) nonlinear σ model. Equations (3.8) then reduce to

$$\nabla\cdot(\sin^2\varphi\,\nabla\alpha) = 0, \qquad (3.21a)$$

$$\Delta\varphi - \sin\varphi\cos\varphi\,(\nabla\alpha)^2 = 0. \qquad (3.21b)$$

In terms of

$$\mu = \cot(\theta/2)\exp(i\alpha) \qquad (3.22)$$

or

$$\xi^1 = \sin\varphi\cos\alpha = (\mu+\mu^*)/(\mu\mu^*+1), \tag{3.23a}$$

$$\xi^2 = \sin\varphi\sin\alpha = (1/i)(\mu-\mu^*)/(\mu\mu^*+1), \tag{3.23b}$$

$$\xi^3 = \cos\varphi = (\mu\mu^*-1)/(\mu\mu^*+1), \tag{3.23c}$$

Eqs.(3.21) are rewritten as

$$(\mu\mu^*+1)\Delta\mu = 2\mu^*(\nabla\mu)^2. \tag{3.24}$$

This is similar in form to the Ernst equation.

(ii) O(1.3) nonlinear σ model

For Eqs.(3.3) a parametrization identical in form to Eqs.(3.4) can be made with

$$\langle\xi,\xi\rangle \equiv (\xi^0)^2-(\xi^1)^2-(\xi^2)^2-(\xi^3)^2 = -1, \tag{3.25}$$

$$\langle\nabla\xi,\nabla\xi\rangle = (\nabla\xi^0)^2-(\nabla\xi^1)^2-(\nabla\xi^2)^2-(\nabla\xi^3)^2. \tag{3.26}$$

Equations (3.6) then hold as they stand. In terms of θ, φ, α the q's here are also parametrized as

$$\xi^0 = \sinh\theta,\quad \xi^1 = \cosh\theta\sin\varphi\cos\alpha,\quad \xi^2 = \cosh\theta\sin\varphi\sin\alpha,\quad \xi^3 = \cosh\theta\cos\varphi. \tag{3.27}$$

Equations (3.6) then rewritten as

$$\nabla\cdot(\cosh^2\theta\sin^2\varphi\,\nabla\alpha) = 0, \tag{3.28a}$$

$$\nabla\cdot(\cosh^2\theta\,\nabla\varphi) - \cosh^2\theta\sin\varphi\cos\varphi(\nabla\alpha)^2 = 0, \tag{3.28b}$$

$$\Delta\theta + \cosh\theta\sinh\theta(\nabla\varphi)^2 + \cosh\theta\sinh\theta\sin^2\varphi(\nabla\alpha)^2 = 0. \tag{3.28c}$$

As in the case (i), we consider in the following several specific cases.

(ii,) O(1,2) nonlinear σ model

For α=const and $\varphi=\pi/2$, Eqs.(3.28) reduce to

$$\nabla\cdot(\cosh^2\theta\,\nabla\varphi) = 0, \tag{3.29a}$$

$$\Delta\theta + \cosh\theta\sinh\theta(\Delta\varphi)^2 = 0, \tag{3.29b}$$

and the same equations with φ replaced by α , respectively. Equations (3.29) can be regarded as equations for O(1,2) nonlinear σ model satisfying Eqs.(3.12) with the inner product $\langle\phi, \phi\rangle$ and $\langle\nabla\phi, \nabla\phi\rangle$, however, defined by

$$\langle\phi,\phi\rangle = (\phi^1)^2-(\phi^2)^2-(\phi^3)^2 = -1, \tag{3.30}$$

$$\langle\nabla\phi,\nabla\phi\rangle \equiv (\nabla\phi^1)^2-(\nabla\phi^2)^2-(\nabla\phi^3)^2. \tag{3.31}$$

Here the ϕ's are parametrized as follows

$$\phi^1 = \sinh\theta, \quad \phi^2 = \cosh\theta\sin\varphi, \quad \phi^3 = \cosh\theta\cos\varphi. \tag{3.32}$$

The case $\alpha = \pi/2$ amounts to parametrizing f' and σ' as

$$f' = \operatorname{cosech}\theta, \quad \sigma' = \coth\theta\exp(i\alpha). \tag{3.33}$$

In this specific case Eqs.((3.3) reduce to

$$(\sigma'\bar{\sigma}'-1)\Delta\sigma' - \bar{\sigma}'(\nabla\sigma')^2 - \sigma'\nabla\sigma'\cdot\nabla\bar{\sigma}' = 0. \tag{3.34}$$

Equations (3.29) have the same form as the equations obtained by Matzner and Misner in the formulation of the axisymmetric gravitational field problem.[8)]

(ii,2) O(3) nonlinear σ model

As in the case (i) Eq.(3.28) for the specific case (3.18) reduce to Eq.(3.21) corresponding to the equations for the O(3) nonlinear model.

We have thus shown that the Yang equations (2.6) are very generic, which include nonlinear field equations for the axisymmetric gravitational field problem, spin problems in solid state physics, differential geometry, etc. as specific cases. Thus, it is convenient to use a correspondence between these problems and the Yang-Mills theory. Some of such correspondences has been exploited to solve, say, the Yang equations for the purpose of obtaining pseudo-particle solutions.[9)]

§4. Dimensional reduction and pseudo-particle resonances

Let us assume in this section that the quantities f, σ, $\bar{\sigma}$ appearing in Eqs.(3.6) depend on the coordinates y, $\bar{y}$, z, $\bar{z}$ only through a pair of variables α and β, namely

$$f = f(\alpha, \beta), \qquad \sigma = \sigma(\alpha, \beta), \qquad \bar{\sigma} = \bar{\sigma}(\alpha, \beta) \tag{4.1}$$

with

$$\alpha = \alpha(y, \bar{y}, z, \bar{z}), \qquad \beta = \beta(y, \bar{y}, z, \bar{z}). \tag{4.2}$$

Equations (3.2a) then reduce to

$$\begin{aligned} &f\left[f_\alpha \tilde{\Delta}\alpha + f_\beta \tilde{\Delta}\beta + f_{\alpha\alpha}(\tilde{\nabla}\alpha)^2 + f_{\beta\beta}(\tilde{\nabla}\beta)^2 + f_{\alpha\beta}(\tilde{\nabla}\alpha\cdot\hat{\nabla}\beta + \hat{\nabla}\beta\cdot\tilde{\nabla}\alpha)\right] \\ &- f_\alpha^2\, \tilde{\nabla}\alpha\cdot\hat{\nabla}\alpha - f_\beta^2\, \tilde{\nabla}\beta\cdot\hat{\nabla}\beta + \sigma_\alpha \bar{\sigma}_\alpha\, \hat{\nabla}\alpha\cdot\tilde{\nabla}\alpha + \sigma_\beta \bar{\sigma}_\beta\, \hat{\nabla}\beta\cdot\tilde{\nabla}\beta \\ &- f_\alpha f_\beta(\tilde{\nabla}\alpha\cdot\hat{\nabla}\beta + \hat{\nabla}\beta\cdot\tilde{\nabla}\alpha) + \sigma_\alpha \bar{\sigma}_\beta\, \tilde{\nabla}\alpha\cdot\hat{\nabla}\beta + \bar{\sigma}_\alpha \sigma_\beta\, \tilde{\nabla}\beta\cdot\hat{\nabla}\alpha = 0, \end{aligned} \tag{4.3a}$$

where

$$\tilde{\Delta}\alpha \equiv \alpha_{y\bar{y}} + \alpha_{z\bar{z}}, \qquad (\hat{\nabla}\alpha)^2 \equiv \alpha_y \alpha_{\bar{y}} + \alpha_z \alpha_{\bar{z}}, \tag{4.4}$$

$$\tilde{\nabla}\alpha\cdot\hat{\nabla}\beta \equiv \alpha_y \beta_{\bar{y}} + \alpha_z \beta_{\bar{z}}, \qquad \hat{\nabla}\beta\cdot\tilde{\nabla}\alpha \equiv \beta_y \alpha_{\bar{y}} + \beta_z \beta_{\bar{z}}. \tag{4.5}$$

Equations (4.3a) reduce to a two-dimensional form of Eq.(3.2a):

$$f(f_{\xi\xi} + f_{\eta\eta}) - f_\xi^2 - f_\eta^2 + \sigma_\xi \bar{\sigma}_\xi + \sigma_\eta \bar{\sigma}_\eta = 0 \tag{4.6a}$$

with

$$\xi = \ln \alpha \quad \text{and} \quad \eta = \ln \beta. \tag{4.7}$$

provided the quantities α and β satisfy the equations

$$\tilde{\Delta}\alpha = \alpha, \qquad (\hat{\nabla}\alpha)^2 = \alpha^2; \tag{4.8a}$$

$$\tilde{\Delta}\beta = \beta, \qquad (\hat{\nabla}\beta)^2 = \beta^2. \tag{4.8b}$$

under the restriction

$$\tilde{\nabla}\alpha\cdot\hat{\nabla}\beta = \hat{\nabla}\beta\cdot\tilde{\nabla}\alpha = 0. \tag{4.9}$$

By the use of the same procedure, Eq.(3.2b) reduces to

$$(f^{-2}\sigma_\zeta)_\zeta + (f^{-2}\sigma_y)_y = 0, \quad (f^{-2}\bar{\sigma}_\zeta)_\zeta + (f^{-2}\sigma_y)_y = 0, \tag{4.6b}$$

Particular solutions to Eqs.(4.8) and (4.9) are given by

$$\alpha = \sum_i \exp(p_i y + \bar{p}_i \bar{y} + \delta_i z + \bar{\delta}_i \bar{z}), \quad \beta = \sum_i \exp(p_i' y + \bar{p}_i' \bar{y} + \delta_i' z + \bar{\delta}_i' \bar{z}) \tag{4.10}$$

with

$$p_i \bar{p}_i + \delta_i \bar{\delta}_i = 1, \qquad p_i \bar{p}_j + \delta_i \bar{\delta}_j = 1, \tag{4.11a}$$

$$p_i' \bar{p}_i' + \delta_i \bar{\delta}_i = 1, \qquad p_i' \bar{p}_j' + \delta_i' \bar{\delta}_j' = 1, \tag{4.11b}$$

$$p_i \bar{p}_j' + \delta_i \bar{\delta}_j' = 0, \qquad \bar{p}_i p_j' + \bar{\delta}_i \delta_j' = 0. \tag{4.11c}$$

Thus, we have reduced the Yang equation (2.6) defined in 4d Euclidean space to the corresponding effective 2d equation given by Eqs.(4.6). Here the first of Eqs.(4.11a) and (4.11b) represent dispersion relations of pseudo-particles, while the second of the same equations can be considered as representing resonance interactions, in analogy with the problem of resonance in conventional soliton problems.[10)]

If the field variables f, σ, $\bar{\sigma}$ are taken to depend only on the single variable α or ζ, Eqs.(2.6) can be integrated. Eqs.(4.6) then reduce to

$$f f_{\zeta\zeta} - f_\zeta^2 + c\bar{c} f^4 = 0, \qquad \sigma_\zeta = c f^2, \quad \bar{\sigma}_\zeta = \bar{c} f^2 \tag{4.12}$$

where c and c are constants. A particular solution to Eqs.(4.12) is obtained as follows

$$f = \operatorname{sech}(A\zeta + B), \quad \sigma = C \tanh(A\zeta + \beta), \quad \bar{\sigma} = \bar{C} \tanh(A\zeta + \beta) \tag{4.13}$$

where A, B and C(C) are constants, the first and the third being dependent on the constants c and c. This is a kind of instanton solution, the examination of which is, however, worth separate discussion.

§5. Examples of pseudo-particle solutions

By the use of the Backlund transformation (2.17) or (2.18) and the method described in § 4, we can obtain various types of particular solutions to the SU(2)-gauge field equations. For the sake of simplicity, however, we give here only few of such solutions obtainable from Eqs. (4.6a) and (4.6b).

(i) vortex-like solutions

By the use of the parametrization (3.4) and (3.7), Eqs. (4.6a) and (4.6b) corresponding to the O(2.1) and O(3) nonlinear σ model given by Eqs. (3.14) and (3.21) take the form

$$\begin{cases} (\sinh^2\theta\,\varphi_\xi)_\xi + (\sinh^2\theta\,\varphi_\eta)_\eta = 0 & (5.1a) \\ \theta_{\xi\xi} + \theta_{\eta\eta} - \sinh\theta\cosh\theta\,(\varphi_\xi^{\,2} + \varphi_\eta^{\,2}) = 0 & (5.1b) \end{cases}$$

$$(\sin^2\theta\,\varphi_\xi)_\xi + (\sin^2\theta\,\varphi_\eta)_\eta = 0 \tag{5.2a}$$

$$\theta_{\xi\xi} + \theta_{\eta\eta} - \sin\theta\cos\theta\,(\varphi_\xi^{\,2} + \varphi_\eta^{\,2}) = 0 \tag{5.2b}$$

There exist multi-vortex solutions to these two equations which are characterized by the Laplace equation

$$\Delta\varphi = 0 \tag{5.3}$$

with the solution given by

$$\varphi = \sum_i q_i \tan^{-1}\left(\frac{\eta - \eta_i}{\xi - \xi_i}\right) \tag{5.4}$$

with

$$q_i = \pm 1, \pm 2, \ldots\ldots$$

under the assumption that the quantity θ depend on ξ and η only through a quantity ψ identified as a stream function in hydrodynamics, namely

$$\theta = \theta(\psi) \quad \text{with} \quad \partial\varphi/\partial\xi = \partial\psi/\partial\eta, \quad \partial\varphi/\partial\eta = -\partial\psi/\partial\xi \tag{5.5}$$

$$\psi = -\sum_i q_i \ln\left[(\xi - \xi_i)^2 + (\eta - \eta_i)^2\right]^{1/2} \tag{5.6}$$

The solutions for θ is then given by

$$\theta = 2\tan^{-1}(\exp\psi) \qquad \text{and} \qquad \theta = 2\tanh(\exp\psi) \tag{5.7}$$

for Eqs.(5.1) and (5.2), respectively. The solutions so obtained are entirely identical in form to the multi-instanton solutions for the O(3) nonlinear σ model obtained by Belavin and Polyakov.[11] Several arguments have been put forward to the similarity of the 4d-gauge theory and the 2d O(3) nonlinear σ model. The above result may show a direct connection between these two cases.

By means of dimensional reduction, we can even consider a partially integrable SU(2) gauge field by studying an integrable 2d O(3.1) nonlinear σ model. Here, instead of Eqs.(3.10), we are concerned with the equations

$$(\varphi_\xi \tanh\theta)_\eta + (\varphi_\eta \tanh\theta)_\xi = 0, \tag{5.8a}$$

$$\theta_{\xi\eta} - \sinh\theta\cosh\theta - \frac{\sinh\theta}{\cosh^3\theta}\varphi_\xi\varphi_\eta = 0. \tag{5.8b}$$

A solution to Eqs.(5.8) is being studied.

(ii) string-like solutions

The method described in §4 can be generalized to the case of three variables. Namely, we assume the field variables f, σ, $\bar{\sigma}$ depend on y,y,z,z only through the quantities α, β and γ. Then, under the condition similar to Eqs.(4.10) and (4.11), we can consider, instead of Eqs.(5.1), the following equations

$$(\sinh^2\theta\,\varphi_\xi)_\xi + (\sinh^2\theta\,\varphi_\eta)_\eta + (\sinh^2\theta\,\varphi_\zeta)_\zeta = 0 \tag{5.9a}$$

$$\theta_{\xi\xi} + \theta_{\eta\eta} + \theta_{\zeta\zeta} - \sinh\theta\cosh\theta(\varphi_\xi^2 + \varphi_\eta^2 + \varphi_\zeta^2) = 0 \tag{5.9b}$$

with

$$\zeta = \ln\gamma. \tag{5.10}$$

By string-like solutions we mean here the solution of the form

$$\theta = \theta(\xi, \eta), \quad \varphi = v_0 \zeta \qquad v_0\text{: const.} \tag{5.11}$$

Equation (5.9a) is then automatically satisfied, while Eq.(5.9b) reduces to the 2d static sinh-Gordon equation:

$$(2\theta)_{\xi\xi} + (2\theta)_{\eta\eta} = v_0^2 \sinh 2\theta \tag{5.12}$$

For Eq.(5.12) at least two kinds of solution exist, one is the multi-soliton solution due to Hirota[12] and another is the one somewhat similar to vortex solutions in 2d hydrodynamics. The former and the latter are given, respectively, by

$$\theta = 2 \tanh^{-1} \left(\frac{e^{X_1'} + e^{X_2'} + \cdots + a_{12} a_{23} a_{31} e^{X_1' + X_2' + X_3'} + \cdots}{1 + a_{12} e^{X_1' + X_2'} + a_{23} e^{X_2' + X_3'} + \cdots} \right) \tag{5.13}$$

$$\theta = 2 \tan^{-1} \left[\sqrt{\frac{K^2}{1-K^2}} \cdot \frac{\sinh (1-K^2)^{1/2} \eta'}{\sinh K \xi'} \right] \tag{5.14}$$

where

$$X_i' = K_i \xi' + L_i \eta' \qquad \text{with} \qquad \xi' = v_0 \xi, \quad \eta' = v_0 \eta \tag{5.15}$$

$$a_{ij} = \left[(K_i - K_j)^2 + (L_i - L_j)^2 \right] \Big/ \left[(K_i + K_j)^2 + (L_i + L_j)^2 \right]. \tag{5.15'}$$

Besides these particular cases, we can consider many nonlinear partial differential equations of physico-mathematical interest, a detailed discussion on which is entirely omitted. Among these, most interesting in particle physics is of course finite-action solutions. Such a solution is out of the scope of the present investigation.

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