**56**

# Perturbations of compact foliations II

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Introduction.    A compact foliation F is one in which every leaf is compact.    The problem we wish to consider concerns foliations F' whose plane fields are close, in some $C^r$-topology, to the plane field tangent to the leaves of F.    Such an F' is called a $C^r$-perturbation.    Then the following question arises: When does F' have a compact leaf ?    The first result of this nature is due to H.Seifert [11].    He proved that any $C^0$-perturbation of the Hopf fibration $S^3 \longrightarrow S^2$ has a compact leaf.    In the same paper, he showed that the theorem is also true for orientable $S^1$-bundles over surfaces B of $\chi(B) \neq 0$, where $\chi(B)$ is the euler characteristic number of B. The result was generalized by F.Fuller [5] to orientable circle bundles over arbitrary closed manifolds B with $\chi(B) \neq 0$. R.Langevin and H.Rosenberg [9] considered a fibration p : E$\longrightarrow$B with fibre L, B a closed 2-manifold, E closed.    They proved that any $C^0$-perturbation of this fibration has a compact leaf when $\pi_1(L)$ is isomorphic to Z, B is a sueface with $\chi(B) \neq 0$ and $\pi_1(B)$ acts trivially on $\pi_1(L)$.    Furthermore the author [4] generalized the above result to compact codimension two foliations.    The purpose of this note is to give the proofs of the results in [4] and generalize the above result to fibrations with B of general dimensions.

§1.    Compact Hausdorff foliations and the generalized first return map.

Let M be a compact m-manifold without boundary and F a compact foliation of codimension q such that the leaf space $M/F$ is Hausdorff. Such a foliation F is called a compact Hausdorff foliation. Then we have a nice picture of the local behavior of F as follows.

Proposition 1 (D.B.A.Epstein[3]). There is a generic leaf $L_0$ with property that there is an open dense subset of M, where the leaves have all trivial holonomy and are all diffeomorphic to $L_0$. Given a leaf L, we can describe a neighborhood U(L) of L, together with the foliation on the neighborhood as follows. There is a finite group G(L) of O(q). G(L) acts freely on $L_0$ on the right and $L_0/G(L) \cong L$. Let $D^q$ be the unit q-disk. We foliate $L_0 \times D^q$ with leaves of the form $L_0 \times \{pt\}$. This foliation is preserved by the diagonal action of G(L), defined by $g(x,y) = (x \cdot g^{-1}, g \cdot y)$ for $g \in G(L)$, $x \in L_0$ and $y \in D^q$. So we have a foliation induced on $U = L_0 \underset{G(L)}{\times} D^q$. The leaf corresponding to $y = 0 \in D^q$ is $L_0/G(L)$. Then there is a $C^\infty$-imbedding $\varphi : U \longrightarrow M$ with $\varphi(U) = U(L)$, which preserves leaves and $\varphi(L_0/G(L)) = L$.

Definition 2. A leaf L is called singular if G(L) is not trivial. The order of G(L) is called the order of holonomy of L.

Definition 3. A singular leaf L is called isolated if the action of G(L) has only the origin of $D^q$ as fixed point.

From Proposition 1, we see that each isolated singular leaf is isolated, hence there are finitely many isolated singular leaves in F because of the compactness of M. Let S be the set of all non-isolated singular leaves of F. The leaf space $M/F$, which we denote by B, is a compact V-manifold of dimension q and the quotient map $\pi : M \longrightarrow B$ is a V-bundle( for definitions see I.Satake [10] ).

2

Let $L_1,\ldots,L_n$ denote all the isolated singular leaves of F with holonomy of order $k_1,\ldots,k_n$ respectively. Put $p_i = \pi(L_i)$ $(i=1,\ldots,n)$ and $S_B = \pi(S)$. Note that $\pi: M - S\cup\{L_i,\ldots,L_n\} \longrightarrow M - S\cup\{L_1,\ldots,L_n\}/F \cong B - S_B\cup\{p_1,\ldots,p_n\}$ is a natural fibration with generic leaf L as fibre. Thus $\pi_1( B - S_B\cup\{p_1,\ldots,p_n\} )$ acts on $\pi_1(L)$.

We assume that F satisfies the following conditions:

($C_1$) $\pi_1(L) \cong Z$ for every leaf L of F,

($C_2$) The codimension of each component of S in M is not equal to two.

Remark 4. From $(C_1)$, G(L)(L$\in$F) is isomorphic to a finite cyclic group. Hence S is a compact submanifold of M.

By Proposition 1, for each isolated singular leaf $L_i$, the restriction $\pi: \partial U(L_i) \cong L\widetilde{\underset{G(L_i)}{\times}}\partial D^q \longrightarrow \partial D^q/G(L_i)$ is a fibration with compact fibre L and

$$\pi_1( \partial D^q/G(L_i)) = \begin{cases} G(L_i) & ( q \geq 3), \\ Z & ( q = 2). \end{cases}$$

Thus we can see that $\pi_1(\partial D^q/G(L_i))$ acts trivially on $\pi_1(L)$ because that $G(L_i)$ is abelian. Furthermore we have $\pi_1(B - S_B) \cong \pi_1(B)$ from $(C_2)$, hence we may consider that $\pi_1(B)$ also acts on $\pi_1(L)$ under the conditions $(C_1)$ and $(C_2)$.

Proposition 5. Let M be a compact manifold without boundary and F a compact Hausdorff foliation of codimension q with leaf space B, satisfying the conditions $(C_1)$ and $(C_2)$. Furthermore we assume that $\pi_1(B)$ acts trivially on $\pi_1(L)$. Let F' be a $C^0$-perturbation of F. Then there exists a vector field X on M satisfying the following;

i) X is orthogonal to F except for singular points of X and

ii) if there is a point $x \in M$ such that $X(x) = 0$, then the leaf

3

L' of F' through x is compact.

Proof.   We shall associate to F' a diffeomorphism f : M⟶M

(a generalized first return map) as follows( see §1 of [9] ).   Fix a

Riemannian metric on M so that F is a Riemannian foliation.   Choose

$\varepsilon > 0$ so that for each x∈ M, the geodesics through x, of length $\varepsilon$ ,

and orthogonal to L(x), form a smoothly imbedded q-disk D(x), where

L(x) is a leaf through x.   We can suppose that for each leaf L,the

disks D(x), x∈ L, form a tubular neighborhood T(L) of L.   Fix a

generic leaf L and a point x∈ L.   Let $\alpha$ be a loop in L at x repre-

senting a generator of $\pi_1(L)$.   Then for F' close to F, $\alpha$ can be

lifted to a path on the leaf of F' through x, to a path starting at

x and ending at a point of D(x).   This end point is denoted by

H(F',$\alpha$ ).   H is the perturbed holonomy map(cf. M.W.Hirsch[6] ).

We define  f(x) = H(F',$\alpha$)(x).   Now if y is another point of L,

let $\beta$ be any path in L from x to y(the length of $\beta$ less than the

diameter of L) and define  f(y) = H(F',$\beta \cdot \alpha \cdot \beta^{-1}$)(y).   This definition

does not depend on $\beta$ and defines a smooth map f : L⟶M.

Next we extend f to a map f : T(L)⟶M  by using the product

structure in T(L) and transporting $\alpha$ to each leaf in T(L).   Since

$\pi$: M - S∪{$L_1$,...,$L_n$}⟶B - $S_B$∪{$p_1$,...,$p_n$} is a fibration with fibre

L and  $\pi_1(B)$ acts trivially on $\pi_1(L)$, we can extend f to a map

f : M - S∪{$L_1$,...,$L_n$}⟶M.   Now we shall extend to each $L_i$(i=1,...,

n) and S.   Let L be a generic leaf in U($L_i$)(resp. U($L_s$), $L_s$∈ S)

and $\alpha$ a loop at x in L representing a generator of $\pi_1(L)$.   We

have a natural projection $j_i$ : L⟶$L_i$ which is a $k_i$-fold covering

(i=1,...,n and s; $k_s$=2).   Let $\bar{\alpha}_i = j_i(\alpha)$ be a loop at $\bar{x} = j_i(x)$

in $L_i$.   Then we define  f($\bar{x}$) = H(F', $\bar{\alpha}_i$)($\bar{x}$)  for $\bar{x}$∈$L_i$.   This is

well-defined since $\pi_1(L_i) \cong Z$. Then we see that the extended map

f : M⟶M is a smooth diffeomorphism. We can see that if f(x)= x

for some x ∈ M, then the leaf of F' through x is compact(see [8] ).

We associate to F a vector field X whose zero's give compact leaves.

We have x and f(x) in the geodesic disk D(x) for x ∈ M. Let X(x)

be the vector tangent to the geodesic in D(x) from to f(x). Note

that X is orthogonal to F. We easily see that if f has no fixed

point, then X is never zero. This completes the proof.

§2. Statement of results for compact codimension two foliations.

Let F be a compact codimension two foliation with isolated sin-

gular leaves $L_1,\ldots,L_n$ of holonomy order $k_1,\ldots,k_n$ respectively.

Furthermore we assume that $\pi_1(L) \cong Z$ for every leaf L of F. We

let F' be a small perturbation of F. Then by the result of

M.W.Hirsch( [6] , Theorem 1.1), we have the following: For each $\overline{U(L_i)}$, $(i=1,\cdots,n)$,

$F'|_{U(L_i)}$ has a compact leaf $L'_i$ in $U(L_i)$ such that there is a diffeo-

morphism $h_i : L_i \longrightarrow L'_i$. We remark that F' has at least n compact

leaves. Let $\alpha$ be a loop in a generic leaf L representing a gener-

ator of $\pi_1(L)$ and $\alpha_i$(resp. $\alpha'_i = h_i(\alpha_i)$) a loop in $L_i$(resp. $L'_i$)

representing a generator of $\pi_1(L_i)$(resp. $\pi_1(L'_i)$) such that $j_i(\alpha)$

$= k_i\alpha_i$, where $j_i : L \longrightarrow L_i$ is the canonical projection. Let $H(\alpha_i)$

(resp. $H(\alpha'_i)$) be the holonomy map of $\alpha_i$(resp. $\alpha'_i$) for F(resp. F'),

which is a local diffeomorphism of $(R^2,0)$. Thus if $H(\alpha_i)$ has no

fixed point except for the origin O, we can define the fixed point

index of $H(\alpha_i)$ at O in the usual way. We denote it by $I(H(\alpha_i),L_i)$.

Now we are in a position to state our main theorem.

Theorem 6. Let M be a compact manifold without boundary and F

a compact codimension two foliation of M with leaf space B.    We

assume that the fundamental groups of all leaves of F are isomorphic

to Z and $\pi_1(B)$ acts trivially on $\pi_1(L)$.    Let F' be a $C^0$-perturb-

ation of F.    If F' has exactly n isolated compact leaves, then we

have a following relation;

$$\chi(B) + \sum_{i=1}^{n} \left( \frac{1}{k_i} - 1 \right) = \sum_{i=1}^{n} \frac{1}{k_i} I(H(\alpha_i')^{k_i}, L_i').$$

The following corollary is an immediate consequence of Theorem 6

and this result is an extension of results of Seifert [11] and

Langevin and Rosenberg [9].

Corollary 7.    Let M be a compact manifold without boundary and

F a compact codimension two foliation of M with leaf space B, which

has no isolated singular leaves.    Suppose that

1)    $\pi_1(L) \cong Z$   for every leaf L of F,

2)    $\pi_1(B)$ acts trivially on  $\pi_1(L)$ and

3)    $\chi(B) \neq 0$.

Then any $C^0$-perturbation of F has a compact leaf.

Example 8.    The Klein bottle $K^2$ is an $S^1$-bundle over $S^1$ with

structure group $Z_2$.    Then we can construct a compact codimension one

foliation G of $K^2$ such that G is transverse to the fibres and has

two isolated singular leaves.    We foliate $K^2 \times S^1$ with leaves of the

form  $L \times \{pt\}$ , $L \in G$.    This foliation F is a compact codimension two

foliation with no isolated singular leaves and the leaf space

$K^2 \times S^1/F$ is homeomorphic to a cylinder $S^1 \times [0,1]$.    Thus the euler

characteristic number of $K^2 \times S^1/F$ is equal to zero.    Furthermore there

exists a $C^0$-perturbation F' of F such that F' has no compact leaves.

This example shows that the condition 3) of Corollary 7 is essential.

Corollary 9.   Under the assumption of Theorem 6, we suppose that $H(\alpha_i')$ is expanding or contracting for $i=1,\ldots,n$.   If $\chi(B) \neq n$, then F' has at least n+1 compact leaves.

Proof.   We assume that F' has exactly n compact leaves.   If $H(\alpha_i')$ is expanding or contracting, we have $I(H(\alpha_i'),L_i') = 1$.   Thus from Theorem 6, we have $\chi(B) \neq n$, which contradicts the assumption.

Corollary 10.   Under the assumption of Theorem 6, we suppose that 1 is not an eigenvalue of the linear holonomy $LH(\alpha_i')^{k_i} \in GL(2,R)$ for $i=1,\ldots,n$.   If $\chi(B) < 0$, then F' has at least n+1 compact leaves.

Proof.   We assume that F' has exactly n compact leaves.   From the assumption, we can easily see that $I(H(\alpha_i')^{k_i},L_i')$ is equal to 1 or -1.   Thus from Theorem 6, we have $\chi(B) \geq 0$, which is a contradiction.

Remark 11.   Let $M^3$ be a closed manifold and F a foliation induced from a non-trivial $S^1$-action on $M^3$ with leaf space B.   Then every leaf L of F is homeomorphic to $S^1$ and $\pi_1(B)$ acts trivially on $\pi_1(L)$.

Proof of Theorem 6.   By the results of D.B.A.Epstein[2] and R.Edwards, K.Millett and D.Sullivan[1], every compact codimension two foliation is Hausdorff.   S is a codimension one submanifold of M.   Hence we can apply Proposition 5 for a small perturbation F' of F and the vector field X is defined.   Note that S may be empty, but now we consider the case $S \neq \phi$.   For simplicity, we assume that S is connected.   It is proved similarly when S is not connected.

Let $U(L_i)$ be the saturated neighborhood of $L_i$ (as in Proposition 2) and $V_1(S)$ the total space of the normal disk bundle of S in M.

Put $V_i = \pi(U(L_i))(i=1,\ldots,n)$ which is a neighborhood of $p_i$ and $V_0 = \pi(\nu_1(S))$ which is a neighborhood of $S_B = \partial B$. Let D be a 2-disk in B such that $\pi$ is trivial over and $D \cap \left\{ \bigcup_{i=0}^{n} V_i \right\} = \emptyset$. We identify D with the unit disk $D^2 \subset R^2$ and $\pi^{-1}(D) = T \cong D \times L$. Let k be the least common multiple of $2, k_1, \ldots, k_n$. Noting that $U(L_i)$ is the total space of the normal disk bundle of $L_i$ in M, choose disjoint $k/k_i$ disk fibres $W_i^j (j=1,\ldots,k/k_i)$ for $i=1,\ldots,n$. Then the restriction

$$\pi: \bigcup_{j=1}^{k/k_i} \partial W_i^j \longrightarrow \partial V_i \cong S^1 \text{ is a k-fold covering for } i=1,\ldots,n.$$ Take

a point $\bar{b}_i \in \partial V_i$ for each i. Put $\pi^{-1}(\bar{b}_i) = \left\{ b_i^1, \ldots, b_i^k \right\}$, $b_i^\ell \in W_i^{j(\ell)}$ for some $j(\ell)(1 \leq j(\ell) \leq k/k_i)$. On the other hand, the restriction $\pi: S \longrightarrow \partial B = S^1$ is a fibration. Thus we can construct $k/2$ disjoint sections of this fibration. We let these sections denote by $T_1$, $\ldots, T_{k/2}$, which are identified with those images. The restricted bundle of $\nu_1(S)$ to $T_i$ is denoted by $\bar{T}_i(i=1,\ldots,k/2)$. Then the restriction $\pi: \bigcup_{i=1}^{k/2} \partial \bar{T}_i \longrightarrow \partial V_0 \cong S^1$ is also a k-fold covering.

Take a point $\bar{b}_0 \in \partial V_0$. Put $\pi^{-1}(\bar{b}_0) = \left\{ b_0^1, \ldots, b_0^k \right\}$, $b_0 \in \bar{T}_{r(\ell)}$ for some $r(\ell)(1 \leq r(\ell) \leq k/2)$. Since B is a compact topological 2-manifold with boundary, there is a cell complex K of $B - \bigcup_{i=0}^{n} \text{int}(V_i)$ such that 1) D is contained in a 2-cell of K, 2) $\left\{ \partial V_i \right\}$, $\left\{ \bar{b}_i \right\}(i=0,1,\ldots,k)$ are 1-cells, 0-cells of K respectively and 3) $B - \bigcup_{i=1}^{n} \text{int}(V_i) - \text{int}(D)$ is homotopy equivalent to $\left| K^{(1)} \right|$, where $\left| K^{(1)} \right|$ is the geometric realization of the 1-skeleton of K. Remark thet $\text{int}(V_0)$ is homeomorphic to $\partial B \times [0,1)$. Then we can construct disjoint k sections

8

$s_1, \ldots, s_k$ over $\left|K^{(1)}\right|$ such that $s_\ell(\bar{b}_i) = b_i^\ell$ ($\ell=1,\ldots,k; i=0,1,\ldots,n$). In fact, for the case of dim $L \geq 2$, it is trivial. For the case of dim $L = 1$, that is, $L = S^1$, orienting $\{b_i^\ell\}$ ($\ell=1,\ldots,k; i=0,1,\ldots,n$) along the orientations of the fibres, we can construct disjoint $k$ sections. Furthermore we extend these sections $s_1, \ldots, s_k$ to a tubular neighborhood $N$ of $\left|K^{(1)}\right|$ in $B - \bigcup_{i=0}^{n} \text{int}(V_i)$. We denote these sections by the same letters. Then we may assume, modifying $s_1, \ldots,$ $s_k$ if necessary, that each $s_\ell(N)$ meets $W_i^{j(\ell)}$ and $\bar{T}_{r(\ell)}$ along a segment in $\partial W_i^{j(\ell)}$ and $\partial \bar{T}_{r(\ell)}$ respectively. Thus the union

$$\bigvee_{i,\ell} \left\{ s_\ell(N) \cup W_i^{j(\ell)} \cup \bar{T}_{r(\ell)} \right\}$$ is a compact 2-manifold, transverse to $F$

over $N \cup \left\{ \bigcup_{i=0}^{n} V_i \right\}$. Since $B - \text{int}(D)$ is homotopy equivalent to

$N \cup \left\{ \bigcup_{i=0}^{n} V_i \right\}$, we have the following proposition.

Proposition 12. There exists a compact connected 2-manifold $B^*$, transverse to $F$ over $B - \text{int}(D)$ such that 1) $\pi : B^* \longrightarrow B - \text{int}(D)$ is a $k$-fold covering except for $p_1, \ldots, p_n$ and $\partial B$(if $\partial B \neq \emptyset$) and 2) $B^*$ meets $\partial T$ in simple closed curves $C_i$($i=1,\ldots,r$), where $r$ is a divisor of $k$.

The vector field $X$ projects naturally a vector field $X^*$ tangent to $B^*$ since $X$ and $B^*$ are transverse to $F$. If $F'$ has exactly $n$ compact leaves, $X^*$ has $k \sum_{i=1}^{n} 1/k_i$ isolated singular points in $B^* \cap \left( \bigcup_{i=1}^{n} U(L_i) \right)$ and $X^*$ is never zero outside of $B^* \cap \left( \bigcup_{i=1}^{n} U(L_i) \right)$. Note that the sum of indices of singular points of $X^*$ is equal to

$k \sum_{i=1}^{n} \frac{1}{k_i} I(H(\alpha_i')^{k_i}, L_i')$ since $H(F', \alpha_i) = H(\alpha_i')$ on $D(\bar{x})(\bar{x} \in L_i)$.

Let $L_0 = \pi^{-1}(0)$, $0 \in D$ and $\alpha$ be a loop in $L_0$ representing a generator of $\pi_1(L_0) \cong Z$. R.Langevin and H.Rosenberg have proved the following proposition which is essentially due to Seifert[11].

Proposition 13. $I(F',\alpha) = 0$ (for definition and proof, see [9]).

We construct a closed 2-manifold $\bar{E}^*$ pasting $r$ disks $D_\ell^2$ ($\ell=1,\ldots,r$) to $B^*$ along $\partial B^* = \bigcup_{\ell=1}^{r} C_\ell$ : $\bar{B}^* = B^* \cup D_1^2 \cup \ldots \cup D_r^2$. The vector field $\pi_*(X(C_i))$ on $\partial D$ is homotopic to the vector field $\pi_*(X(n_i \cdot \alpha))$, where $n_i$ is the integer defined by $[C_i] = n_i [\alpha]$ and $[\ ]$ denotes homotopy class in $\pi_1(T)$. From Proposition 13, $\pi_*(X(\alpha))$ is homotopic to the constant vector field, hence deforming $X^*$ along $\partial B^*$ in the homotopy class, we may assume that $\pi_*(X^*|_{\partial B^*})$ is a constant vector field on $\partial D$. Since $\pi: C_i \longrightarrow \partial D \cong S^1$ is a $k/r$-fold covering, we easily see that the vector field $X^*$ is extended to a vector field $\bar{X}$ on $\bar{B}^*$ with exactly $r$ singular points of index $-(\frac{k}{r}-1)$ in $\bigcup_{\ell=1}^{r} D_\ell^2$.

Now the euler characteristic number of $\bar{B}^*$ is equal to

$$\chi(\bar{B}^*) = -r(\frac{k}{r}-1) + k\sum_{i=1}^{n}\frac{1}{k_i} I(H(\alpha_i')^{k_i}, L_i').$$ On the other hand,

$$\chi(\bar{B}^*) = \chi(B^* \cup D_1^2 \cup \ldots \cup D_r^2) = k\left\{\chi(B) + \sum_{i=1}^{n}(\frac{1}{k_i}-1) -1\right\} + r.$$ Hence we have the required relation.

§4. Remarks.

Theorem 14. Let $\pi: M \longrightarrow B$ be a fibration with $B$ a closed oriented $k$-manifold and compact fibre $L$. Suppose that 1) $\pi_1(L) \cong Z$, 2) $\pi_1(B)$ acts trivially on $\pi_1(L)$, 3) $\pi^*: H^k(B;Z) \longrightarrow H^k(M;Z)$ is injective and 4) $\chi(B) \neq 0$. Then any $C^0$-perturbation of the foliation induced from this fibration has a compact leaf.

Proof.    Let F' be a small perturbation of F.    We assume that F' has no compact leaves.    Applying Proposition 5 for such an F', there exists a vector field $X$ on M such that X is orthogonal to F.

Let $\chi \in H^k(B;Z)$ be the euler class of T(B).    The image $\pi^*\chi \in H^k(M;Z)$ is the primary obstruction of a section of the induced bundle $\pi^*T(B)$ over M.    $\pi^*T(B)$ has the non-zero section X, so $\pi^*\chi = 0$. From the condition 3), $\chi = 0$, hence $\chi(B) = 0$ which contradicts the condition 4).

Now we shall apply Main theorem of Hirsch and Thurston [7] to Theorem 14.    First we prepare the following terminology(see [7]).

Definition 15.    A smooth bundle $\xi = (\pi, M, B)$ is called a foliated bundle if $\xi$ has a foliation whose leaves are transverse to the fibres and of complementary dimension.

As well-known, a smooth bundle $\xi$ with compact fibre L is a foliated bundle with holonomy group $\Gamma$ if and only if $\xi$ is a $(\Gamma,L)$-bundle with discrete structure group $\Gamma$.

Definition 16.    A group $\Gamma$ is called amenable if there exists a left-invariant linear function A on $B(\Gamma)$, the bounded functions on $\Gamma$ , such that $A(f) \geq 0$ when $f \geq 0$ and $A(1) = 1$.

Remark 17.    The class of amenable groups is closed under the operations of taking quotients, subgroups, extensions of amenable groups by amenable groups and direct limits; and contains all abelian groups and finite groups.

Notation 18.    We denote by $\mathcal{C}$ the smallest class of groups that contains all amenable groups and is closed under finite extensions and free products.

Remark 19.        $\Gamma \in \mathcal{C}$ if $\Gamma$ is solvable, free or of subexpo-

nential growth(see [7]).

Definition 20. A content on a space V is by definition a linear functional S on the space of continuous real functions on V such that S(c) = c for any constant function c.

Theorem 21. Let $\xi = (\pi, M, B)$ be a foliated bundle with B a closed k-manifold compact fibre L and holonomy group $\Gamma$.

Suppose that 1) $\pi_1(L) \cong Z$,

2) $\pi_1(B)$ acts trivially on $\pi_1(L)$,

3) (a) $\Gamma$ preserves a content on L or (b) $\Gamma \in \mathcal{C}$ and

4) $\chi(B) \neq 0$.

Then any $C^0$-perturbation of $\xi$ has a compact leaf.

Proof. Passing to a double covering of B, if necessary, we may assume that B is orientable. From the condition 3) we have Main theorem of Hirsch and Thurston [7]. Hence it follows that $\pi^* : H^k(B;R) \longrightarrow H^k(M;R)$ is injective. Since B is oriented, $H^k(B;Z) \cong Z$, therefore $\pi^* : H^k(B;Z) \longrightarrow H^k(M;Z)$ is injevtive. Hence Theorem 21 is a corollary to Theorem 14.

Remark 22. $\pi_1(B) \in \mathcal{C}$ implies $\Gamma \in \mathcal{C}$, for $\Gamma$ is the image of the holonomy homomorphism $\pi_1(B) \longrightarrow \text{Diff}(L)$ of the foliated bundle $\xi$.

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