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On Beppo Levi spaces

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Let R^n be the n-dimensional Euclidean space. We write $x = (x_1,\cdots,x_n)$, $y = (y_1,\cdots,y_n),\cdots$ for the elements of R^n. The inner product of $x,y\in R^n$ is the number $(x,y) = \Sigma_{j=1}^n x_j y_j$; the norm of $x\in R^n$ is the number $|x| = (x,x)^{1/2}$. If $\alpha = (\alpha_1,\cdots,\alpha_n)$ is an n-tuple of nonnegative integers α_j, we call α a multi-index and denote by x^α the monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, which has degree $|\alpha| = \Sigma_{j=1}^n \alpha_j$. Similarly, if $D_j = \partial/\partial x_j$ for $1\le j\le n$, then $D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ denotes a differential operator of order $|\alpha|$. We shall use the following notations of L.Schwarz[5]: $\mathcal{D}(R^n) = \mathcal{D}$, $\mathcal{D}'(R^n) = \mathcal{D}'$. For a positive integer m and $p > 1$, the Beppo Levi space $L_m^p(R^n) = L_m^p$ is defined as follows:

$$L_m^p = \{u\in\mathcal{D}' ;\ D^\alpha u\in L^p(R^n) \text{ for any } \alpha \text{ with } |\alpha| = m\}.$$

Furthermore $u_k\to 0\,(k\to\infty)$ in L_m^p means that $u_k\to 0\,(k\to\infty)$ in $\mathcal{D}'$ and $|u_k|_{m,p} = \Sigma_{|\alpha|=m} \|D^\alpha u\|_p \to 0\,(k\to\infty)$. We note that L_m^p is contained in L_{loc}^p ([2]).

Our purpose is to investigate the aspects of the space L_m^p.

First we give a remark about the topology in L_m^p.

Remark. The following three conditions are mutually equivalent:

i) $u_k\to 0\,(k\to\infty)$ in $\mathcal{D}'$ and $|u|_{m,p}\to 0\,(k\to\infty)$,

ii) $u_k\to 0\,(k\to\infty)$ in L_{loc}^p and $|u|_{m,p}\to 0\,(k\to\infty)$, iii)

$$\left(\int_{|x|\leq 1} |u_k|^p dx\right)^{1/p} + |u|_{m,p} \to 0\,(k\to\infty).$$

We also note that L^p_m is a Banach space. We shall give some observations on the space L^p_m in case of $m - (n/p) < 0$. We assume $m - (n/p) < 0$. For $u \in \mathcal{D}$, u has the following integral representation ([2],[4]):

$$u(x) = \Sigma_{|\alpha|=m}\, a_\alpha \int_0^\infty \xi^\alpha t^{m-1} D^\alpha u(x-t\xi)\,dt$$

$$= \Sigma_{|\alpha|=m}\, a_\alpha \int \frac{(x-y)^\alpha}{|x-y|^n} D^\alpha u(y)\; dy$$

where ξ is an arbitrary point on the unit sphere. By the Hardy-Littlewood-Sobolev's inequality, we see

$$\left(\int \left| \int \frac{(x-y)^\alpha}{|x-y|^n} D^\alpha u(y)\,dy\right|^{p_m} dx\right)^{1/p_m}$$

$$\leqq \left(\int \left(\int |x-y|^{m-n} |D^\alpha u(y)|\,dy\right)^{p_m} dx\right)^{1/p_m}$$

$$\leqq C\,\|D^\alpha u\|_p,$$

where $1/p_m = (1/p)-(m/n)$, and hence we have $\|u\|_{p_m} \leqq C|u|_{m,p}$. Therefore for a seqeunce $\{u_k\}$ in $\mathcal{D}$ it fololows from $|u_k|_{m,p} \to 0$ $(k\to\infty)$ that u_k converges to 0 in L^{p_m} and hence $u_k \to 0\,(k\to\infty)$ in L^p_m. Moreover from Theorem B* in [6] we obtain

$$\left(\int |x|^{-mp} \left|\int \frac{(x-y)^\alpha}{|x-y|^n} D^\alpha u(y)\,dy\right|^p dx\right)^{1/p}$$

$$\leqq \left(\int |x|^{-mp} \left(\int |x-y|^{m-n} |D^\alpha u(y)|\,dy\right)^p dx\right)^{1/p}$$

$$\leq C\|D^\alpha u\|_p,$$

so that

$$\left(\int |x|^{-mp}|u(x)|^p\,dx\right)^{1/p} \leq C|u|_{m,p}. \qquad (*)$$

Remark. In case of $m - (n/p) < 0$, from $(*)$ we have the following estimate:

$$\left(\int (1+|x|)^{-mp}|u(x)|^p dx\right)^{1/p} \leq C|u|_{m,p} \qquad \text{for } u \in \mathcal{D}.$$

In case of $m - (n/p) \geqq 0$, the above estimate is not valid. However, in case of $m - (n/p) > 0$ and $\neq$ integer, we have the following estimates:

i) $$\left(\int (1+|x|)^{-mp}|u(x)|^p dx\right)^{1/p} \leq C(1+r^{m-(n/p)})|u|_{m,p}$$

for $u \in \mathcal{D}$ and $\operatorname{supp} u \subset \{|x| \leq r\}$.

ii) $|D^\beta u(0)| \leq Cr^{m-(n/p)-|\beta|}|u|_{m,p}$ for $u \in \mathcal{D}$, $\operatorname{supp} u \subset \{|x| \leq r\}$ and $|\beta| \leq [m-(n/p)]$.

Next we state the following proposition which has independent interest. We denote by e_j the multi-index $(0,\cdots,\overset{j}{1},\cdots,0)$.

Proposition 1. We assume $m - (n/p) < 0$. Let $\{f_\alpha\}_{|\alpha|=m}$ be a family of functions in L^p and assume $D_i f_\alpha = D_j f_\beta$ for any α, β with $|\alpha| = |\beta| = m$ and $\alpha + e_i = \beta + e_j$. Then $D^\alpha F = f_\alpha$ for any α with $|\alpha| = m$ if we put

$$F(x) = \Sigma_{|\alpha|=m}\, a_\alpha \int \frac{(x-y)^\alpha}{|x-y|^n} f_\alpha(y)\,dy.$$

For $u \in L^p_m$, $D^\alpha v = D^\alpha u$ for any α with $|\alpha| = m$ if we put

$$v(x) = \Sigma_{|\alpha|=m} a_\alpha \int \frac{(x-y)^\alpha}{|x-y|^n} D^\alpha u(y)dy$$

from Proposition 1. Hence there exists a polynomial P of degree $\leqq m - 1$ such that $u = v + P$. We note that

$$(\int |x|^{-mp}|v(x)|^p dx)^{1/p} \leqq C|u|_{m,p}.$$

When $m - (n/p) \geqq 0$, the aspects of the space L^p_m are rather different

Remark. Let $m - (n/p) \geqq 0$ and $[m-(n/p)] = d$.

i) There exists a sequence $\{\psi_k\}$ in $\mathcal{D}$ such that $\psi_k(x) \to \infty$ $(k \to \infty)$ for all $x \in R^n$ and $|\psi_k|_{m,p} \to 0$ $(k \to \infty)$.

ii) (see [3]) For any polynomial P of degree $\leqq d$, there exists a sequence $\{\phi_k\}$ in $\mathcal{D}$ such that $\phi_k \to P$ $(k \to \infty)$ in $\mathcal{D}'$ and $|\phi_k|_{m,p} \to 0$ $(k \to \infty)$.

A general proposition may be formulated as follows.

Proposition 2. (cf[1]) Let m be a positive integer and $p > 1$. Suppose that u belongs to L^p_m. Then

$$\int |u(x)|^p (1+|x|)^{-mp}(\log(e+|x|))^{-p} dx < \infty$$

if and only if there exists a sequence $\{u_k\}$ in $\mathcal{D}$ such that $u_k \to u$ $(k \to \infty)$ in L^p_m.

Now we shall study the space L^p_m in case of $m - (n/p) > 0$ and $\ne 1,2,\cdots,m-1$. Let $[m-(n/p)] = d$. For $u \in C^\infty(R^n)$ from the Taylor's formula we have

$$u(x) - \Sigma_{|\beta|\leqq d} (D^{\beta}u(0)/\beta!)x^{\beta}$$

$$= \Sigma_{|\gamma|=d+1}((d+1)/\gamma!)\int_0^{|x|}(|x|-t)^d x'^{\gamma} D^{\gamma}u(tx')\ dt$$

where $x' = (x/|x|)$. In order to estimate the right side we establish the following integral inequalities.

Proposition 3. i) Let $m - (n/p) > d$ and h be a nonnegative measurable function on $(0,\infty)$. Then we have

$$\left(\int_0^{\infty} r^{-mp+n-1}\left(\int_0^r (r-t)^d t^{m-d-1-((n-1)/p)} h(t)dt\right)^p dr\right)^{1/p}$$

$$\leqq C\left(\int_0^{\infty} h(r)^p dr\right)^{1/p}.$$

ii) Let $m - (n/p) > d$ and w be a nonnegative continuous function on R^n. Then we obtain

$$\left(\int |x|^{-mp}\left(\int_0^{|x|}(|x|-t)^d w(tx')dt\right)^p dx\right)^{1/p}$$

$$\leqq C\left(\int |x|^{-(m-d-1)p}\ w(x)^p\ dx\right)^{1/p}.$$

It follows from Proposition 3 that

$$\left(\int |x|^{-mp}|u(x)-\Sigma_{|\beta|\leqq d} (D^{\beta}u(0)/\beta!)x^{\beta}|^p\ dx\right)^{1/p}$$

$$\leqq C\Sigma_{|\gamma|=d+1} \left(\int |x|^{-(m-d-1)p}|D^{\gamma}u(x)|^p\ dx\right)^{1/p}$$

for $u \in C^{\infty}(R^n)$. When in particular u belongs to $\mathcal{D}$, we get

$$\left(\int |x|^{-(m-d-1)p}|D^{\gamma}u(x)|^p\ dx\right)^{1/p}$$

$$\leqq C\Sigma_{|\delta|=m-d-1} \left(\int |D^{\delta}D^{\gamma}u(x)|^{p}dx\right)^{1/p}$$

since $(m-d-1)p < n$, and hence we have

$$\left(\int |x|^{-mp}|u(x)-\Sigma_{|\beta|\leqq d} (D^{\beta}u(0)/\beta!)x^{\beta}|^{p}dx \leqq C|u|_{m,p}.\right.$$

Thus, if we put $P(x) = \Sigma_{|\beta|\leqq d} (D^{\beta}(0)/\beta!)x^{\beta}$ and $v(x) = u(x) - p(x)$ for $u \in \mathfrak{D}$, P and v satisfy the following conditions:

(i) P can be approximated by a sequence in $\mathfrak{D}$,

(ii) $D^{\alpha}v = D^{\alpha}u$ for any α with $|\alpha| = m$,

(iii) $D^{\beta}v(0) = 0$ for any β with $|\beta| \leq d$,

(iv) $\left(\int |x|^{-mp}|v(x)|^{p}dx\right)^{1/p} \leqq C|u|_{m,p}.$

Next, in order to give the decomposition of u in L^p_m we state the following proposition about a primitive of functions. We require several notations. For $x \in R^n$, Let $L = \{\xi\in R^n;\ (\xi,x) = 0\}$ and $M = L \cap B$, where B is the unit ball, centered at the origin. Furthermore we put $M_1^x = M + (x/|x|)$, $M_2^x = M + x - (x/|x|)$,

$$D_1^x = \{t(\xi+(x/|x|));\ \xi \in M_1^x,\ 0\leqq t\leqq |x|/2\},$$

and

$$D_2^x = \{x-(|x|-t)((x/|x|)-\xi);\ \xi \in M_2^x,\ |x|/2\leqq t\leqq |x|\}.$$

Proposition 4. We assume that $p > n$. Let $\{f_j\}_{j=1,\cdots,n}$ be a family of functions in L^p and assume $D_if_j = D_jf_i$ for any i,j. If we put

$$G(x) = a\left(\Sigma_{j=1}^{n}\int_{D_1^x} \frac{y_j}{|y|^n\cos^n\theta_1} f_j(y)dy + \int_{D_2^x} \frac{x_j-y_j}{|x-y|^n\cos^n\theta_2} f_j(y)dy\right),$$

then $G(x)$ is continuous, $G(0) = 0$, $D_jG = f_j$ for $1 \leqq j \leqq n$ and

$$\left(\int |x|^{-\sigma p}|G(x)|^p dx\right)^{1/p} \leqq C\Sigma_{j=1}^{n}\left(\int |x|^{-(\sigma-1)p}|f_j(x)|^p\, dx\right)^{1/p}$$

for $\sigma > (n/p)$, where θ_1 (resp. θ_2) is the angle between x and y (resp. $-x$ and $y-x$).

Let $u \in L^p_m$. For γ with $|\gamma| = d + 1$, we put

$$u^\gamma(x) = \Sigma_{|\delta|=m-d-1}\, a_\delta \int \frac{(x-y)^\delta}{|x-y|^n} D^\delta D^\gamma u(y)\,dy.$$

From $(m-d-1)p < n$, we have $D^\delta u^\gamma = D^\delta D^\gamma u$ for δ with $|\delta| = m-d-1$ by Proposition 1. Furthermore we see

$$\left(\int |x|^{-(m-d-1)p}|u^\gamma(x)|^p dx\right)^{1/p} \leqq C\Sigma_{|\delta|=m-d-1}\|D^\delta D^\gamma u\|_p$$

and $u^\gamma \in L^{p_{m-d-1}}$, where $1/p_{m-d-1} = (1/p) - ((m-d-1)/n)$. We note that $p_{m-d-1} > n$ from $m - (n/p) > d$. For γ with $|\gamma| = d$, we put

$$u^\gamma(x) = a\Sigma_{j=1}^{n}\int_{D_1^x} \frac{y_j}{|y|^n \cos^n\theta_1} u^{\gamma+e_j}(y)\,dy + \int_{D_2^x} \frac{x_j-y_j}{|x-y|^n\cos\theta_2} u^{\gamma+e_j}(y)\,dy.$$

On account of Proposition 4 it follows that $u^\gamma \in C^0$, $u^\gamma(0) = 0$, $D_j u^\gamma = u^{\gamma+e_j}$ and

$$\left(\int |x|^{-(m-d)p}|u^\gamma(x)|^p dx\right)^{1/p} \leqq C\Sigma_{j=1}^{n}\left(\int |x|^{-(m-d-1)p}|u^{\gamma+e_j}(x)|^p dx\right)^{1/p}.$$

Repeating this argument, we get the function v satisfying the following conditions:

(i) $v \in C^d(R^n)$,

(ii) $D^\beta v(0) = 0$ for any β with $|\beta| \leqq d$,

(iii) $D^\gamma v = u^\gamma$ for any γ with $|\gamma| = d + 1$,

(iv) $\left(\int |x|^{-mp}|v(x)|^p dx\right)^{1/p} \leqq C\Sigma_{|\gamma|=d+1}\left(\int |x|^{-(m-d-1)p}|u^\gamma(x)|^p dx\right)^{1/p}$

so that

(v) $D^\alpha v = D^\alpha u$ for any α with $|\alpha| = m$,

(vi) $\left(\int |x|^{-mp}|v(x)|^p\, dx\right)^{1/p} \leqq C|u|_{m,p}$.

If we put $P = u - v$, then $P(x) = \Sigma_{|\gamma|\leqq m-1}\, a_\gamma x^\gamma$. From $D^\beta v(0) = 0$ for any β with $|\beta| \leqq d$, it follows that $a_\beta = (D^\beta u(0)/\beta!)$ for any β with $|\beta| \leqq d$. Thus we have the following theorem.

Theorem. Let $m - (n/p) \neq 0,1,\cdots, m-1$ and $d = [m - (n/p)]$. Then every function $u \in L^p_m$ has the following unique decomposition:

$$u = P_1 + P_2 + v$$

where $P_1(x) = \Sigma_{d+1\leqq|\gamma|\leqq m-1}\, a_\gamma x^\gamma$, $P_2(x) = \Sigma_{|\beta|\leqq d}\, (D^\beta u(0)/\beta!)x^\beta$, $v \in C^d(R^n)$, $D^\beta v(0) = 0$ for any β with $|\beta| \leqq d$,

$$\left(\int (1+|x|)^{-mp}|P_2(x)|^p dx \leqq C\left(\left(\int_B |u(x)|^p dx\right)^{1/p} + |u|_{m,p}\right),\right.$$

and

$$\left(\int |x|^{-mp}|v(x)|^p dx\right)^{1/p} \leqq C|u|_{m,p}.$$

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