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On the Axiom of Multiple Choice

By

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§1. Introduction

By the axiom of multiple choice (MC) we mean the statement:

MC For every family S of nonempty sets, there exists a function f on S such that f(X) is a nonempty finite subset of X for each $X \in S$.

This paper discusses the strength of MC in Zermelo-Fraenkel set theory with atoms(ZFA). Atoms are objects which differ from the empty set and which have no elements. The language of ZFA consists of = and $\in$ and of two constant symbols 0(the empty set) and A(the set of all atoms). The axiom of ZFA are as follows:

0. Empty set. $\neg\exists x(x\in 0)$.

A. Atoms. $\forall z(z\in A \leftrightarrow \neg z=0 \wedge \neg\exists x\, x\in z)$.

Atoms are the elements of A; sets are all objects which are not atoms.

A1. Extensionality. $\forall$ set X $\forall$ set Y($\forall u(u\in X \leftrightarrow u\in Y)\rightarrow X=Y$).

A2. Pairing. A3. Complehension. A4. Union. A5. Power set. A6. Replacement.

A7. Infinity.

A8. Regularity. $\forall$ nonempty set $S\,\exists x\in S(x\cap S=0)$.

If we add to ZFA the axiom A=0, then we get the usual Zermelo-Fraenkel set theory(ZF).

For a set S P(S) denotes the power set of S : $P(S)=\{X \mid X \text{ is a set} \wedge X\subset S\}$. For any set S let $P^{\alpha}(S)$ be defined as follows: $P^{0}(S)=S$, $P^{\alpha+1}(S)=P^{\alpha}(S)\cup PP^{\alpha}(S)$, $P^{\alpha}(S)=\bigcup_{\beta<\alpha}P^{\beta}(S)$ for limit α; and let $P^{\infty}(S)=\bigcup_{\alpha\in On}P^{\alpha}(S)$. Then we have $V=P^{\infty}(A)$. Let $U=P^{\infty}(0)$. Then U is a model for ZF.

Clearly the Axiom of choice (AC) implies MC. In ZF, MC implies AC. But in ZFA, MC does not imply even the axiom of choice for pairs (AC_2) [Jech, Theorem 9.1,9.2,4.3]. So at a glance we feel that MC is very week in ZFA. This feeling is correct in some sense , but not in another. To state these senses and our results, we enumerate notations. In order to make the strength of MC clearer, we give also some interesting statements not discussed in this paper.

AP. Antichain Principle. Each partially ordered set has a maximal antichain (i.e., a maximal subset of mutually incomparable elements).

LW. Every linearly ordered set can be well ordered.

PW. The power set of every well ordered set can be well ordered.

CP. Cofinality Principle. Every linearly ordered set has a cofinal subset which is well ordered by the induced ordering.

Reg($\aleph$). $\aleph$ is a regular cardinal.

NW. Every linearly ordered set which is not well ordered has an infinite descending sequence.

DC. Principle of Depending Choices. If R is a relation on a nonempty set S such that for every $x \in S$ there exists $y \in S$ with $x R y$, there is a sequence $\langle x_n \mid n < \omega \rangle$ of elements of S such that $\forall n < \omega \; x_n R x_{n+1}$.

CAC. Countable axiom of Choice. Every countable family of nonempty sets has a choice function.

UC.. The union of countably many countable sets is countable.

UC($\mathbb{R}$). The union of countably many countable sets of reals is countable.

IC. Every infinite set has a countable subset.

IC($\mathbb{R}$). Every infinite set of reals has a countable subset.

CL. Every cluster point of a topological space is the limit of a sequence of elements of the space.

CL($\mathbb{R}$). Every cluster point of a set of reals is the limit of a sequence of elements of the set.

SM. Every subspace of a separable metric space is separable.

PI. Prime Ideal Theorem for Boolean Algebras. Every Boolean algebra has a maximal ideal.

OE. Order Extension Principle. Every partially ordered set can be extended to a linearly ordered set.

OP. Ordering Principle. Every nonempty set can be linearly ordered.

ACF. Axiom of Choice for Finite Sets. Every family of nonempty finite sets has a choice function.

ACn. Every family of n-element sets has a choice function.

M(ω). P(ω) has a measure which is 0 on finite sets.

2VM(ω). P(ω) has a 2-valued measure which is 0 on finite sets.

BP. Every set of reals has the Baire property.

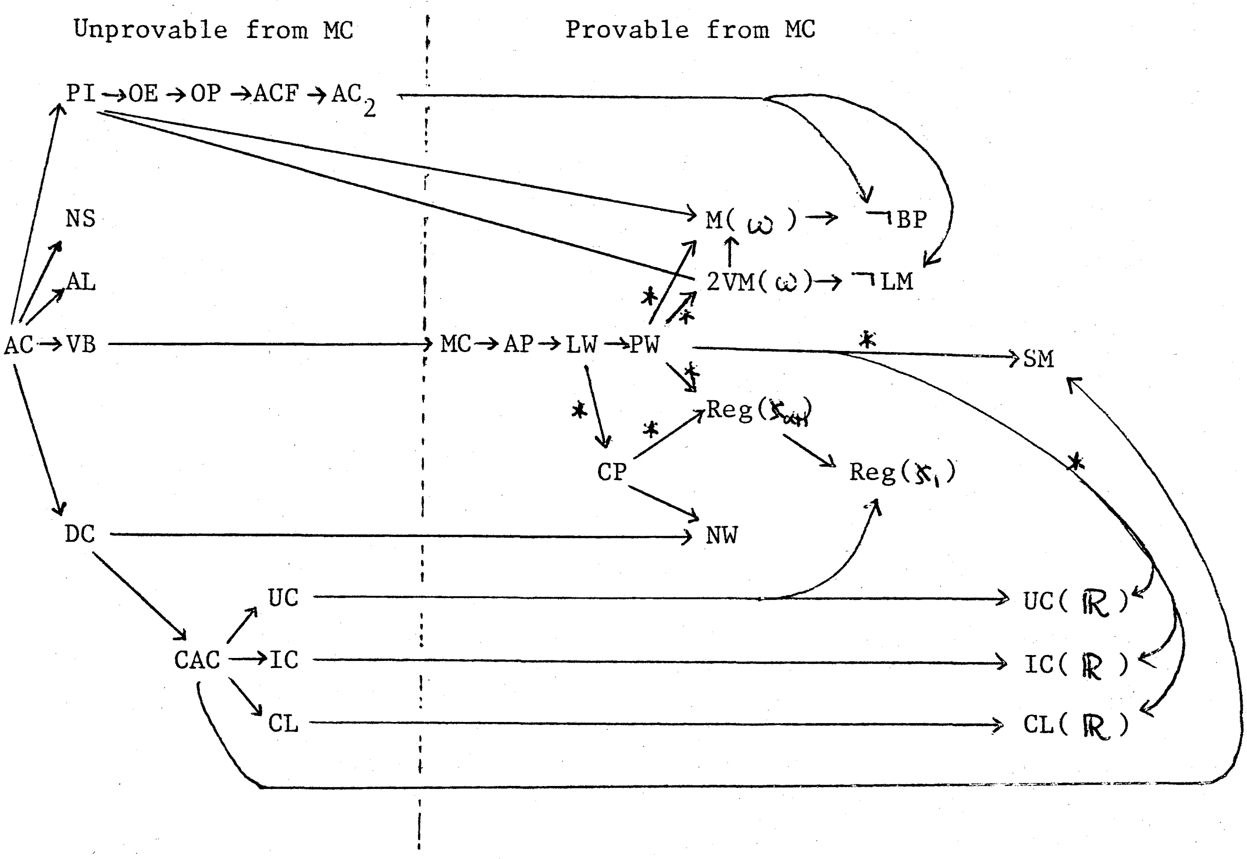
LM. Every set of reals is Lebesgue measurable.

VB. Every linearly independent subset of a vector space can be extended to a basis.

NS. Nielsen- Schreier Theorem. Every subgroup of a free group is a free group.

AL. For any field F, an algebraic closure of F exists and is unique upto isomorphism.

Most of our results are shown in the following diagram.



Arrows → in the diagram are either found in [Jech] or clear. Arrows $\xrightarrow{*}$ will be shown in § 2.

Looking at the diagram, we feel MC is very week in the sense that MC does not imply weak statements AC_2 and UC and important statements PI, VB, NS, and DC.These results will be obtained in § 3.

Next observe that the most of interesting statements in mathematics concerns only on sets independent of atoms (i.e.sets in U). Theorem 2.1 shows that if φ is such a statement and $AC \rightarrow \varphi$ is provable in ZF, $MC \rightarrow \varphi$ is provable in ZFA. In this sense MC is strong. Moreover MC implies CP and CP implies some interesting statements. These are the contents of § 2.

§ 2. Provable Statements from MC

For a formula φ in the language of ZF(i.e. without the constant symbol A) let φ^U denote the formula whose quantifiers are restricted to U.

Theorem 2.1 Let φ be a sentence in the language of ZF such that

$$(*) \qquad AC \to \varphi \text{ is provable in ZF}$$

and such that

$$(**) \qquad \varphi^U \to \varphi \text{ is provable in ZFA.}$$

Then $PW \to \varphi$ is provable in ZFA.

Proof. We use the following sentences are provable in ZFA.

(1) ψ^U for each theorem ψ in ZF.

(2) $PW \to PW^U$.

(3) $PW^U \to AC^U$.

(1) is because U is a model for ZF. (2) is because $P(S) = P^U(S)$ for $S \in U$.
(3) Since $PW \to AC$ is provable in ZF, from (1) $PW^U \to AC^U$ is provable in ZFA.

Now let φ be a sentence in the theorem. Then from $(*)$ and (1)

(4) $AC^U \to \varphi^U$ is provable in ZFA.

Combining (2),(3),(4) and $(**)$ we have that $PW \to \varphi$ is provable in ZFA.

Corollary 2.2 In ZFA, PW implies the following statements:

(1) $M(\omega)$.

(2) $\neg BP$.

(3) $2VM(\omega)$.

(4) $\neg LM$.

(5) $Reg(\aleph_{\omega+1})$.

(6) SM.

Proof. (1) $\sim$ (5) are direct consequences of Theorem 2.1.

(6) Let $\langle S,d \rangle$ be a separable metric space, and $\{x_n \mid n < \omega\}$ be a dence subset of S. Since the function $f: S \to {}^\omega\mathbb{R}$ defined by $f(x) = \langle d(x_n,x) \mid n < \omega \rangle$ is injective and ${}^\omega\mathbb{R} \in U$ we can apply Theorem2.1.

Using Theorem 2.1 we can enumerate many other statements which are provable in ZFA+PW. Here we write only three such statements in contrast to Theorem 3.3.

Corollary 2.3 In ZFA, PW implies UC($\mathbb{R}$), IC($\mathbb{R}$) and CL($\mathbb{R}$).

Theorem 2.4 MC implies CP in ZFA.

Proof. Since LW $\rightarrow$ CP is trivial and MC $\rightarrow$ LW is known, it follows that MC $\rightarrow$ CP. But here we give a direct proof. Let $\langle L, \prec \rangle$ be a linearly ordered set. Using a multiple choice function f on P(L), we can define $l_\alpha \in L$ by induction on α :

$$l_\alpha = \text{the } \prec\text{-least element of } f(\{ l \in L \mid \forall \beta < \alpha \ l_\beta < l \}).$$

Then the set $\{ l_\alpha \mid l_\alpha \text{ is defined} \}$ is cofinal subset of $\langle L, \prec \rangle$ and well ordered by $\prec$.

Remarks 1. CP $\rightarrow$ LW is not provable in ZFA. Since in ZF LW $\rightarrow$ AC, if CP $\rightarrow$ LW in ZFA, then CP $\rightarrow$ AC in ZF, which contradicts a results of [Morris].
2. PW $\rightarrow$ CP is not provable in ZFA. Since both PI $\rightarrow$ OE and OE $\wedge$ CP $\rightarrow$ AC, are provable, if PW $\rightarrow$ CP were provable, PW $\wedge$ PI $\rightarrow$ AC would be so. But in the ordered Mostowski model PW $\wedge$ PI $\wedge \neg$AC holds [Jech, Theorem 7.1, 9.2(iv)], a contradiction.

[Morris] described without proof a number of implications from CP in ZF. They hold also in ZFA eventually with a slight modification.

Definition W^0 = the class of all well orderable sets,

$$W^\lambda = \bigcup_{\xi<\lambda} W^\xi \text{ if } \lambda \text{ is limit,}$$

$$W^{\alpha+1} = \{ \bigcup_{\xi<\nu} X_\xi \mid \nu \in On \wedge X_\xi \in W^\alpha \},$$

$$W = \bigcup_{\alpha \in On} W^\alpha .$$

Theorem 2.5 In ZFA, CP implies the following statements:

(1) Each linearly orderable W-set is well orderable.

(2) If for some α $P^\alpha(A)$ is not well orderable, the least such α is not limit.

(3) Every linearly ordered set has a maximal well orderable initial segment.

(4) No infinit Dedekind finite set is linearly orderable.

(5) For every linearly ordered set L, the least $\aleph$ not $\leq$ L is a succesor

cardinal.

(6) $\mathrm{Reg}(\aleph_{\alpha+1})$.

Proof. (1) By induction on α we prove that if $\langle L, <_L \rangle$ is a linearly ordered set and $L \in W^\alpha$, then L is well orderable.

This is trivial when α is 0 or a limit, so assume $\alpha = \beta + 1$.

Let $L = \bigcup_{\xi<\nu} L_\xi$, $L_\xi \in W^\alpha$. By induction on ν.

If $\nu = 0$, $L = 0$ so L is well orderable. If $\nu = \mu+1$, $L = (\bigcup_{\xi<\mu} L_\xi) \cup L_\mu$.

By the induction hypothesis on ν, $\bigcup_{\xi<\mu} L_\xi$ is well orderable, and the induction hypothesis on α L_μ is well orderable, thus so is L.

Let ν be limit. Set $L'_\xi = \bigcup_{\eta<\xi} L_\eta$ for $\xi < \nu$. By the induction hypothesis on ν each L'_ξ is well orderable. Set

$$\mathcal{L} = \{ \langle L'_\xi, W \rangle \mid \xi < \nu \wedge \langle L'_\xi, W \rangle \text{ is a well ordered set} \} .$$

For $\langle L'_\xi, W \rangle$, $\langle L'_\eta, Z \rangle \in \mathcal{L}$ set

$$\langle L'_\xi, W \rangle \prec \langle L'_\eta, Z \rangle \leftrightarrow \xi < \eta \vee (\xi = \eta \wedge \exists \zeta (\forall \theta < \zeta\, 1^W_\theta = 1^Z_\theta \wedge 1^W_\zeta < 1^Z_\zeta)),$$

where 1^W_ζ is the ζ-th element in $\langle L'_\xi, W \rangle$. Then $\langle \mathcal{L}, \prec \rangle$ is a linearly ordered set. By CP there is a cofinal $\mathcal{L}_0 \subseteq \mathcal{L}$ such that $\langle \mathcal{L}_0, \prec \rangle$ is a well ordered set. We may assume for each $\xi < \nu$ at most one $\langle L'_\xi, W \rangle$ is in $\mathcal{L}_0$. For each $1 \in L$ set $\xi_1 = \bigwedge \{ \xi \mid \exists \langle L'_\xi, W \rangle \in \mathcal{L}_0\ 1 \in L'_\xi \}$. Define $\lessdot$ by

$$1 \lessdot 1' \leftrightarrow \xi_1 < \xi_{1'} \vee (\xi_1 = \xi_{1'} \wedge 1 W_{\xi_1} 1').$$

Then $\langle L, \lessdot \rangle$ is a well ordered set.

(2) Let α be limit and assume that $P^\beta(A)$ is well ordered for all $\beta < \alpha$. Set

$$\mathcal{L} = \{ \langle P^\beta(A), W \rangle \mid \beta < \alpha \wedge \langle P^\beta(A), W \rangle \text{ is a well ordered set such that } \forall x,y \in P^\beta(A) (x \in y \rightarrow xWy) \} .$$

$\mathcal{L}$ is not empty. Recall that every well orderd set $\langle P^\beta(A), W \rangle$ induces a linear ordering $\underline{W}$ on $P^{\beta+1}(A)$ by

$$S\, \underline{W}\, T \leftrightarrow \exists \xi (\forall \eta < \xi (x^W_\eta \in S \leftrightarrow x^W_\eta \in T) \wedge x^W_\xi \notin S \wedge x^W_\xi \in T),$$

where x^W_ξ is the ξ-th element of $\langle P^\beta(A), W \rangle$. For $\langle P^\beta(A), W \rangle$, $\langle P^\beta(A), W' \rangle \in \mathcal{L}$

let $\langle P^{\beta}(A), W\rangle \prec \langle P^{\beta'}(A), W'\rangle$

$$\leftrightarrow \beta<\beta' \vee (\beta=\beta' \wedge \exists\delta<\beta \text{ (W and W' agree on } P^{\delta}(A) \wedge$$
$$\exists\xi(\forall\eta<\xi\ x^{W}_{\eta}=x^{W'}_{\eta} \wedge x^{W}_{\xi}\in P^{\delta+1}(A)\wedge x^{W}_{\xi}\ \underline{W}\ x^{W'}_{\xi}))).$$

Then $\langle\mathcal{L},\prec\rangle$ is a linearly ordered set. Applying CP and using a similar argument as in (1) we can construct a well ordering on $P^{\alpha}(A)$.

(3) Let $\langle L,<_L\rangle$ be a linearly ordered set. Set

$\mathcal{L}=\{\langle I,W\rangle \mid I$ is an initial segment of L and $\langle I,W\rangle$ is a well ordered set $\}$.

For $\langle I,W\rangle, \langle I',W'\rangle\in\mathcal{L}$ set

$$\langle I,W\rangle\prec\langle I',W'\rangle\leftrightarrow I\subsetneq I'\vee (I=I'\wedge\exists\xi\forall\eta<\xi(i^{W}_{\eta}=i^{W'}_{\eta}\wedge i^{W}_{\xi}<_L i^{W'}_{\xi})).$$

Then $\langle\mathcal{L},\prec\rangle$ is a linearly ordered set. Applying CP to $\langle\mathcal{L},\prec\rangle$ and using a similar argument as in (1) we can construct a maximal well orderable initial segment.

(4) Let $\langle L,<_L\rangle$ be an infinite linearly ordered set. By (3) take $I\subseteq L$ a maximal well orderable initial segment. By the maximality I is not finite. So we can enumerate the elements of I without repetitions as $I=\{l_\xi\mid\xi<\alpha\}$ for some $\alpha\geq\omega$. Since $\{l_{\xi 2}\mid\xi 2<\alpha\}\subsetneq L$, L is not Dedekind finite.

(5) Let $\langle L,<_L\rangle$ be a linearly ordered set and $\aleph_\alpha$ be the least aleph not $\leq$ L. Consider the set

$$\mathcal{L}=\{\varphi\mid\varphi \text{ is an injection from an ordinal to L}\}.$$

For $\varphi,\varphi'\in\mathcal{L}$ set

$$\varphi\prec\varphi'\leftrightarrow \mathrm{dom}(\varphi)<\mathrm{dom}(\varphi')\vee$$
$$(\mathrm{dom}(\varphi)=\mathrm{dom}(\varphi')\wedge\exists\gamma<\mathrm{dom}(\varphi)(\varphi|\gamma=\varphi'|\gamma\wedge\varphi(\gamma)<_L\varphi'(\gamma))).$$

Then $\langle\mathcal{L},\prec\rangle$ is a linearly ordered set. By CP there is a cofinal sequence $\{\varphi_\xi\mid\xi<\beta\}$. Set $L_0=\bigcup_{\xi<\beta}\mathrm{rng}(\varphi_\xi)$. For each $l\in L_0$ let ξ_l be the least ξ such that $l\in\mathrm{rng}(\varphi_\xi)$. For $l,l'\in L_0$ set

$l\prec_0 l'\leftrightarrow\xi_l<\xi_{l'}\vee(\xi_l=\xi_{l'}\wedge\varphi^{-1}_{\xi_l}(l)<_L\varphi^{-1}_{\xi_{l'}}(l')$. Then $\langle L_0,\prec_0\rangle$ is a well ordered set. Assume α is limit. Since every $\mathrm{rng}(\varphi_\xi)$ is embeddable into L_0 and for each $\aleph_\eta<L$ there is a ξ such that $\mathrm{dom}(\varphi_\xi)\geq\aleph_\eta$,

the order type of $\langle L_0, <_0 \rangle \geq \bigcup_{\xi<\beta} \mathrm{dom}(\varphi_\xi) \geq \aleph_\alpha$, which is a contradiction. So α is not limit.

(6) Assume that $\mathrm{cf}(\aleph^+) \leq \aleph$ and let $f: \aleph \to \aleph^+$ be a cofinal map such that $\forall \xi < \aleph \, |f(\xi)| = \aleph$. Set

$$L = \{ g \mid g \text{ is a bijection from } \aleph \text{ to } f(\xi) \text{ for some } \xi < \aleph \},$$

$$g \prec g' \leftrightarrow \mathrm{rng}(g) < \mathrm{rng}(g') \vee (\mathrm{rng}(g) = \mathrm{rng}(g')$$
$$\wedge \exists \eta < \aleph (g|\eta = g'|\eta \wedge g(\eta) < g'(\eta))).$$

Then $\langle L, g \rangle$ is a linearly ordered set. By CP there is a cofinal sequence $\langle g_\xi \mid \xi < \alpha \rangle \in {}^{\alpha}L$. We may assume $\alpha \leq \aleph$ since instead of this sequence we can take a subsequence such that $\mathrm{rng}(g_\xi)$ is an increasing function of ξ. Let $J : \aleph \times \aleph \to \aleph$, $K_i : \aleph \to \aleph$ $(i=1,2)$ be the standard pairing functions. Define $F : \aleph \to \aleph^+$ by

$$F(\xi) = \begin{cases} g_{K_1(\xi)}(K_2(\xi)) & \text{if } K_1(\xi) < \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then F is surjective, a contradiction.

§ 3. Unprovable Statements from MC

First we recall simply how to construct permutation models and some notations and definitions. For details refer [Jech, Chapter 4].

We work in the theory ZFA+AC.

(1) Let π be a permutation of the set A. Using $\in$-induction we can define $\pi(x)$ for every x : $\pi(x) = \{ \pi(y) \mid y \in x \}$. Under this definition π becomes an $\in$-automorphism of the universe.

(2) Let $\mathcal{G}$ be a group of permutations of A. For each finite set E of A set $\mathrm{fix}(E) = \{ \pi \in \mathcal{G} \mid \forall e \in E \ \pi(e) = e \}$. Let $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ generated by $\{ \mathrm{fix}(E) \mid E \text{ is a finite subset of } A \}$.

(3) For each x let $\mathrm{sym}(x) = \{ \pi \in \mathcal{G} \mid \pi(x) = x \}$. When $\mathrm{sym}(x) \in \mathcal{F}$ we call x is symmetric. If x is symmetric, then there is a finite subset E of A

such that $\text{fix}(E) \subseteq \text{sym}(x)$. We call such an E a support of x.

(4) Define the class $\mathcal{V} = \{ x \mid \text{sym}(x) \in \mathcal{F} \wedge x \subset \mathcal{V} \}$ consisting of all hereditarily symmetric objects. We call $\mathcal{V}$ a permutation model determined by $\mathcal{G}$. Then $\mathcal{V}$ is a transitive model of ZFA and contains all the elements of U and also A .

To show that the permutation models used below satisfy MC it suffices to prove the following lemma,which is a general description of the principle used to prove that the second Fraenkel model satisfies MC [Jech, Theorem 9.2(i)].

Lemma 3.1 Assume the set A is divided into disjoint finite sets: $A = \bigcup_{i \in I} A_i$. Let $\mathcal{G}_i$ be a subgroup of the symmetric group of A_i,and $\mathcal{G}$ be the direct sum of $\mathcal{G}_i$'s. Then the permutation model determined by $\mathcal{G}$ satisfies MC.

Proof. Let $X \in \mathcal{V}$ be a set of nonempty sets. For each $x \in X$ set $o(x) = \{ \pi(x) \mid \pi \in \text{sym}(X) \}$. Let $y \in x$,then

(1) $\{ \rho(y) \mid \rho \in \text{sym}(x) \} \in \mathcal{V}$

(2) $\pi\{ \rho(y) \mid \rho \in \text{sym}(x) \} = \{ \rho(y) \mid \rho \in \text{sym}(\pi(x)) \}$.

(3) $\{ \rho(y) \mid \rho \in \text{sym}(x) \}$ is finite.

(1) $y \in x$ implies $y \in \mathcal{V}$, so $\{ \rho(y) \mid \rho \in \text{sym}(x) \} \subset \mathcal{V}$. Since $\text{sym}(x) \in \mathcal{F}$ and $\text{sym}(x) \leq \{ \rho(y) \mid \rho \in \text{sym}(x) \}$, $\{ \rho(y) \mid \rho \in \text{sym}(x) \} \in \mathcal{V}$.

(2) follows from a simple computation.

(3) Since each A_i is finite,we can take $i_1, \ldots, i_n \in I$ such that $A_{i_1} \cup \cdots \cup A_{i_n}$ is a support of y. If $\pi, \pi' \in \mathcal{G}$ and agree on $A_{i_1} \cup \cdots \cup A_{i_n}$, then $\pi(y) = \pi'(y)$. Thus

$$0 < |\{ \pi(y) \mid \pi \in \text{smy}(x) \}| \leq |[\text{smy}(y) : \{ \pi \in \mathcal{G} \mid \pi | A_{i_1} \cup \cdots \cup A_{i_n} = 1 \}]|$$
$$\leq |\textstyle\prod_{k=1}^{n} A_{i_k}| < \omega.$$

Using AC there are choice functions f, g on X, $\{ o(x) \mid x \in X \}$ respectively.

From (2) we can define a function on X by

$F(\pi(g(o(x))))=\pi\{\rho f(g(o(x))) \mid \rho \in sym(g(o(x)))\}$ for each $x\in X$ and $\pi\in sym(X)$. (2) also implies that F is in $\mathfrak{N}$. (1) and (3) show each $F(x)$ is a nonempty finite subset of x for each $x\in X$.

It is clear that $MC\wedge ACF \rightarrow AC$ is provable in ZFA. But we have

Theorem 3.2. $MC\wedge\forall n<\omega\ AC_n \rightarrow AC$ is not provable in ZFA.

Proof. Assume that the set A is countable and let $A=\bigcup_{n=0}^{\infty}A_n$, where $A_n=\{a_1^n,\dots,a_{p_n}^n\}$, p_n being the n-th prime number. Let $\mathcal{G}$ be the group generated by the following permutations of A_i: $\pi_n=(a_1^n,\dots,a_{p_n}^n)$. The model $\mathfrak{N}$ determined by $\mathcal{G}$ satisfies $\forall n<\omega\ AC_n$ but not ACF [Jech, Theorem 7.11]. By Lemma 3.1 in $\mathfrak{N}$ MC holds.

Next consider the corresponding statements to Corollary 2.3.

Theorem 3.3 MC does not imply IC,UC,CL in ZFA.

Proof. Let the set A be countable and divide it into countably many disjoint pairs: $A=\bigcup_{n=0}^{\infty}A_n, A_n=\{a_{n0},a_{n1}\}, a_{n0}\neq a_{n1}$. Let $\mathcal{G}$ be the group of all those permutations of A which preserve the pairs. The permutation model $\mathfrak{N}$ determined by this $\mathcal{G}$ is the second Fraenkel model. It is known that (1) The sequence $\langle A_n \mid n<\omega\rangle$ is in $\mathfrak{N}$, thus the set $\{A_n \mid n<\omega\}$ is countable in $\mathfrak{N}$.

(2) If $f:\omega\rightarrow A$ is in $\mathfrak{N}$ then rng(f) is finite.

(3) MC holds in $\mathfrak{N}$.

From (1) and (2) IC and UC are false in $\mathfrak{N}$. Consider $A\cup\{0\}$ as a topological spase letting

$$\{\{a_{ni}\} \mid n<\omega\wedge i<2\}\cup\{\bigcup_{n<i<\omega}A_i\cup\{0\} \mid n<\omega\}$$

as open basis. Then $0\in\overline{A-\{0\}}$ but 0 is not a limit of any sequence of elements of A.

Theorem 3.4 MC does not imply VB,NS,AL in ZFA.

Proof. (1) Assume that the set A is countable and divide it into countably many triples: $A=\bigcup_{n=0}^{\infty}A_n$, $A_n=\{a_{-1}^n, a_0^n, a_1^n\}$. Consider A_n as a vector space over Z_3, a_0^n as the zero vector of A_n, and A as the direct sum of A_n's. Let $\mathcal{G}_n$ be the automorphism group of A_n as a vector space and $\mathcal{G}$ be the direct sum of $\mathcal{G}_n$'s. In the permutation model $\mathcal{V}$ determined by this $\mathcal{G}$ MC holds by Lemma 3.1. To show VB is false in $\mathcal{V}$ it suffices to prove that every linearly independent subset $B \in \mathcal{V}$ of A is finite.
Let E be a common support B and the vector space A. By [E] we denote the subspace generated by E. Assume $b \in B - [E]$. Then there is a least n such that $b_n \notin E$, where b_n is the n- th coordinate of b. Let π be the permutation of A defined by $\pi(b_n) = -b_n$, $\pi(-b_n)=b_n$ and $\pi(x)=x$ otherwise. Then $\pi \in \mathcal{G}$ and $\pi \in \mathrm{fix}(E)$. Since $b_n \notin [E]$, $b_n \neq 0$, so $\pi(b) \neq b$. From $\pi \in \mathrm{sym}(B)$ $\pi(b) \in B$, hence $\{b, \pi(b)\}$ is linearly independent. Set $b^*=2b+\pi(b)$. Then from the linear independence of $\{b, \pi(b)\}$, also $\{b^*, \pi(b^*)\}$ is linearly independent. But $b^*+\pi(b^*)=3(b+\pi(b))=0$, which is a contradiction. So $B \subseteq [E]$ and B is finite.

(2) Let A be countable Consider the commutator subgroup C of the free group whose generators are the elements of A. In [Jech, Theorem 10.12] it is shown that C is not free in the basic Fraenkel model. The proof is based on the following facts:

(*) For any finite subset E of A, there are two distinct elements u,v of $A - E$ and a permutation $\pi \in \mathcal{G}$ of A such that $\pi(u)=v$, $\pi(v)=u$ and $\pi(a)=a$ otherwise.

In the second Fraenkel, too, (*) is the case. So C is not free also in the second Fraenkel model.

(3) In [Jech, Theorem 10.13] it is shown that in the basic Fraenkel model the field F of fractions of the polynomial ring $\mathbb{R}[A]$ has no algebraic closure. The proof is again based on (*), so in the second

Fraenkel model F has no algebraic closure.

The following theorem implies that in ZFA MA adds no generalrestriction to the order of cardinals.

Theorem 3.5 Assume that A is infinite. Let $\langle I, \preceq \rangle$ be a partially ordered set with $|A| = |I|$ and $I \in U$. Then there is a permutation model in which MC and the following statement hold:

$$\exists \langle S_i \mid i \in I \rangle \ \forall i,j \in I (i \preceq j \leftrightarrow |S_i| \leq |S_j|)$$

Proof. Divide A into $I \times \omega$ disjoint pairs:

$$A = \bigcup_{i\in I} \bigcup_{n<\omega} \{ a_{ni0}, a_{ni1} \} .$$

Let $\mathcal{G}$ be the group of those permutations of A which preserve the pairs. In the model $\mathfrak{N}$ determined by this $\mathcal{G}$, MC holds by Lemma 3.1. Since $i \in I \longrightarrow \{ j \in I \mid j \preceq i \}$ is an order monomorphism, it suffices to represent $\langle P(I), \subseteq \rangle$ in the order of cardinals. For each $p \in I$ set $S_p = \{ a_{ink} \mid i \in p \wedge n < \omega \wedge k < 2 \}$. $\langle S_p \mid p \subseteq I \rangle \in \mathfrak{N}$ is easily checked. If $p \subseteq q \subseteq I$, then $S_p \subseteq S_q$ and so $|S_p| \leq |S_q|$ in $\mathfrak{N}$. Assume $p \not\subseteq q$ and take $i \in p - q$. Let $g : S_p \rightarrow S_q$ be a function in $\mathfrak{N}$. We show that g is not injective. Let E be a support of g. Since E is finite we can take an n such that $\{ a_{in0}, a_{in1} \} \cap E = 0$. Let π be a permutation of A such that $\pi(a_{in0}) = a_{in1}$, $\pi(a_{in1}) = a_{in0}$ and $\pi(a) = a$ otherwise. Since $\pi \in \mathrm{fix}(E)$, $\pi \in \mathrm{sym}(g)$. Since $g(a_{in1}) \in S_q$ and $i \notin q$ $\pi(g(a_{in1})) = g(a_{in1})$ by the choice of π. So $g(a_{in0}) = (\pi g)(\pi(a_{in1})) = \pi g(a_{in1}) = g(a_{in1})$.

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