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Contractions on Hilbert space

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Let T be a contraction ,that is $\|T\| \leq 1$, on a separable Hilbert space $\mathcal{R}$. Then $D_T = (I - T^*T)^{1/2}$ is well defined, which is called defect operator of T. In this case we have $\sigma(T) \subset \tilde{D}$, where D and $\tilde{D}$ denote the open unit disc and its closure respectively. Contractions which have defect operators of finite ranks have been studied by many mathematicians. For investigations of contraction T with $D_T \in (\sigma,c)$, that is $I - T^*T \in (\tau,c)$, where (σ,c) and (τ,c) denote the Hilbert Schmidt class and the trace class respectively, some mathematicians added a condition $\sigma(T) \neq \tilde{D}$. Such a contraction T was called weak contraction by M.G.Krein. The spectral decomposition for weak contraction T or accretive operator

$$(I+T)(I-T)^{-1}$$

were obtained by Sz.-Nagy and Foias,Brodskii and Ginzburg (cf .[7]).

Since T is a contraction,$\|T^n x\|$ is decreasing for each x. Sz.-Nagy and Foias defined contractions' classes as following:

$$C_{1\cdot} = \{T: \lim_{n \to \infty} \|T^n x\| > 0 \text{ for each } x \neq 0 \},$$

$$C_{0\cdot} = \{T: \lim_{n \to \infty} \|T^n x\| = 0 \text{ for each } x \},$$

$$C_{\cdot 1} = \{T: T^* \in C_{1\cdot} \} , \quad C_{\cdot 0} = \{T: T^* \in C_{0\cdot} \} ,$$

$$C_{ij} = C_{i\cdot} \cap C_{\cdot j} \quad (0 \leq i,j \leq 1).$$

These formal notations are playing important roles in the studies of contraction. In particular they showed that every weak contraction in C_{00} belongs to C_0 (about this notation see [7]),

and every weak contraction is decomposed to direct sum of the contraction in C_0 and the contraction in C_{11} . The Jordan models for weak contractions were constructed by P.Y.Wu [10].

In [9] the author applied the results of Bercovici and Voiculescu's paper [1] to investigate a contraction T satisfying $\sigma(T) = \widetilde{D}$ and $D_T \in (\sigma, c)$, in particular, showed that T belongs to C_{10} iff there is a quasi-affinity X such that

$$X T = S_E X ,$$

where E is a Hilbert space with dim E = - index T (this "index" is Fredfolm index) and S_E is the unilateral shift on $\ell_+^2(E)$. From the results of [9], he conjectured that contraction in C_{00} with (σ, c)-defect operator belongs to C_0 . In [8] Takahashi and Uchiyama showed that this was true.

In this note we will clear the structure of a contraction T with D_T in (σ, c). In particular, setting

$$\alpha = \min \{\dim N(T-\lambda) : \lambda \in D \}, \quad \beta = \min \{\dim N(T^*-\lambda) : \lambda \in D \},$$

where $N(T) = \{x : Tx = 0\}$, we will show that there are vector valued holomorphic functions $h_i(\lambda)$, $f_j(\lambda)$ $(1 \leq i \leq \alpha, 1 \leq j \leq \beta)$ defined on D satisfying

$$(T-\lambda) h_i(\lambda) \equiv 0 \quad, \quad (T^*-\lambda) f_j(\lambda) \equiv 0$$

, and that if $\alpha = \beta = 0$, then T is a weak contraction.

In section 4, we will study the weighted shifts with finite matrices' weights.

From now on, we use the symbol D(T) instead of D_T for convenience

1. Upper triangulation

Let T be a contraction on $\mathcal{H}$ with $D(T) \in (\sigma, c)$. Then, since $\sum_i (1 - \|Te_i\|^2) < \infty$ for a C.O.N.B. $\{e_i\}$ of $\mathcal{H}$, we have $\dim N(T) < \infty$. Let $T = V|T|$ be the polar decomposition of T . Then there is a isometric (or co-isometric) extension V_1 of V such that $V_1 - V$ is of finite rank. In this case $\dim N(V_1 - \lambda)$ is constant on D and finite , also $\dim N(V_1^* - \lambda)$ is constant on D. Since range$(V_1 - \lambda)$ is closed $(V_1 - \lambda)$ is a semi-Fredholm operator, and index$(V_1 - \lambda)$ is constant on D. Since $T - \lambda = V_1 - \lambda + (V - V_1) - V(I - |T|)$, $T - \lambda$ is a semi-Fredholm operator, and index$(T - \lambda)$ is constant on D and less than ∞ . Thus we have

$$(1.1) \qquad \sigma(T) \cap D = \{\sigma_p(T) \cup \overline{\sigma_p(T^*)}\} \cap D \qquad .$$

Now we notice that if $\dim N(T^*)$ is finite , then $(T - \lambda)$ is a Fredholm operator for each $\lambda \in D$.

From the definition of C_1. it follows that

$$(1.2) \qquad \sigma_p(T) \cap D = \phi \qquad \text{for } T \in C_1. \qquad .$$

In this section we obtain an upper triangulation of T whose diagonal elements were already studied.

The next lemma is trivial, but for the sake of the completeness we prove it.

Lemma 1.1. Let Y be a bounded operator and F a Fredholm

operator such that $FY \in (\tau,c)$. Then we have $Y \in (\tau,c)$.

Proof. There are bounded operators F' and P such that

$$F'F = I-P \quad , \text{ range } P = N(F).$$

Thus $(I-P)Y = F'FY \in (\tau,c)$ implies $Y = (I-P)Y+PY \in (\tau,c)$. Q.E.D.

Lemma 1.2. Let T be a contraction with $D(T) \in (\sigma,c)$ and let

$$(1.3) \qquad T = \begin{bmatrix} T_0. & B \\ 0 & T_1. \end{bmatrix}$$

be the decomposition of T such that $T_0. \in C_0.$, $T_1. \in C_1.$ (see [7]). Then $D(T_0.)$ and $D(T_1.)$ are in (σ,c) and B in (τ,c).

Proof. Since $I-T^*T \in (\tau,c)$,

$$I-T_0.^* T_0. \quad , \quad B^*T_0. \quad \text{and} \quad I-(B^*B+T_1.^*T_1.)$$

belong to (τ,c), where I of "$I-T_0.^* T_0.$" is the identity on the space where $T_0.$ is defined. From next lemma , it follows that $T_0.$ is a Fredholm operator. Thus , by Lemma 1.1, we have $B \in (\tau,c)$ and hence $I-T_1.^*T_1. \in (\tau,c)$.　　　　　　Q.E.D.

Lemma 1.3. Suppose $T_0. \in C_0.$ and $D(T_0.) \in (\sigma,c)$, then $T_0.$ is a Fredholm operator.

Proof. Let

$$(1.4) \qquad T_0. = \begin{bmatrix} T_{01} & A \\ 0 & T_0 \end{bmatrix}$$

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be the decomposition of T_0. satisfying $T_{01} \in C_{01}$ and $T_0 \in C_{00}$ ([7]).

Since $I - T_0. * T_0. \in (\tau,c)$, $I - T_{01} * T_{01}$, $A * T_{01}$ and $I - (A*A + T_0 * T_0)$

are in (τ,c) too. From (1.2) we have $\sigma_p(T_{01}*) \cap D = \phi$, hence T_{01}

is a Fredholm operator. Consequently, from Lemma 1.1, $A \in (\tau,c)$

and hence $I - T_0 * T_0 \in (\tau,c)$. Since $T_0 \in C_{00}$, we have $T_0 \in C_0$ [8],

which implies $\dim N(T_0) = \dim N(T_0*) < \infty$ [7]. Therefore T_0 is a

Fredholm operator. Thus

$$T_0. = \begin{bmatrix} T_{01} & 0 \\ 0 & T_0 \end{bmatrix} + \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$$

is a Fredholm operator. Q.E.D.

Lemma 1.4. Suppose $T_1. \in C_1.$ and $D(T_1.) \in (\sigma,c)$ and let

$$T_1. = \begin{bmatrix} T_{11} & F \\ 0 & T._0 \end{bmatrix}$$

be a decomposition of $T_1.$ such that $T_{11} \in C_{11}$, $T._0 \in C._0$ ([7]).

Then $D(T_{11})$ and $D(T._0)$ are in (σ,c) and F in (τ,c), and $T._0 \in C_{10}$.

Proof. $I - T_{11} * T_{11}$, $F * T_{11}$ and $I - (F*F + T._0 * T._0)$ belong to

(τ,c). From (1.2) we have

$$\sigma_p(T_{11}) \cap D = \phi \quad \text{and} \quad \sigma_p(T_{11}*) \cap D = \phi \quad ,$$

and hence, by (1.1) we have

(1.5) $\sigma(T_{11}) \cap D = \phi$.

Thus $F \in (\tau,c)$ and hence $I - T._0 * T._0 \in (\tau,c)$. To show $T._0 \in C_{10}$,

decompose $T._0$ as

$$(1.6) \qquad T._0 = \begin{bmatrix} T_{00} & F_3 \\ 0 & T_{10} \end{bmatrix} \quad ,$$

where $T_{00} \in C_{00}$, $T_{10} \in C_{10}$. Then we have $I - T_{00}*T_{00} \in (\tau, c)$ and hence $T_{00} \in C_0$, from which we get

$$(1.7) \qquad \sigma(T_{00}) \cap D \neq D \quad .$$

Denote the space on which $T_1.$ is defined by $\mathcal{L}$, and let $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ be a decomposition of $\mathcal{L}$ corresponding to

$$T_1. = \begin{bmatrix} T_{11} & F_1 & F_2 \\ 0 & T_{00} & F_3 \\ 0 & 0 & T_{10} \end{bmatrix} \quad ,$$

where $[\, F_1 \, , \, F_2 \,] = F$. Set

$$(1.8) \qquad T_2 = \begin{bmatrix} T_{11} & F_1 \\ 0 & T_{00} \end{bmatrix} \quad .$$

Then, since $T_2 = T_1. \big|_{\mathcal{L}_1 \oplus \mathcal{L}_2}$, we have $T_2 \in C_1.$ and $D(T_2) \in (\sigma, c)$. Above triangulation of T_2 implies that

$$\sigma(T_2) \subset \sigma(T_{11}) \cup \sigma(T_{00}) \quad .$$

From this relation and (1.5),(1.7), it follows that

$$\sigma(T_2) \cap D \neq D \quad .$$

Therefore T_2 is a weak contraction. The $C_0 - C_{11}$ decomposition of T_2 ([7]) implies T_2 has no C_0-part , because $T_2 \in C_1.$, and so $T_2 \in C_{11}$. From (1.8) we have $T_{00}* = T_2* \big|_{\mathcal{L}_2}$,which belongs to $C_0.$ and $C_1.$; this is impossible. Thus $\mathcal{L}_2$ reduces to 0,so that from (1.6) we have $T._0 = T_{10} \in C_{10}$. $\qquad$ Q.E.D.

Theorem 1.5.   Let T be a contraction with $D(T) \in (\sigma,c)$.
Then we have an upper triangulation :

$$
T = \begin{bmatrix}
T_{01} & & & \\
0 & T_0 & * & \\
0 & 0 & T_{11} & \\
0 & 0 & 0 & T_{10}
\end{bmatrix} ,
$$

where $D(T_{01})$, $D(T_0)$, $D(T_{11})$ and $D(T_{10})$ belong to (σ,c), and

$T_{01} \in C_{01}$, $T_0 \in C_0$, $T_{11} \in C_{11}$, $T_{10} \in C_{10}$, and * belongs to (τ,c).

Proof. At first, decompose T as Lemma 1.2 , next decompose
T_0 . as (1.4). In the proof of Lemma 1.3 we showed that T_{01}
and T_0 satisfy the conditions in theorem. At last decompose
T_1 . as Lemma 1.4 and set $T_{10} = T._0$.   Q.E.D.

Definition. Above upper triangulation is called the
canonical triangulation for T with $D(T) \in (\sigma,c)$.

Remark.We showed that T_{01} and T_0 are Fredholm operators
and T_{11} is invertible. But dim $N(T_{10}*)$ may be infinite.

2. Eigenvectors

Let T be a contraction on $\mathcal{H}$ with $D(T) \in (\sigma, c)$. Set

$$\alpha = \min \{\dim N(T-\lambda) : \lambda \in D\} \ , \quad \beta = \min \{\dim N(T^*-\lambda) : \lambda \in D\} \ ,$$

$$i(\lambda) = \dim N(T-\lambda) - \alpha \quad (<\infty) \ , \quad \Lambda = \{\lambda \in D : i(\lambda) > 0\} \ .$$

Now we note that if a bounded operator A is decomposed as

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \ , \text{ where } A_1 \text{ is a surjection,}$$

then $\dim N(A) = \dim N(A_1) + \dim N(A_3)$. In fact, we have

$$N(A) = N(A_1) + \{(- B^{-1} A_2 x \ , \ x) : x \in N(A_3)\} \ ,$$

where B is the restriction of A_1 to $N(A_1)^{\perp}$.

Theorem 2.1. Let T be a contraction with $D(T) \in (\sigma, c)$. And consider the canonical triangulation of T. Then

$$\alpha = \dim N(T_{01}) \quad \text{and} \quad \beta = \dim N(T_{10}^*) \ .$$

Proof. At first, we notice (1.3). Since $\sigma_p (T_1.) \cap D = \phi$, it is not difficult to show $N(T-\lambda) = N(T_0.-\lambda)$ for $\lambda \in D$. Next we notice (1.4). Since $D(T_{01}) \in (\sigma, c)$ and $\sigma_p (T_{01}^*) \cap D = \phi$, $(T_{01}-\lambda)$ is a surjection for each $\lambda \in D$. Thus we have

(2.1) $\quad \dim N(T-\lambda) = \dim N(T_0.-\lambda) = \dim N(T_{01}-\lambda) + \dim N(T_0-\lambda)$

$$= \text{index } (T_{01}-\lambda) + \dim N(T_0-\lambda) = \text{index } T_{01} + \dim N(T_0-\lambda).$$

$T_0 \in C_0$ implies that $\sigma(T_0) \cap D$ is countable. Hence we have

$$\alpha = \text{index } T_{01} = \dim N(T_{01}) \ .$$

To show $\beta = \dim N(T_{10}^*)$, take the adjoint of (1.3), that

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is

$$T^* = \begin{bmatrix} T_1 \cdot^* & B^* \\ 0 & T_0 \cdot^* \end{bmatrix} \ .$$

Since $\sigma_p(T_1 \cdot) \cap D = \phi$ and $D(T_1 \cdot) \in (\sigma, c)$, $(T_1 \cdot^* - \lambda)$ is a surjection

for each $\lambda \in D$. Thus we have

$$\dim N(T^* - \lambda) = \dim N(T_1 \cdot^* - \lambda) + \dim N(T_0 \cdot^* - \lambda) \ .$$

From (1.4) , it follows that $N(T_0 \cdot^* - \lambda) = N(T_0^* - \lambda)$ for $\lambda \in D$,

because $\sigma_p(T_{01}^*) \cap D = \phi$. Now we notice the decomposition of

$T_1 \cdot$ in Lemma 1.4 and remark that we set T_{10} instead of $T_{\cdot 0}$ in

the canonical triangulation of T . Since $\sigma_p(T_{11}^*) \cap D = \phi$,

it is clear that $N(T_1 \cdot^* - \lambda) = N(T_{10}^* - \lambda)$ for $\lambda \in D$, so that

$$\dim N(T^* - \lambda) = \dim N(T_{10}^* - \lambda) + \dim N(T_0^* - \lambda) \ .$$

Consequently we have $\beta = \dim N(T_{10}^*)$. Q.E.D.

Corollary 2.2. Let T be a contraction with $D(T) \in (\sigma, c)$.

Then $\sum_{\lambda \in \Lambda} (1 - |\lambda|) \cdot i(\lambda) < \infty$.

Proof. From (2.1), we have $i(\lambda) = \dim N(T_0 - \lambda)$. Thus , by [7]

we can conclude the proof. Q.E.D.

Theorem 2.3. Let T be a contraction with $D(T) \in (\sigma, c)$.

Then there are holomorphic vector valued functions $h_i(\lambda)$, $f_j(\lambda)$

, $(1 \leq i \leq \alpha$, $1 \leq j \leq \beta$) defined on D such that

$$(T - \lambda) h_i(\lambda) \equiv 0 \qquad (T^* - \lambda) f_j(\lambda) \equiv 0 \ ,$$

and for each $\lambda \in D$ $\{h_1(\lambda), \ldots, h_\alpha(\lambda)\}$ are linearly independent,

also $\{f_1(\lambda), \ldots, f_\beta(\lambda)\}$ are . In this case , setting

$\mathcal{L}^{\perp} = \bigvee \{h_i(\lambda), f_j(\lambda): i, j, \lambda\}$, $P_{\mathcal{L}} T|_{\mathcal{L}}$ is a weak contraction.

Proof. We showed that T_{01} in the canonical triangulation of T is a Fredholm operator. Hence

$$T_{01}{}^*(I - T_{01}T_{01}{}^*) = (I - T_{01}{}^*T_{01})T_{01}{}^* \in (\tau, c)$$

implies, by Lemma 1.1, $D(T_{01}{}^*) \in (\sigma, c)$. Therefore there is a quasi-affinity X such that $X T_{01}{}^* = S_E X$, where $\dim E = -\text{index } T_{01}{}^* = \dim N(T_{01}) = \alpha < \infty$ [9]. Let $\{e_1, \ldots, e_\alpha\}$ be a C.O.N.B. of E. Then $g_i(\lambda) = \{e_i, \lambda e_i, \lambda^2 e_i, \ldots\}$ $(1 \le i \le \alpha)$ is holomorphic function defined on D with value in $\ell_+^2(E)$. And for each $\lambda \in D$ $\{g_1(\lambda), \ldots, g_\alpha(\lambda)\}$ are orthogonal each other. It is trivial to show that

$$(S_E{}^* - \lambda) g_i(\lambda) \equiv 0, \quad \bigvee_{i\lambda} g_i(\lambda) = \ell_+^2(E) .$$

Since $T_{01} X^* = X^* S_E{}^*$,

$$h_i(\lambda) = \begin{bmatrix} X^* g_i(\lambda) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (1 \le i \le \alpha)$$

satisfy the conditions given in the theorem. Since $T_{10} \in C_{10}$ and $D(T_{10}) \in (\sigma, c)$, there is a quasi-affinity Y such that

$$Y T_{10} = S_F Y , \quad \text{where} \quad \dim F = \beta \le \infty .$$

We can show the existence of $f_j(\lambda)$ with the same way as above ; hence we omit it. We must show the last assertion. To this end, we notice that $\{h_i(\lambda): 1 \le i \le \alpha, \lambda \in D\}$ and $\{f_j(\lambda): 1 \le j \le \beta, \lambda \in D\}$ span the spaces on which T_{01} and T_{10}, respectively, are defined . Thus , by Theorem 1.5 we have

$$(2.2) \qquad P_{\mathcal{L}} \ T|_{\mathcal{L}} = \begin{bmatrix} T_0 & * \\ 0 & T_{11} \end{bmatrix} .$$

In this case * clearly belongs to (τ,c). Now we set $T_{\mathcal{L}}=P_{\mathcal{L}} T|_{\mathcal{L}}$. From (2.2) , $D(T_0) \in (\sigma,c)$ and $D(T_{11}) \in (\sigma,c)$ imply that $D(T_{\mathcal{L}}) \in (\sigma,c)$. Since T_{11} is invertible , we have

$$\sigma_p(T_{\mathcal{L}})= \sigma_p(T_0) \qquad \sigma_p(T_{\mathcal{L}}{}^*) =\sigma_p(T_0{}^*) .$$

$T_0 \in C_0$ implies that $\sigma_p(T_0{}^*)= \overline{\sigma_p(T_0)} \neq D$ [7]. Thus by (1.1) we have $\sigma(T_{\mathcal{L}}) \cap D = \sigma_p(T_0) = \Lambda \neq D$. Thus $T_{\mathcal{L}}$ is a weak contraction.

                                                                    Q.E.D.

Theorem 2.4. Let T be a contraction with $D(T) \in (\sigma,c)$; then the following are equivalents:

(a) $\alpha = \beta = 0$;

(b) T is a weak contraction ;

(c) T is decomposable (about definition see [2]).

Proof. (a)$\Rightarrow$(b): From Theorem 2.1. $N(T_{01})= 0$, which implies T_{01} is a weak contraction. Therefore there is a C_0-C_{11} decomposition of T_{01} , but it is impossible , because $T_{01} \in C_{01}$. Thus the space on which T_{01} is defined reduces to 0. Similarly the space on which T_{10} is defined reduces to 0. Thus $\mathcal{L}$ in Theorem 2.3 is $\mathcal{H}$. Therefore T is a weak contraction.

(b) $\Rightarrow$ (c): This was shown by Jafarian [5].

(c) $\Rightarrow$ (a): Since decomposable T has the single valued extension

property, $\alpha=0$ follows . Thus for $\lambda \notin \Lambda$, $(T-\lambda)$ is injective

semi-Fredholm operator. Hence $\sigma_{\ell}(T) \cap D \subset \Lambda$. Thus we have

$\sigma(T) \cap D \subset \Lambda$ (see p.30 of [2]). Consequently $\beta = 0$. Q.E.D.

Proposition 2.5. Let T be a contraction on $\mathcal{H}$ with

$D(T) \in (\sigma,c)$. Then $T \in C_{10}$ if and only if there are vector valued

holomorphic functions $h_i(\lambda)$ such that

$$(T^* -\lambda)h_i(\lambda) \equiv 0 , \qquad \bigvee_{i,\lambda} h_i(\lambda) = \mathcal{H} .$$

Proof. "Only if" part follows from Theorem 2.3 and its

proof. We must show "if" part. Since

$$T^{*n}h_i(\lambda) = \lambda^n h_i(\lambda) \to 0 \quad \text{as } n \to \infty,$$

T^{*n} strongly converges to 0 on linear spann of $\{h_i(\lambda):i,\lambda\}$.

Suppose

$$T^{*n} x_i \to 0 \quad (n \to \infty) \text{ and } x_i \to x \ (i \to \infty).$$

Since $\| T^{*n} x \| \leq \| T^{*n} x_i \| + \| T^{*n}(x-x_i) \| \leq \| T^{*n}x_i \| + \| x-x_i \|$,

we have $\overline{\lim_{n \to \infty}} \| T^{*n}x \| \leq \| x-x_i \|$. Since we can make the right

side arbitrary small, $T^{*n}x \to 0$ $(n \to \infty)$. Thus T belongs to $C._0$,

therefore the canonical triangulation of T becomes

$$T = \begin{bmatrix} T_0 & * \\ 0 & T_{10} \end{bmatrix} .$$

Let P be the orthogonal projection to the space which T_0 is

defined on . Then we have

$$0 = P (T^*-\lambda) h_i(\lambda) = P (T^*-\lambda)P h_i(\lambda) = (T_0^*-\lambda)P h_i(\lambda) .$$

Since $\sigma_p(T_0^*)$ are countable , $Ph_i(\lambda) \equiv 0$. Consequently $P\mathcal{R} = 0$ and hence $T = T_{10}$.   Q.E.D.

Alternately we have

Proposition 2.6.   Let T be a contraction on $\mathcal{R}$ with $D(T) \in (\sigma, c)$. Then $T \in C_{01}$ iff there are vector valued holomorphic functions $f_j(\lambda)$ defined on D   such that

$$(T-\lambda)\, f_j(\lambda) \equiv 0 \qquad \bigvee_{j,\lambda} f_j(\lambda) = \mathcal{R} \qquad .$$

3. m-accretive operators

Let A be an m-accretive operator densely defined in $\mathcal{R}$ (about the definition see [6]). Then

(3.1) $T = (A-I)(A+I)^{-1}$

is a contraction defined on $\mathcal{R}$ and

$$\sigma_p(T) \not\ni 1 \qquad \text{and} \qquad T^* = (A^*-I)(A^*+I)^{-1}$$

(see Chap IV of [7]). It is trivial to show that

$$((I-T^*T)h,h) = 4\,\mathrm{Re}\,(A(A+I)^{-1}h,\, (A+I)^{-1}h) \quad \text{for } h \in \mathcal{R}.$$

Since $A(A+I)^{-1}$ and $(A+I)^{-1}$ are bounded , we have a relation:

$$I-T^*T \in (\tau,c) \Longleftrightarrow u(A) \in (\tau,c) \quad ,$$

where $u(A) = \mathrm{Re}\,((A^*+I)^{-1}A(A+I)^{-1})$. In this section we denote the

open right half plane by Ω . The mapping

$$\psi: \mu \longrightarrow \frac{\mu -1}{\mu +1}$$

transforms Ω onto D. It is clear that

(3.2) $(A-\mu)x=0 \Longleftrightarrow (T- \psi(\mu))(A+I)x=0$.

Set

$\alpha= \min\{\dim N(A-\mu):\mu\in\Omega\}$, $\beta= \min\{\dim N(A^{*}-\mu): \mu\in\Omega\}$,

$i(\mu)= \dim N(A-\mu)-\alpha$, $\Gamma=\{\mu : i(\mu)>0\}$.

Proposition 3.1. Let A be an m-accretive operator densely
defined in $\mathcal{H}$. If $u(A) \in (\tau,c)$, then it follows that

$$\sum_{\mu\in\Gamma} (\frac{Re \mu}{1 +|\mu|^2})\cdot i(\mu) < \infty .$$

Proof. Since range of (A+I) is $\mathcal{H}$, by (3.2), we have

$\dim N(A-\mu) = \dim N(T-\psi(\mu))$, $\alpha=\min\{\dim N(T-\lambda):\lambda\in D\}$,

$\dim N(T-\lambda)-\alpha = \dim N(A-\psi^{-1}(\lambda))-\alpha = i(\psi^{-1}(\lambda))$,

$\{\lambda: i(\psi^{-1}(\lambda))> 0\} = \psi(\Gamma)$.

Thus from Corollary 2.2 , it follows that

$$\sum_{\lambda\in\psi(\Gamma)} (1-|\lambda|)\cdot i(\psi^{-1}(\lambda))< \infty$$

so that $\sum_{\mu\in\Gamma} (1- |\psi(\mu)|)\cdot i(\mu)< \infty$.

Therefore we have

$$\sum_{\mu\in\Gamma} \frac{Re \mu}{1 +|\mu|^2}\cdot i(\mu)< \infty \qquad (cf. p.132 of [4]) .$$

Theorem 3.2.   Let A be an m-accretive operator densely
defined in $\mathcal{H}$. If $u(A) \in (\tau,c)$, then there are vector valued

holomorphic functions $x_i(\mu)$, $y_j(\mu)$, $(1 \leq i \leq \alpha, 1 \leq j \leq \beta)$ defined on Ω such that

$$(A-\mu) \, x_i(\mu) \equiv 0 \quad \text{and} \quad (A^*-\mu) \, y_j(\mu) \equiv 0 \quad .$$

Proof. From Theorem 2.3, for T defined by (3.1) there are holomorphic functions $h_i(\lambda)$ $(1 \leq i \leq \alpha)$ such that

$$(T-\lambda) \, h_i(\lambda) \equiv 0 \quad .$$

Then
$$x_i(\mu) = (A+I)^{-1} h_i(\psi(\mu))$$

is a holomorphic function defined on Ω , and for each $\mu \in \Omega$ $x_i(\mu)$ belongs the domain of A. From (3.2), we have

$$(A-\mu) \, x_i(\mu) \equiv 0.$$

We can similarly show the existence of $y_j(\mu)$ from the alternate relation of (3.2), that is

$$(A^*-\mu)x = 0 \quad \Longleftrightarrow \quad (T^*-\psi(\mu))(A^*+I)x = 0 \quad . \qquad Q.E.D.$$

4. Weighted unilateral shifts

In this section we study weighted unilateral shifts with $(\sigma,c)-$ defect operators. Let E be an N-dimensional finite Hilbert space, and A_n (n=0.1.2...) invertible contraction on E. Let T be a weighted unilateral shift on $\ell_+^2(E)$ defined by

$$T \{x_0,x_1,\ldots \} = \{0,A_0 x_0, A_1 x_1,\ldots\}$$

Lemma 4.1.   Let B be an invertible operator on E. Then we have

$$\|B^{-1}\| \leq \frac{\|B\|^{N-1}}{|\det B|} \quad , \quad \frac{1}{|\det B|} \leq \|B^{-1}\|^N \quad .$$

Proof. Let $\lambda_1 \geq \ldots \geq \lambda_N > 0$ be eigen values of B*B. Then we have

$$\|B^{-1}\|^2 = \|(B^*B)^{-1}\| = \frac{1}{\lambda_N} \leq \frac{\lambda_1^{N-1}}{\lambda_1 \ldots \lambda_N} = \frac{\|B^*B\|^{N-1}}{\det (B^*B)} \quad .$$

Thus we have

$$\| B^{-1}\| \leq \frac{\|B\|^{N-1}}{|\det B|} \quad .$$

The second inequality similarly follows (cf. p.200 of [3]).Q.E.D.

Now we remember next fact:

for scalar a_n such that $0 < |a_n| < 1$, $\prod_{n=0}^{\infty} |a_n|$ converges iff $\sum_{n=0}^{\infty} (1 - |a_n|) < \infty$.

Theorem 4.2.   Let T be a contractive weighted shift defined above. Then the following are equivalents :

(a) $T \in C_{10}$;

(b) $D(T) \in (\sigma, c)$;

(c)   T is similar with simple sift S_E ;

(d)   there is a $\delta > 0$ such that

$$\| A_n \cdots A_0 x \| \geq \delta \|x\| \quad \text{for every } x \in E \text{ and every } n .$$

Proof. (d) $\Rightarrow$ (c): For each m we have

$$\|A_{m+n} \cdots A_m x\| = \|A_{m+n} \cdots A_m A_{m-1} \cdots A_0 (A_{m-1} \cdots A_0)^{-1} x\|$$

$$\geq \delta \| (A_{m-1} \cdots A_0)^{-1} x \| \geq \delta \frac{1}{\|A_m \cdots A_0\|} \|x\| \geq \delta \|x\| \quad ,$$

because each A_i is a contraction. Thus for each $f \in \ell_+^2(E)$, we have

$$\| T^n f \| \geq \delta \|f\| \qquad \text{for every n.}$$

By the well known Sz.-Nagy's theorem, T is similar with an isometry V . Since T belongs to $C_{\cdot 0}$, so do V, hence V is a unilateral shift . Since

$$\dim N(V^*) = \dim N(T^*) = \dim E = N$$

dimension of the wandering space for V is N. Thus V is unitarily equivalent with S_E .

(c)$\Longrightarrow$(a): This is obvious .

(a)$\Longrightarrow$(d): Set $\quad \ell(x) = \lim_{n \to \infty} \| T^n \{x,0,0,..\} \| \qquad$ for $x \in E$.

Since ℓ is continuous and $\ell(x) \neq 0$ for $x \neq 0$, there is a $\delta > 0$ such that

$$\ell(x) \geq \delta \qquad \text{for x in the unit surface of E.}$$

Since $\ell(\alpha x) = |\alpha| \ell(x)$, we have

$$\lim_{n \to \infty} \| A_n \cdots A_0 \, x \| = \ell(x) \geq \delta \| x \| \quad \text{for } x \in E.$$

(b)$\Longrightarrow$(d): From

$$\infty > \| I - T^* T \|_1 = \sum_{n=0}^{\infty} \| I - A_n^* A_n \|_1 \geq \sum_{n=0}^{\infty} \| I - A_n^* A_n \| \, ,$$

it follows that

$$\prod_{n=0}^{\infty} (1 - \| I - A_n^* A_n \|)$$

converges and we denote its limit by δ^2 . In view of

$$\| A_i^{-1} \|^2 = \| (A_i^* A_i)^{-1} \| = \| (I - (I - A_i^* A_i))^{-1} \| \leq \frac{1}{1 - \| I - A_i^* A_i \|} \, ,$$

we have

$$\| A_n \cdots A_0 \ x \ \|^2 \geq \frac{\|x\|^2}{\|(A_n \cdots A_0)^{-1}\|^2} \geq \frac{\|x\|^2}{\|A_n^{-1}\|^2 \cdots \|A_0^{-1}\|^2}$$

$$\geq \prod_{i=0}^{n} (1-\|I-A_i{}^*A_i\|) \ \|x\|^2 \geq \delta^2 \|x\|^2 \ \text{for every } n.$$

(d) $\Longrightarrow$ (b): Since each A_n is an invertible contractive matrix,

we have $\quad \| 1-A_n{}^*A_n \| = 1- \min \{\lambda: \lambda \in \sigma_p (A_n{}^*A_n)\}$

$$= 1- \frac{1}{\|(A_n{}^*A_n)^{-1}\|} = 1- \frac{1}{\|A_n^{-1}\|^2} \leq 2(1- \frac{1}{\|A_n^{-1}\|})$$

from Lemma 4.1,

$$\leq 2(1- \frac{|\det A_n|}{\|A_n\|^{N-1}}) \leq 2(1- |\det A_n|).$$

From (d) and Lemma 4.1, we have

$$|\det A_n| \ \cdots\cdots \ |\det A_0| = |\det (A_n \cdots A_0)|$$

$$\geq \|(A_n \cdots A_0)^{-1}\|^{-N} \geq \delta^N,$$

which implies that $\quad \prod_{n=0}^{\infty} |\det A_n| \quad$ converges, and hence

$$\sum_{n=0}^{\infty} \| I- A_n{}^*A_n \| \leq 2 \sum_{n=0}^{\infty} (1- |\det A_n|) < \infty \qquad . \text{Q.E.D.}$$

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