

BIMEROMORPHIC GEOMETRY OF NORMAL GORENSTEIN SURFACES

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The purpose of this note is to provide a "bimeromorphic geometry" in the category of normal Gorenstein surfaces. Details will be discussed elsewhere.

§1. Normal Gorenstein Surfaces

An isolated singularity y of a surface Y is said to be Gorenstein if there exists a neighborhood U of y satisfying (i) the restriction map $O(U) \rightarrow O(U \setminus y)$ is bijective (i.e., y is normal), (ii) \exists a nowhere vanishing holomorphic 2-form on $U \setminus y$.¹⁾

Example. Hypersurface singularities are always Gorenstein. In particular, the following singularities are called rational double points.

1) In the 2-dimensional case, we have the equivalence:

$$\left\{ \begin{array}{l} \text{an isolated Gorenstein} \\ \text{singularity} \end{array} \right\} = \left\{ \begin{array}{l} \text{a normal Gorenstein} \\ \text{singularity} \end{array} \right\}$$

For ring theoretic aspects, we refer to the expository article by S.Goto: On Gorenstein rings (in Japanese), Sugaku 31(1979).

$$\begin{aligned}
A_n &: x^2 + y^2 + z^{n+1} = 0 & n \geq 1 \\
D_n &: x^2 + y^2 + z + z^{n-1} = 0 & n \geq 4 \\
E_6 &: x^2 + y^3 + z^4 = 0 \\
E_7 &: x^2 + y^3 + yz^3 = 0 \\
E_8 &: x^2 + y^3 + z^5 = 0
\end{aligned}$$

In what follows, by a normal Gorenstein surface we shall mean a 2-dimensional, reduced, irreducible compact complex space with isolated Gorenstein singularities.

Let Y be a normal Gorenstein surface. The singular locus $\text{Sing}(Y)$ consists of finite points $\{y_1, \dots, y_s\}$. The sheaf of holomorphic 2-forms Ω^2 on $Y \setminus \text{Sing}(Y)$ extends naturally as an invertible sheaf ω_Y to Y . This sheaf is nothing but the Grothendieck dualizing sheaf of Y . We denote by K_Y a canonical divisor (bundle, in general) of Y corresponding to the sheaf ω_Y . For positive integers m , we define the arithmetic m -genus by

$$\bar{P}_m(Y) = \dim H^0(Y, \omega_Y^{\otimes m}). \quad 2)$$

2) We borrowed the bar notation from Wilson, P.M.H.: The arithmetic plurigenera of surfaces. Math. Proc. Camb. Phil. Soc. 85 (1979), 25-31. If there is no danger of confusion, we may omit the bar as in a paper of Brenton, L.: On singular complex surfaces with vanishing geometric genus and pararational singularities. Compositio Math. 43 (1981), 297-315.

§2. Local Properties (For instance, see [8])

Let $\pi: X \rightarrow Y$ be the minimal resolution in the sense that there exist no exceptional curves of the first kind in $\pi^{-1}(\text{Sing}(Y))$. Then there exists a unique effective divisor Δ such that

$$\pi^*\omega_Y \cong \mathcal{O}(K_X + \Delta).$$

Furthermore if $\Delta > 0$, we easily find the property:

$$\omega_\Delta \cong \mathcal{O}_\Delta. \quad 3)$$

Let $\Delta = \sum \Delta_i$ be the decomposition so that each Δ_i is supported in $\pi^{-1}(y_i)$. We have $\Delta_i = 0$ if and only if y_i is a rational double point. In case $\Delta_i > 0$, we have $\text{Supp}(\Delta_i) = \pi^{-1}(y_i)$ and we find

$$\dim(\mathcal{R}^1 \pi_* \mathcal{O}_X)_{y_i} = \dim H^1(\Delta_i, \mathcal{O}_{\Delta_i}).$$

So y_i is a rational singularity if and only if it is a rational double point.

Remark. The converse process is possible. Let Δ be a connected curve on a non-singular surface X satisfying

(i) the intersection matrix of irreducible components of Δ is negative definite,

(ii) $\omega_\Delta \cong \mathcal{O}_\Delta$.

Then Δ can be contracted to an isolated Gorenstein singularity (Laufer[6], Bădescu[4]).

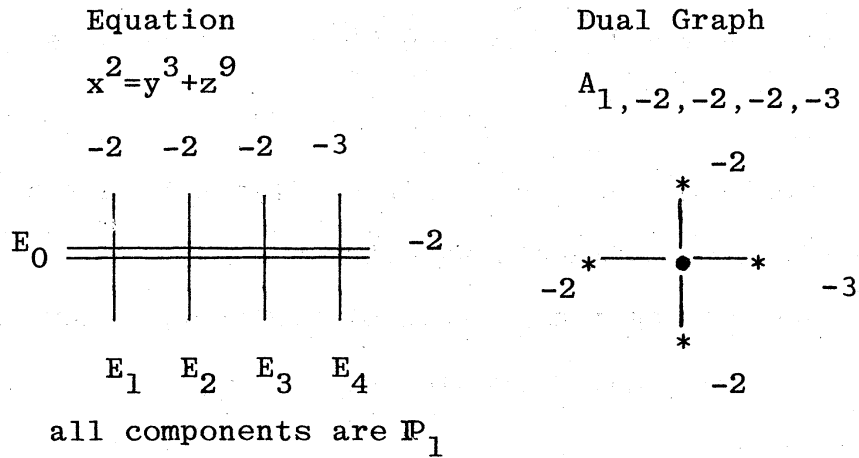
3) By definition $\omega_\Delta = \text{Ext}_{\mathcal{O}_X}^1(\mathcal{O}_\Delta, \omega_X)$ which is isomorphic to the sheaf $\mathcal{O}(K_X + \Delta) \otimes \mathcal{O}_\Delta$.

The divisors Δ_i can be calculated as follows. Let E_1, \dots, E_r be the irreducible components of $\pi^{-1}(y_i)$. If $\Delta_i = \sum n_j E_j$, then (n_1, \dots, n_r) is a solution of the equations:

$$(K_X + \sum n_j E_j) E_k = 0 \quad \text{for } k=1, \dots, r.$$

Since the intersection matrix $(E_j E_k)$ is negative definite, the solution is unique. We also get the non-negativeness of each n_j from the fact that the resolution is minimal. Furthermore if y_i is Gorenstein, the solution must be integral.

Example. For a minimally elliptic singularity (i.e., the case where $\dim H^1(\Delta_i, \mathcal{O}_{\Delta_i}) = 1$), the divisor Δ_i coincides with the fundamental cycle of y_i (cf. [6]).



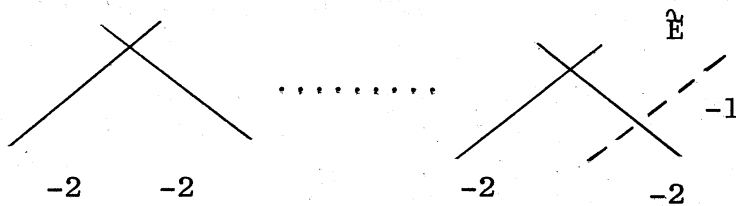
In this example, we have

$$\Delta = 2E_0 + E_1 + E_2 + E_3 + E_4.$$

§3. Generalized Blowing Ups

We shall introduce (generalized) blowing ups for normal Gorenstein surfaces. Namely, we define a blowing up \tilde{Y} of Y at a point y in the following way.

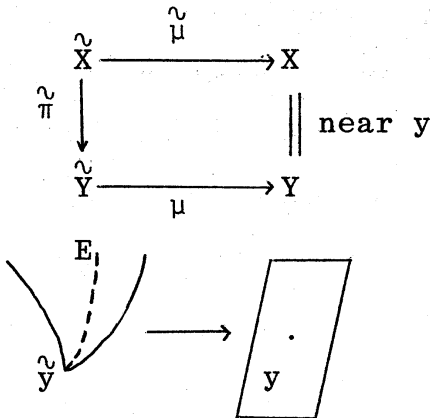
Case(i). y is a non-singular point. Take a sequence of usual blowing ups n -times so that the exceptional curves form



By contracting the (-2) curves ⁴⁾ to a rational double point \tilde{y} (type A_{n-1}), we obtain a blowing up $\mu: \tilde{Y} \rightarrow Y$. In case $n=1$, this is nothing but the usual one. If \tilde{E} denotes the (-1) curve, then μ contracts the curve $E = \pi(\tilde{E})$ to y . We find

$$\omega_{\tilde{Y}} \cong \mu^* \omega_Y \otimes \mathcal{O}(E).$$

In this case E is a Cartier divisor.



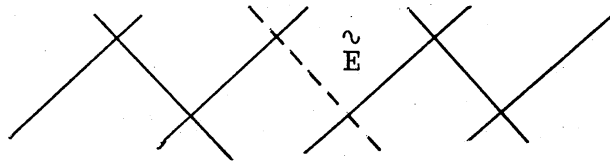
⁴⁾ For simplicity's sake, we use the terminology:

$$(-k) \text{ curve} = \left\{ \begin{array}{l} \text{a non-singular rational curve} \\ \text{with self-intersection number } -k \end{array} \right\}$$

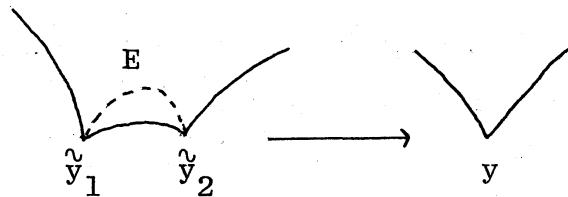
Case(ii). y is a rational double point. Then $\pi^{-1}(y)$ is a union of (-2) curves. Let $\tilde{\pi}:X \rightarrow \tilde{Y}$ be the contraction of these curves except one curve \tilde{E} . We thus obtain a blowing up $\mu:\tilde{Y} \rightarrow Y$ which contracts the curve $E=\tilde{\pi}(\tilde{E})$ to y . In this case E need not be a Cartier divisor. We have the formula:

$$\omega_{\tilde{Y}}^{\vee} \cong \mu^* \omega_Y.$$

Example. Suppose $\pi^{-1}(y)$ is of type A_7 . Let \tilde{E} be the central curve.



In Y , we have two points \tilde{y}_1, \tilde{y}_2 of type A_3 .

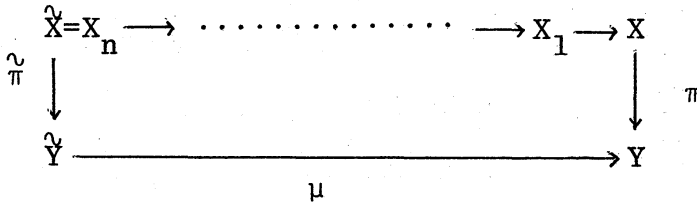


Case(iii). y is a non-rational singularity. Choose a point x_1 in Δ . Let $\tilde{\mu}_1:X_1 \rightarrow X$ be the usual blowing up at x_1 . Put $\Delta_1 = \tilde{\mu}_1^* \Delta - E_1$ where the E_1 is the exceptional curve. If the multiplicity of Δ at x_1 is greater than or equal to two, we have $E_1 \subset \Delta_1$. Choose again a point x_2 in E_1 and let $\tilde{\mu}_2:X_2 \rightarrow X_1$ be the usual blowing up at x_2 . We continue in this way so that $v_1 \geq 2, \dots, v_{n-1} \geq 2$, but $v_n = 1$ where each v_i denotes the multiplicity of Δ_{i-1} at x_i . Then $\tilde{\Delta} = \tilde{\mu}_n^* \Delta_{n-1} - E_n$ does not contain $\tilde{E} = E_n$, from which we infer that $\tilde{\Delta}$ contains no (-1) curves. By construction $\omega_{\tilde{\Delta}} \cong \mathcal{O}_{\tilde{\Delta}}$. Therefore $\tilde{\Delta}$ can be contracted to a Gorenstein singularity \tilde{y} . Thus we obtain a blowing up $\mu:\tilde{Y} \rightarrow Y$ where the curve $E = \tilde{\pi}(\tilde{E})$ is

contracted to y . We find

$$\omega_{\tilde{Y}} \cong \mu^* \omega_Y.$$

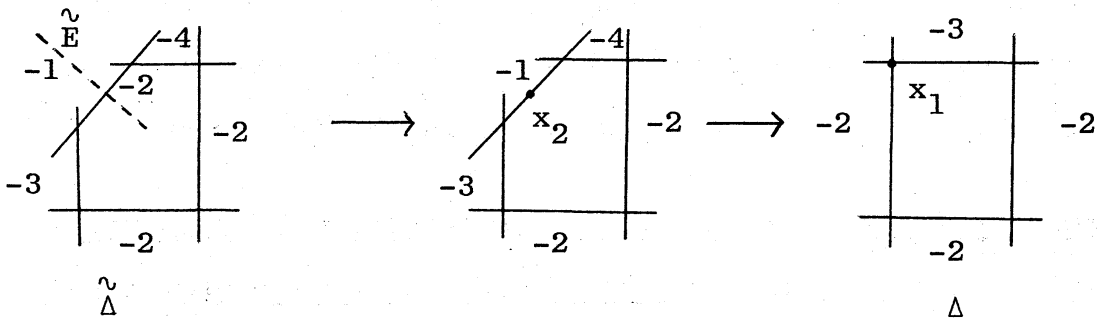
Note that E need not be a Cartier divisor.



Remark. The above defined blowing ups are characterized by the following properties (cf. §4):

- (i) \tilde{Y} is again a normal Gorenstein surface,
- (ii) μ contracts an irreducible curve E to a point y such that $\mu: \tilde{Y} \setminus E \rightarrow Y \setminus y$ is an isomorphism,
- (iii) $\mu_* \omega_{\tilde{Y}} \cong \omega_Y$.

Example.



$$z(xy+z^2) = x^5 + y^6$$

$$z^2 = (y+x^2)(y^2+x^8)$$

We borrowed the equations from [6]. All curves are non-singular rational curves. In this example, $v_1=2$, $v_2=1$.

§4. Factorization Theorem

Let Y, Y' be two normal Gorenstein surfaces. A bimeromorphic morphism $f: Y \rightarrow Y'$ is said to be canonical if $f_*\omega_Y \cong \omega_{Y'}$.

Lemma. Let $f: Y \rightarrow Y'$ be a canonical bimeromorphic morphism between normal Gorenstein surfaces. Then

$$f_*\omega_Y^{\otimes m} \cong \omega_{Y'}^{\otimes m} \quad \text{for } m > 0.$$

Proof. We infer from the hypothesis that $\omega_Y \cong f^*\omega_{Y'} \otimes \mathcal{O}(G)$ where the effective Cartier divisor G is contracted to points by f . It follows that $\omega_Y^{\otimes m} \cong f^*\omega_{Y'}^{\otimes m} \otimes \mathcal{O}(mG)$, which deduces the required result. Q.E.D.

Corollary. We have

$$\bar{P}_m(Y) = \bar{P}_m(Y').$$

In order to develop a geometry with invariants \bar{P}_m , it is therefore natural to restrict ourselves to canonical bimeromorphic morphisms. Our main observation is the following

Theorem. Any canonical bimeromorphic morphism between normal Gorenstein surfaces can be factored into a sequence of blowing ups (as defined in §3).

Remark. Between non-singular surfaces, a bimeromorphic morphism is automatically canonical. So the above theorem generalizes the well known fact that any bimeromorphic morphism between non-singular surfaces can be factored into a sequence of blowing ups.

As a consequence, we propose the following

Definition. Two normal Gorenstein surfaces Y and Y' are in the same "bimeromorphic class" if they are connected by blowing ups:

$$\mu_r \circ \dots \circ \mu_1(Y) = \mu'_s \circ \dots \circ \mu'_1(Y'),$$

where μ_i, μ'_j are blowing ups defined in §3.

Remark. The following conditions are equivalent for bimeromorphic morphisms $f: Y \rightarrow Y'$ between normal Gorenstein surfaces.

- (i) f is canonical,
- (ii) $R^1 f_* \mathcal{O}_Y = 0$.

This follows from the Grothendieck duality (Kempf's argument in [5], p.49) via the Grauert-Riemenschneider vanishing theorem for f (See §5).

§5. Generalities

Finally we collect some general results concerning normal Gorenstein surfaces. Let L be a line bundle on a normal Gorenstein surface Y .

I Riemann-Roch Theorem ([3])

$$\chi(\mathcal{O}(L)) = \frac{1}{2}(L^2 - KL) + \chi(\mathcal{O}_Y),$$

II Serre Duality

$$H^i(Y, O(K+L)) \cong H^{2-i}(Y, O(-L))^\vee,$$

III Kodaira-Ramanujam Vanishing Theorem

If (i) $L^2 > 0$, (ii) $LC \geq 0$ for all curves C on Y , then 5)

$$H^i(Y, O(K+L)) = 0 \text{ for } i > 0,$$

IV Grauert-Riemenschneider Vanishing Theorem (cf. [4])

If $f: Y \rightarrow Y'$ is a surjective morphism between normal Gorenstein surfaces, then

$$R^i f_* \omega_Y = 0 \text{ for } i > 0, \quad 6)$$

V Classification Theory

In the Moisëzon case (i.e., Y has two algebraically independent meromorphic functions, cf. [1]), we have an Enriques type classification. We refer the reader to [8].

5) If such a bundle L exists, then Y is a Moisëzon surface.

6) In general, Vanishing Theorems III, IV are valid for normal surfaces. We can employ the fact that the injective trace map

$$\pi_* \omega_X \hookrightarrow \omega_Y$$

has a cokernel supported on isolated singularities. Here X denotes the minimal resolution of Y . Hence, for instance

$$H^i(Y, \pi_* \omega_X \otimes O(L)) \rightarrow H^i(Y, \omega_Y \otimes O(L)) \rightarrow 0$$

for $i > 0$.

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