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THE OBSTACLE PROBLEM AND ITS APPLICATION

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We shall be concerned with one of the application of the obstacle problem. For the obstacle problem and results relating to quadrature domains, see [5] and [6].

Our subject is the force of gravity. It was the main subject in classical potential theory. In this note, we take up two phenomena. Both of them go back to Newton.

I. A homogeneous ball attracts a particle as if the ball were concentrated at its center.

According to Newton's law of universal gravitation, two particles attract each other with a force whose direction is that of the line joining the two, and whose magnitude is directly as the product of their masses, and inversely as the square of their distance.

For continuously distributed bodies, for example, a homogeneous ball B with centered at c (or, solid sphere), we use the method of the integral calculus to know what the whole force attracts a particle.

For this homogeneous ball, it attracts a particle as if the

ball were concentrated at its center. It is well known that this phenomenon was discovered by Newton. We write this by the equation

$$\int_B \frac{y-x}{|y-x|^3}dy = m(B)\frac{c-x}{|c-x|^3} \qquad \text{for } x \in \mathbb{R}^3 \setminus B,$$

where $m(B)$ denotes the volume of B and we have assumed the density is equal to 1 everywhere on B.

The second we shall treat is the following:

II. In the cavity of an ellipsoidal homoeoid, the force of gravity is null.

It is natural to consider what the whole force attracts a particle when it is inside of the ball. In the above, the particle x is outside of the ball. It is not difficult to show that

$$\int_B \frac{y-x}{|y-x|^3}dy = \begin{cases} m(B_x)\dfrac{c-x}{|c-x|^3} & \text{for } x \in B \setminus \{c\} \\ 0 & \text{for } x = c, \end{cases}$$

where B_x denotes a closed ball with centered at c and radius $|c-x|$. The point x is on the boundary ∂B_x of B_x.

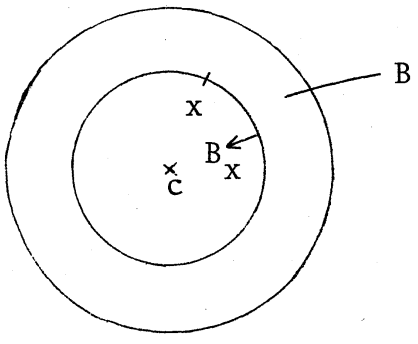

Fig. 1

From this we see that if we fix the inner and outer balls B, B' with the same center, then a particle inside of the inner ball is not attracted to any direction by the whole force produced by B'\B. Namely, a homogeneous body bounded by concentric spheres exercises no attraction in the cavity. Newton discovered another example of a body which has a cavity of null force of gravity. It is an ellipsoidal homoeoid. For the sake of simplicity, we take the origin as its center and write

$$E = \{ x = (x_1,x_2,x_3) \mid k^2 \leqq \sum_{i=1}^{3} (\frac{x_i}{a_i})^2 \leqq 1 \},$$

where a_i are positive constants and k is a constant satisfying $0<k<1$.

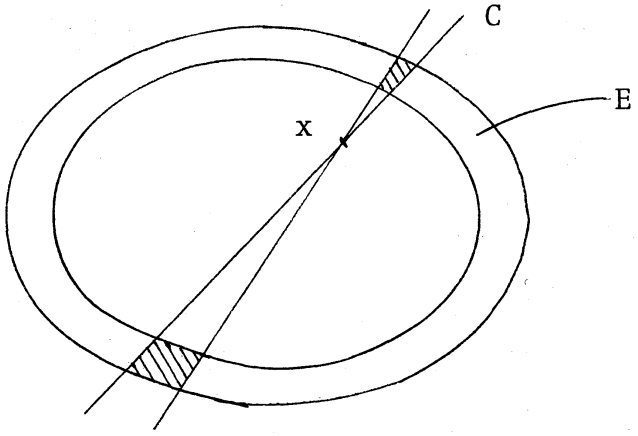

Fig. 2

What was proved by Newton is expressed by

$$\int_{E} \frac{y-x}{|y-x|^3} dy = 0 \quad \text{if} \quad \Sigma(\frac{x_i}{a_i})^2 < k^2.$$

It is natural to ask "Are there any domains other than balls which attract a particle as if they were concentrated at one point?" and "Are there any domains other than ellipsoidal

homoeoids having a cavity of null attraction?".

We shall discuss these problems in this note. It is somewhat surprising that the answers were given quite recently whereas the questions seem to be open for a long time. After giving a brief explanation on the known results, we give new results. Explaining roughly, the answer for the first question is "No." and that for the second is "Yes.". There are several interesting problems still open in the area.

The known results

We summarise here the known results.

I. Let us consider an open set Ω satisfying

$$(*) \qquad \int_{\Omega} \frac{y-x}{|y-x|^3} dy = m(\Omega) \cdot - \frac{x}{|x|^3} \qquad \text{for} \quad x \in \mathbb{R}^3 \setminus \Omega.$$

We have taken the origin for c and from now, we consider open sets instead of closed sets. From a result given by Kuran [3], we can easily deduce the following theorem:

THEOREM A. If Ω satisfies (*), Ω is bounded, the exterior Ω^e of Ω is connected and $\partial\Omega = \partial(\Omega^e)$, then $\Omega = B$ and B is a ball with centered at 0 and $m(\Omega) = m(B)$.

Aharonov, Schiffer and Zalcman [1] deleted superfluous conditions.

THEOREM B. If Ω satisfies (*) and $\int_{\Omega} 1/|x| dx < \infty$, then Ω is

a ball as in Theorem A.

They also treated the case that Ω has a smooth radial density.

THEOREM C. If Ω satisfies

$$\int_{\Omega} \frac{y-x}{|y-x|^3} w(|y|)\,dy = \int_{\Omega} w(|y|)\,dy \cdot -\frac{x}{|x|^3} \qquad \text{for } x \in \mathbb{R}^3 \setminus \Omega$$

and

$$\int_{\Omega} \frac{w(|x|)}{|x|}\,dx < \infty,$$

where $w \in C^1(\mathbb{R})$ satisfies $w \geq w_0$ for some positive constant w_0, then $\Omega = B$ and B is a ball with centered at 0 and $\int_{\Omega} w(|y|)\,dy = \int_{B} w(|y|)\,dy$.

II. For the second phenomenon, I do not know what results are known. But I guess that someone found out the following example:

THEOREM D. An infinite tube whose cross section is a domain surrounded by similar ellipses with the same axes has a cavity of null force of gravity.

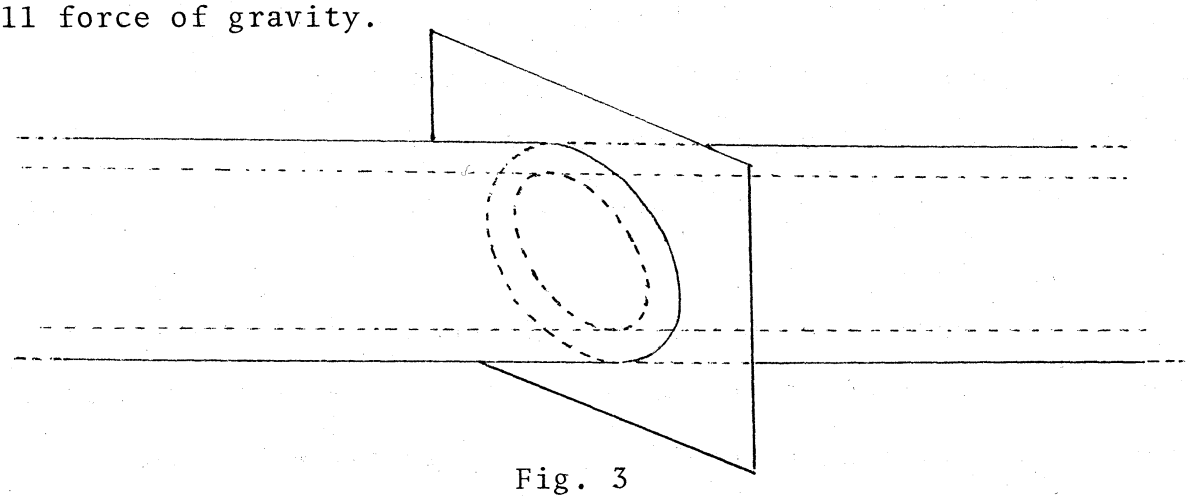

Fig. 3

There are two proofs on the fact that an ellipsoidal homoeoid has a cavity of null force field. Newton's proof is the following. Let x be a point in the cavity. Let C be a small cone with vertex at x and consider a body $C \cap E$, the intersection of C and E. Here "small" means small solid angle. Then $\{-(C-x)+x\} \cap E$ produces the force whose direction is opposite that produced by $C \cap E$ and whose magnitude is the same as that produced by $C \cap E$. And so, two forces are cancelled out each other and we see that the force field in this cavity is null.

By the same reason, we see that in the cavity of the infinite tube in Theorem D, the force field is null. It seems to be difficult to construct another example than ellipsoidal homoeoids and infinite tubes by applying the previous argument.

The second proof on an ellipsoidal homoeoid is done by calculating the Newton potential of an ellipsoid which has no cavites and subtruct the potentail of a similar and smaller ellipsoid with the same axes from it. Then we see that the resulting potentail is constant in the cavity, see Kellogg [2]. But, the calculation of the Newton potentail of an ellipsoid is not easy. To calculate the Newton potential of a given domain is difficult in general, so it is difficult to construct another example by calculating the Newton potential.

It seems that Newton was very lucky, because it looks like as if there are no elementary domains with null-gravitational cavities except ellipsoidal homoeoids. Here "an elementary domain" means a bounded domain surrounded by closed surfaces expressed by

elementary functions. The proof is not given yet, but it seems to be true.

Formulations and new results

Let w be a measurable function satisfying $w_1 \leqq w \leqq w_2$ in $\mathbb{R}^3$ for some positive constants w_1 and w_2. We shall call this the density function or the weight function.

I. Let $\nu(\neq 0)$ be a finite positive measure on $\mathbb{R}^3$ with compact support. Now we define an equi-gravitational domain and give its example.

DEFINITION. An equi-gravitational domain Ω of ν with weight w is an open subset of $\mathbb{R}^3$ satisfying

(1) $\nu(\mathbb{R}^3 \setminus \Omega) = 0$,

(2) $\int_{\Omega} \frac{w(y)}{|y|^2} dy < \infty$,

(3) $\int_{\Omega} \frac{y-x}{|y-x|^3} d\nu(y) = \int_{\Omega} \frac{y-x}{|y-x|^3} w(y)\, dy$ for every $x \in \mathbb{R}^3 \setminus \Omega$ with $\int_{\Omega} \frac{d\nu(y)}{|y-x|^2} < \infty$.

EXAMPLE. A ball B with centered at c is an equi-gravitational domain of $m(B) \cdot \delta_c$ with weight 1, where δ_c denotes the point measure at c.

Next we give three theorems on equi-gravitational domains.

THEOREM 1. If

(a) ν is a singular measure with respect to the 3-dimensional Lebesgue measure m

or

(b) $d\nu = fdm$, $f \in L^1(\mathbb{R}^3)$, $f > w$ on a bounded domain D with smooth boundary and $f = 0$ on $\mathbb{R}^3 \setminus D$,

then there exists an equi-gravitational domain of ν with weight w.

An outline of the proof. Take a sufficiently large ball B, and set $\psi = G(w\chi_B m) - G\nu$, where $G\lambda$ denotes the Green potential of λ on B and χ_B denotes the characteristic function of B. Consider the solution u of the obstacle problem with obstacle ψ. Then $\Omega = \{x \in B \mid u(x) - \psi(x) > 0\}$ is an equi-gravitational domain.

THEOREM 2. Equi-gravitational domains of ν with weight w are uniformly bounded.

An outline of the proof. For a positive measure λ on $\mathbb{R}^3$, we define a vector valued function $\hat{\lambda}$ by

$$\hat{\lambda}(x) = \int \frac{y-x}{|y-x|^3} d\lambda(y)$$

in a domain where $\int (1/|y-x|^2) d\lambda(y) < \infty$. Let B(r) be a ball with radius r and centered at the origin. Then

$$\int_{\partial B(r)} \left(\frac{y-x}{|y-x|^3}, \frac{y}{|y|}\right) d\sigma(y) = \begin{cases} 4\pi & x \in B(r) \\ 0 & x \notin \overline{B(r)}, \end{cases}$$

where σ denotes the surface element of $\partial B(r)$. Hence

$$\frac{1}{4\pi}\int_{\partial B(r)} (\hat{\lambda}(y), \frac{y}{|y|})d\sigma(y) = -\lambda(B(r))$$

for $r > 0$ with

$$\int_{y\in\partial B(r)}\int_{z\in\mathbb{R}^3} \frac{1}{|z-y|^2}d\lambda(z)d\sigma(y) < \infty.$$

Therefore

$$\int_{\Omega\cap B(r)} w dm - \int d\nu = \frac{1}{4\pi}\int_{\partial B(r)} (\hat{\nu}(y) - (w\chi_\Omega)\hat{}(y), \frac{y}{|y|})d\sigma(y)$$

for r satisfying supp $\nu \subset B(r)$.

Since $\hat{\nu}(y) = (w\chi_\Omega)\hat{}(y)$ on $(\mathbb{R}^3\setminus\Omega)\cap\partial B(r)$,

$$\left|\int_{\Omega\cap B(r)} w dm - \int d\nu\right| \leq \frac{1}{4\pi}\int_{\Omega\cap\partial B(r)} |\hat{\nu}(y) - (w\chi_\Omega)\hat{}(y)|d\sigma(y).$$

Set $v(r) = \int_{\Omega\cap\partial B(r)} w(y)d\sigma(y)$, then

$$\int_0^r v(t)dt - \int d\nu \leq v(r)^{1+\frac{1}{2}}(A + B\log\frac{r}{v(r)^{1/2}})$$

for some constants A,B and for r with small $r/v(r)^{1/2}$, see [4,pp. 92-94]. Since $\int_1^\infty v(t)/t^2dt < \infty$, we can apply Lemma 11.1 in [4], use the same argument as in [4,pp.94-95] and obtain $\int_0^\infty v(t)dt < \infty$. Hence

$$\left|\int_0^r v(t)dt - \int d\nu\right| \leq v(r)^{1+\frac{1}{2}}(a + b\log\frac{1}{v(r)})$$

for some constants a , b, and so $\int_0^\infty v(t)dt = \int d\nu$,

$$\int_r^\infty v(t)dt \leq v(r)^{1+\frac{1}{2}}(a + b\log\frac{1}{v(r)})$$

for some constants a, b and for r with small v(r). Therefore the theorem follows from the following lemma:

LEMMA. Let v(r) be a nonnegative integrable function on $[0,+\infty)$ satisfying

$$\int_r^\infty v(r)dt \leqq v(r)^{1+c}(a + b\log\frac{1}{v(r)})$$

for every r with $0 \leqq v(r) \leqq 1/e$, where a,b and c are constants with $a \geqq 0$, $b \geqq 0$, $c > 0$, and the right-hand side is equal to 0 if $v(r) = 0$. Then v vanishes almost everywhere on $[e^{1/c}\int_0^\infty vdt+C, +\infty)$, where $C = \{(1+c)a+(1+2c^{-1})b\}c^{-1}e^{-1}$.

THEOREM 3. If Ω is an equi-gravitational domain of ν with radial weight w satisfying

(i) $\operatorname{supp}\nu \subset \Omega$,

(ii) Ω^e is connected, $\partial\Omega$ is smooth and $\partial\Omega = \partial(\Omega^e)$,

(iii) $$\int_\Omega \frac{y-x}{|y-x|^3}d\nu(y) \neq \int_\Omega \frac{y-x}{|y-x|^3}w(|y|)dy \qquad \text{for } x \in \Omega\setminus\operatorname{supp}\nu$$

then Ω is the unique equi-gravitational domain containing supp ν. Here "radial" means $w(y) = w(|y|)$ on $\mathbb{R}^3$.

We omit the proof. We note that the result given by Aharonov, Schiffer and Zalcman can be improved. As we have pointed out in Example, a ball B is an equi-gravitational domain of $m(B)\cdot\delta_c$ with weight 1. It is easy to show that the ball satisfies (i) to (iii) in Theorem 3, and so the ball is the uniqe equi-gravitational domain. Thus we have solved Rubel's problem cited in [1].

COROLLARY. If Ω satisfies (*) and $\int_{\Omega} 1/|x|^2 dx < \infty$, then Ω is a ball as in Theorem A.

II. Next we give the definition of a domain with null cavity and its existence theorem.

DEFINITION. An open subset Ω of $\mathbb{R}^3$ is called a domain with null cavity under weight w if

$$\int_{\Omega} \frac{y-x}{|y-x|^3} w(y)\, dy = 0$$

for every x in some component C of Ω^e. We call this component C a null cavity.

It is somewhat surprising that there are many domains with null cavity.

THEOREM 4. Let D be a bounded connected open set with smooth boundary such that D^e is connected. Then, for every neighborhood N of ∂D, there is a domain Ω with null cavity such that $\Omega \subset N$ and

$$\int_{\Omega} \frac{y-x}{|y-x|^3} w(y)\, dy = 0$$

for every $x \in D \setminus \Omega$.

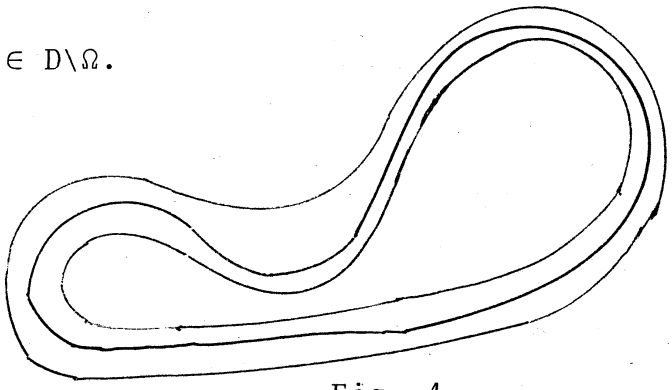

Fig. 4

An outline of the proof. Take an equilibrium mass distribution ν for $\overline{D}$, for example, the capacitary mass distribution for $\overline{D}$. Then it is concentrated on ∂D and its potential is constant in D. Let ε be a positive constant and set $\psi_\varepsilon = G(w\chi_B m) - G(\varepsilon\nu)$ as in the proof of Theorem 1. Then $\Omega_\varepsilon = \{x \in B \mid u_\varepsilon(x) - \psi_\varepsilon(x) > 0\}$, where u_ε is the solution of the obstacle problem with obstacle ψ_ε, satisfies

$$\int_{\Omega_\varepsilon} \frac{y-x}{|y-x|^3} w(y)\,dy = 0 \qquad \text{for every } x \in D\setminus\Omega_\varepsilon .$$

By taking ε so that $\Omega_\varepsilon \subset N$ and setting $\Omega = \Omega_\varepsilon$, we obtain the required domain Ω.

From the proof given here, we can also construct a domain having a finite number of null cavity.

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