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Semisimple degree of symmetry of manifolds with the homotopy type of $S^{n_1} \times \ldots \times S^{n_k}$ ($n_i = 1,2,3$)

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In this talk, I report some results about the degree of symmetry of certain manifolds. Here the degree of symmetry of a manifold M is, by definition, the maximum of dimension of compact connected Lie groups which act on M topologically and almost effectively. We also define the semisimple degree of symmetry of M by the maximum of dimension of compact connected semisimple Lie groups which act on M topologically and almost effectively. Recently many authors studied the degree of symmetry of manifolds with large low homotopy or cohomology such as A_k-manifolds([5]), hyperaspherical manifolds ([6],[7]) or manifolds with the Property $P_{r,s}$(r = 1,2)([1]). Being motivated by these works, I am interested in studying the degree of symmetry of manifolds of following type.

(1) M is a closed topological manifold with a map $f:M \to (S^1)^r \times (S^2)^s$ of non-zero degree.

(2) M has the same integral cohomology ring as $(S^3)^s$.

(3) M is homotopy equivalent to $(S^1)^r \times (S^2)^{s'} \times (S^3)^{s''}$ ($r > 0$, $s'' > 0$). Put $s = s' + s''$.

We obtain the following results.

A. Let M be a topological manifold of type 1,2, or 3. Then S^3 is only the compact simply connected Lie group which acts on M topologically and almost effectively.

B. Let M be as in A. If a compact connected Lie group G acts on M topologically and almost effectively, then G is locally isomorphic to $T^u \times (S^3)^v$, where

i) <u>$u + v \leq r + 2s$</u>

ii) <u>$v \leq s$</u>

iii) <u>$u + v \leq 2s$ if M is of type 1 and the Euler characteristic of M is non-zero.</u>

C. <u>Let M be a topological manifold of type 1 and L an orientable closed manifold which is not a rational homology sphere with dim L = dim M. Then the connected sum L # M has zero semisimple degree of symmetry.</u>

These results are partly based on a joint work with K. Saito. In this talk, I shall give an outline of the proof of results A and B. From now on, I consider only topological manifolds and topological almost effective actions, and H*( ) denotes rational cohomology.

1. Leray spectral sequence.

The proof is based on the Leray spectral sequence of the orbit map. We recall some basic results about the Leray spectral sequence of the orbit map. Let M be a closed manifold or a covering of a closed manifold. We may treat more general space, but it is sufficient for us to consider only manifolds as above. Assume a compact connected Lie group G acts on M. Let $\pi : M \to M/G$ be the orbit map and $\{E_r^{p,q}, d_r\}$ the Leray spectral sequence of the orbit map π. Recall that

(1) $E_2^{p,q} = H^p(M/G, H^q(\pi))$, where $H^q(\pi)$ is the sheaf generated by the presheaf $U^* \to H^q(\pi^{-1}(U^*))$ for open set U^* in M/G.

(2) The edge homomorphism $e: H^q(M) \to E_2^{0,q}$ is given by $e(a)(x^*) = i_x^*(a)$, where $i_x: G(x) \to M$ is the inclusion.

(3) The stalk of $H^q(\pi)$ at x^* is $H^q(G(x))$.

The following Propositions are basic for the proof.

Proposition 1. Let k be the dimension of a principal orbit. If there is a point x in M such that $H^k(G(x)) = 0$, then the edge homomorphism $e:H^k(M) \longrightarrow E_2^{0,k}$ is trivial. In particular, we have $E_\infty^{0,k} = 0$.

Remark. The hypothesis can be replaced by the following;

There is a point x in M such that $i_x^*:H^k(M) \rightarrow H^k(G(x))$ is trivial.

Proposition 2. Let M be a closed manifold with a map $f:M \rightarrow (S^1)^r \times (S^3)^s$ of non-zero degree. Assume $SU(3)$ or $Sp(2)$ acts on M with a finite principal isotropy subgroup. Then there is a singular orbit.

Proposition 3. Let M be a closed manifold of type 2 or 3. If $SU(2)$ acts on M with a finite principal isotropy subgroup and a singular orbit, then there is a point x whose isotropy subgroup is a torus.

2. Outline of the proof.

Now using Propositions above one can prove the result A as follows. It is sufficient to show that $SU(3)$ or $Sp(2)$ cannot acts on M. Assume $SU(3)$ or $Sp(2)$ acts on M.

The case where M is of type 1. Construct a T^s-bundle $\tilde{M}$ over M as follows. Put $N_i = (S^1)^r \times (S^3)^i \times (S^2)^{s-i}$. Then N_i can be considered as a T^1-bundle over N_{i-1}. Let M_1 be the pull-back of the bundle $N_1 \rightarrow N_0$ by the map f and f_1 the bundle map $M_1 \rightarrow N_1$ covering f. Inductively one can define a sequence of manifolds and maps, $M_0 = M, M_1, \ldots, M_s$, $f_o = f, f_1, \ldots, f_s$ such that

(1) $f_i:M_i \rightarrow N_i$ is a map of non-zero degree.

(2) The following diagram is commutative;

$$\begin{array}{ccc} M_i & \xrightarrow{f_i} & N_i \\ \downarrow & & \downarrow \\ M_{i-1} & \xrightarrow{f_{i-1}} & N_{i-1} \end{array}$$

Define $\widetilde{M} = M_s$ and $f = f_s : \widetilde{M} \longrightarrow N_s = (S^1)^r \times (S^3)^s$. It is well known that the action $\phi : G \times M \longrightarrow M$ is lifted over M_i, which is denoted by ϕ_i, where $G = SU(3)$ or $Sp(2)$. Let $K = SU(2)$ or $Sp(1)$ be the standard subgroup of G and ψ or ψ_i the restriction of ϕ or ϕ_i. One can prove the following

<u>Proposition 4.</u> <u>$\widetilde{\psi}$ is almost free, i.e. all isotropy subgroups are finite, where $\widetilde{\psi} = \psi_s$</u> .

This means that the orbit map $\pi : \widetilde{M} \rightarrow \widetilde{M}/K$ behaves as if it were a fiber bundle in rational coefficients. It is not difficult to see that the edge homomorphism $e : H^3(\widetilde{M}) \longrightarrow E_2^{0,3}$ is surjective, where $\{ E_r^{p,q}, d_r \}$ is the Leray spectral sequence of the map π. In fact, assume the contrary. Since $\dim E_2^{0,3}$ is proved to be one, e is trivial. Hence we have $H^3(\widetilde{M}/K) = E_\infty^{3,0} = H^3(\widetilde{M})$ via the homomorphism π^*, which is proved by the standard spectral sequence arguments. It is easy to see that $H^1(M/K) = H^1(M)$ via π^*. Now these lead immediately to a contradiction. It follows that $H^*(\widetilde{M}) = H^*(\widetilde{M}/K) \otimes H^*(S^3)$ and that $i_x^* : H^3(\widetilde{M}) \rightarrow H^3(K(x))$ is non-trivial for every point x in $\widetilde{M}$. Consider the diagram;

$$\begin{array}{ccc} H^3(\widetilde{M}) & \xrightarrow{i^*_x} & H^3(K(x)) \\ & {\scriptstyle j^*_x} \searrow \quad \nearrow & \\ & H^3(G(x)) & \end{array} .$$

It follows that j^*_x is non-trivial. In particular, we have $H^3(G(x)) \neq 0$ for every point x in $\widetilde{M}$. This means that G_x is finite, which contradicts Proposition 2.

The case where M is of type 2 or 3. As before, one can construct a $T^{s'}$-bundle $\overline{M}$ over M, which is homotopy equivalent to $(S^1)^r \times (S^3)^s$. If M admits an action of G = SU(3) or Sp(2) then $\overline{M}$ does also. Then we may assume M is homotopy equivalent to $(S^1)^r \times (S^3)^s$. Let K, ϕ, ψ be as before. It is easy to see that ψ has a finite pricipal isotropy subgroup. It follows from Proposition 3 that there is a point x whose isotropy subgroup is a torus. Let T and C be a maximal torus and the center of K, respectively. Then one can prove the following

Proposition 5. (1) $M^T \subsetneq M^C \subsetneq M$.
(2) $M^C \underset{Q}{\sim} (S^1)^r \times S^{n_1} \times \ldots \times S^{n_s}$ (n_i = 1, 3), where $X \underset{Q}{\sim} Y$ means that X and Y have the same cohomology rings.

Now one can lead to a contradiction as follows. For simplicity, consider the case r = 0. Consider the following diagrams;

$$(1) \quad H^3(M,M-M^C) \longrightarrow H^3(M) \xrightarrow{i^*} H^3(M-M^C) \longrightarrow H^4(M,M-M^C)$$
$$j_x^*: H^3(M) \searrow H^3(K(x)), \qquad i_x^*: H^3(M-M^C) \swarrow H^3(K(x))$$

$$(2) \quad \begin{array}{ccccccc} H^{m-3}(M,M-M^C) & \longrightarrow & H^{m-3}(M) & \xrightarrow{i^*} & H^{m-3}(M-M^C) & \longrightarrow & H^{m-2}(M,M-M^C) \\ \cong & & \cong & & & & \\ H_3(M^C) & \longrightarrow & H_3(M) & & & & \end{array}$$

where m = dim M.

It can be shown that $M-M^C \underset{Q}{\sim} (M-M^C)/K \times S^3$ and hence $i_x^*: H^3(M-M^C) \longrightarrow H^3(K(x))$ is non-trivial for every point x in $M-M^C$. Since K acts on M with a singular orbit, j_x^* is trivial for every point x in M. These arguments lead to a contradiction as follows. If dim $M^C < m - 4$, then $H^3(M,M-M^C) = H^4(M,M-M^C)$

$= 0$ and hence i^* is an isomorphism. This is clearly a contradiction. If $\dim M^C = m - 2$, then $H^4(M, M-M^C) = H^2(M^C) = 0$. It is easy to see that i_x^* is trivial. Assume $\dim M^C = m - 4$. Let $b_1, \ldots, b_s \in H^3(M)$ be generators of $H^*(M)$. It follows from (1) and (2) that $i^*(b_i) = \pi^*(c_i)$, $c_i \in H^3((M-M^C/K)$ and we may assume $i^*(b_1 \ldots b_{s-1}) \neq 0$. Then we have

$$0 \neq i^*(b_1 \ldots b_{s-1}) = \pi^*(c_1 \ldots c_{s-1}) = 0,$$

since $H^{m-3}(M-M^C)/K) = 0$.

Next we shall prove result B. By the argument in [7], one can prove the following

Proposition 6. Let M be a closed manifold of type 1, 2, or 3. Assume a compact connected semisimple Lie group G_1 acts on M. Then we have $\dim M/G_1 \geqq r$.

Let $G_1 = (SU(2))^v$ act on M and H_1 a principal isotropy subgroup. It is clear that $\dim H_1 \leqq v$. It follows from Proposition 6 that $\dim M/G_1 = \dim M - \dim G_1/H_1 \geqq r$. Since $\dim H_1 \leqq v$, we have $r + 2s - 3v + v \geqq r$, and hence $v \leqq s$.

The case where M is of type 1. Assume that M has non-zero Euler characteristic and admits an action of T^n. Since the action has non-empty fixed point set, the map $ev^x : T^n \to M$ defined by $ev^x(g) = gx$ induces the trivial homomorphism ev^x_*: $\pi_1(T^n) \to \pi_1(M)$. Then it follows from the same arguments as in [7] that the action of T^n is lifted over M', where M' is the pull-back of the covering $R^r \to N = (S^1)^r$ by the map $pr. f : M \to (S^1)^r \times (S^2)^s \xrightarrow{pr.} (S^1)^r$ and $\dim M/T^n \geqq r$. This implies that $2s \geqq n$.

The case where M is of type 2 or 3. One can prove the following

Proposition 7. Let X be a closed $(r+3s)$-dimensional manifold with $X \underset{Q}{\sim} (S^1)^r \times (S^3)^s$. If $(r+s)$-dimensional torus T acts on X almost freely, then X/T has non-zero Euler characteristic.

It follows from this Proposition that if an n-dimensional torus T acts on X almost freely, then n is at most $r+s$.

Now assume an n-dimensional torus T^n acts on M. Then we can decompose T^n as a product $T^n = T_1 \times T_2$ such that T_1 acts on M with a fixed point and T_2 acts on M^{T_1} almost freely. Note that the homomorphism $ev_*^x: \pi_1(T_1) \to \pi_1(M)$ induced by the map $ev^x: T_1 \to M$ is trivial for x in M^{T_1}. It is known that the action of T_1 is lifted over $\tilde{M}$ (= the universal covering of M). Then the same argument of the proof of Proposition 5 shows that $M^{T_1} \underset{Z}{\sim} (S^1)^a \times (S^3)^b$, where $a+b = r+s$ and $r < a$. Let $\dim T_1 = t$. Then it follows that $2t \leq \dim M - \dim M^{T_1}$ $= r + 3s - (a+3b) = 2(s-b)$, i.e. $t \leq s - b$. It follows that $n - t \leq a + b$, which means that $n \leq a + s \leq r + 2s - b \leq r + 2s$.

Finally we shall prove that if $G_1 = SU(2)^{(1)} \times \dots \times SU(2)^{(n)}$ acts on M, then we have $n \leq s$. One can prove the following

Proposition 8. Let X be a closed $(r+3s)$-dimensional manifold with $X \underset{Z}{\sim} (S^1)^r \times (S^3)^s$. If $G = (SU(2))^k$ acts on X almost freely, then we have $k \leq s$.

It follows from this Proposition that if G_1 acts on M almost freely, then $n \leq s$. Assume the action is not almost freely. Then there is an index t such that $T = T^{(1)} \times \dots \times T^{(t)}$ has a fixed point and $G_2 = SU(2)^{(t+1)} \times \dots \times SU(2)^{(n)}$ acts on M^T almost freely. Here $T^{(i)}$ is a maximal torus of $SU(2)^{(i)}$.

As noted above, we have $M^T \underset{Z}{\sim} (S^1)^a \times (S^3)^b$ and hence Proposition 8 shows that $n - t \leq b$. This implies that $n \leq s$ as follows. It follows from a result in [9] that $\dim M - \dim M^T \geq 2t$. Since $a + b = r + s$, we have $t \leq s - b$, which implies that $n \leq s$.

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