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Fine limits of logarithmic potentials

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1. Statement of results

Let R^n ($n \geqq 2$) be the n-dimensional euclidean space. For a nonnegative (Radon) measure μ on R^n, we set

$$L\mu(x) = \int \log (1/|x-y|)\, d\mu(y)$$

if the integral exists at x. We note here that $L\mu$ is not identically $-\infty$ if and only if

$$\int \log (1+|y|)\, d\mu(y) < \infty. \tag{1}$$

Denote by $B(x,r)$ the open ball with center at x and radius r. For $E \subset B(0,2)$, define

$$C(E) = \inf \mu(R^n),$$

where the infimum is taken over all nonnegative measures μ on R^n such that S_μ (the support of μ) $\subset B(0,4)$ and

$$\int \log (8/|x-y|)\, d\mu(y) \geqq 1 \qquad \text{for every } x \in E.$$

If $E \subset B(x^0,2)$, then we set

$$C(E) = C(\{x-x^0;\ x \in E\}).$$

We note here that this is well defined.

Fuglede [2] discussed fine differentiability properties of logarithmic potentials in the plane. To state his result, we let

$$L(x) = \log (1/|x|)$$

and set for a nonnegative integer m,

$$L_m(x,y) = L(x-y) - \sum_{|\lambda| \leq m} (\lambda!)^{-1}(x-x^0)^\lambda \left[\left(\frac{\partial}{\partial x}\right)^\lambda L\right](x^0-y),$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a multi-index with length $|\lambda| = \lambda_1 + \ldots + \lambda_n$, $\lambda! = \lambda_1! \ldots \lambda_n!$, $x^\lambda = x_1^{\lambda_1} \ldots x_n^{\lambda_n}$ and $(\partial/\partial x)^\lambda = (\partial/\partial x_1)^{\lambda_1} \ldots (\partial/\partial x_n)^{\lambda_n}$.

Theorem 1 (Fuglede [2; Notes 3]). Let μ be a nonnegative measure on R^n satisfying (1) and

$$\int |x^0-y|^{-1}\log (2+|x^0-y|^{-1})d\mu(y) < \infty,$$

then there exists a set E in R^n which is logarithmically thin at x^0 and satisfies

$$\lim_{x \to x^0,\, x \in R^n - E} |x-x^0|^{-1} \int L_1(x,y)d\mu(y) = 0.$$

Here a set E in R^n is called logarithmically thin at x^0 if

$$\sum_{j=1}^{\infty} jC(E_j') < \infty,$$

where $E_j' = \{x \in B(x^0,2)-B(x^0,1);\ x^0 + 2^{-j}(x-x^0) \in E\}$. For a proof of Theorem 1, see also Davie and Øksendal [1; Theorem 6].

Our main aim in this paper is to establish the following two theorems.

Theorem 2. Let μ be a nonnegative measure on R^n satisfying (1) and

$$\int |x^0-y|^{-m} d\mu(y) < \infty$$

for a positive integer m smaller than n. Then there exists a set E in R^n such that

$$\lim_{x \to x^0,\, x \in R^n - E} |x-x^0|^{-m} \int L_m(x, y) d\mu(y) = 0$$

and

$$\sum_{j=1}^{\infty} C(E_j') < \infty.$$

Theorem 3. Let μ be a nonnegative measure on R^n satisfying (1) and the following two conditions:

(a) $\lim_{r \downarrow 0} r^{-n} |\mu - a\Lambda_n|(B(x^0,r)) = 0$ for some a,

where Λ_n denotes the n-dimensional Lebesgue measure;

(b) $A_\lambda = \lim_{r \downarrow 0} \int_{R^n - B(x^0,r)} \left[\left(\frac{\partial}{\partial x}\right)^\lambda L\right](x^0-y) d\mu(y)$

exists and is finite for any λ with length n.

Then there exists a set E in R^n such that

(i) $$\lim_{x\to x^0,\, x\in R^n-E} |x-x^0|^{-n} \left\{\int L_{n-1}(x,y)d\mu(y) - \sum_{|\lambda|=n} (\lambda!)^{-1} C_\lambda (x-x^0)^\lambda\right\} = 0$$

and

(ii) $$\lim_{j\to\infty} C(E_j') = 0,$$

where $C_\lambda = A_\lambda + aB_\lambda$ for $|\lambda| = n$ and B_λ will be defined later (in Lemma 2).

One may compare these theorems with fine and semi-fine differentiabilities of Riesz potentials investigated by Mizuta [3] and [4].

Remark. Set $E = \{x; \int |x-y|^{-m} d\mu(y) = \infty\}$ for a nonnegative measure μ on R^n satisfying (1). Then $C_{n-m}(E) = 0$, where C_α denotes the Riesz capacity of order α. Further we note that (a) and (b) in theorem 3 hold for almost every x^0 (cf. [5; Chap. III, 4.1]).

2. Proof of Theorem 2

In this section we prove the following generalization of Theorems 1 and 2.

Theorem 2'. Let h and k be positive and nonincreasing functions on the interval $(0, \infty)$ such that

(a) $rh(r)$ is nondecreasing on $(0, \infty)$ and $\lim_{r \downarrow 0} rh(r) = 0$;

(b) $k(r) \leqq \text{const.}\ k(2r)$ for $r > 0$.

Let μ be a nonnegative measure on R^n satisfying (1) and

$$\int |x^0-y|^{-m} H(|x^0-y|)\, d\mu(y) < \infty,$$

for a positive integer m, where $H(0) = \infty$ and $H(r) = h(r)k(r)$ for $r > 0$. Then there exists a set E in R^n such that

(i) $$\lim_{x \to x^0,\, x \in R^n - E} |x-x^0|^{-m} h(|x-x^0|) \int L_m(x,y)\, d\mu(y) = 0;$$

(ii) $$\sum_{j=1}^{\infty} k(2^{-j}) C(E_j') < \infty.$$

Proof. Without loss of generality, we may assume that $x^0 = 0$. Let μ be a nonnegative measure on R^n satisfying (1) and

$$\int |y|^{-m} H(|y|)\, d\mu(y) < \infty.$$

We write

$$\int L_m(x,y)\, d\mu(y) = \int_{R^n - B(0,2|x|)} L_m(x,y)\, d\mu(y)$$
$$+ \int_{B(0,2|x|) - B(x,|x|/2)} L_m(x,y)\, d\mu(y)$$
$$+ \int_{B(x,|x|/2)} L_m(x,y)\, d\mu(y)$$

$$= u_1(x) + u_2(x) + u_3(x).$$

If $y \in R^n - B(0,2|x|)$, then we have by elementary calculations

$$|L_m(x,y)| \leqq \text{const. } |x|^{m+1}|y|^{-m-1},$$

so that Lebesgue's dominated convergence theorem gives

$$\limsup_{x\to 0} |x|^{-m}h(|x|)|u_1(x)|$$

$$\leqq \text{const. } \limsup_{x\to 0} |x|h(|x|) \int_{R^n-B(0,2|x|)} |y|^{-m-1}d\mu(y)$$

$$= \text{const. } \limsup_{x\to 0} |x|h(|x|) \int_{B(0,1)-B(0,2|x|)} |y|^{-m-1}d\mu(y) = 0$$

since $\lim_{r\downarrow 0} rh(r) = 0$ and $rh(r) \leqq k(1)^{-1}sh(s)k(s)$ whenever $0 < r < s < 1$.

If $y \in B(0,2|x|)$ and $|x-y| \geqq |x|/2 > 0$, then

$$|L_m(x,y)| \leqq \text{const. } |x|^m|y|^{-m}.$$

Hence we obtain

$$\limsup_{x\to 0} |x|^{-m}h(|x|)|u_2(x)|$$

$$\leqq \text{const. } \limsup_{x\to 0} h(|x|)\int_{B(0,2|x|)} |y|^{-m}d\mu(y) = 0$$

since $h(r) \leqq h(s) \leqq 2h(2s) \leqq 2k(1)^{-1}h(2s)k(2s)$ whenever $0 < s < r < 1/2$.

As to u_3, we note that

$$|x|^{-m}h(|x|)|u_3(x)|$$
$$\leqq \text{const. } |x|^{-m}h(|x|)\int_{B(x,|x|/2)} \log(|x|/|x-y|)d\mu(y)$$
$$+ \text{const. } \int_{B(x,|x|/2)} |y|^{-m}h(|y|)d\mu(y).$$

The second term of the right hand side tends to zero as $x \to 0$ by the assumption. What remains is to prove that the first term of the right hand side tends to zero as $x \to 0$, $x \in R^n - E$, where E is a set in R^n satisfying property (ii). To prove this, take a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j \int_{B_j} |y|^{-m}H(|y|)d\mu(y) < \infty,$$

where $B_j = B(0, 2^{-j+2}) - B(0, 2^{-j-1})$. Consider the sets

$$E_j = \left\{x \in A_j;\ \int_{B_j} \log(2^{-j+3}/|x-y|)d\mu(y) \geqq 2^{-mj}h(2^{-j})^{-1}a_j^{-1}\right\}$$

for $j = 1,2,\ldots$, and $E = \bigcup_{j=1}^{\infty} E_j$, where $A_j = B(0, 2^{-j+1}) - B(0, 2^{-j})$. By the assumption on h, one sees easily that

$$k(2^{-j})C(E_j') \leqq a_j 2^{mj} H(2^{-j})\mu(B_j) \leqq \text{const. } a_j \int_{B_j} |y|^{-m} H(|y|)d\mu(y).$$

Hence E satisfies property (ii). Further,

$$\limsup_{x\to 0,\, x\in R^n - E} |x|^{-m} h(|x|) \int_{B(x,|x|/2)} \log(|x|/|x-y|)d\mu(y)$$

$$\leqq \text{const. } \limsup_{j\to\infty} \sup_{x\in A_j - E_j} 2^{mj} h(2^{-j}) \int_{B_j} \log(2^{-j+3}/|x-y|)d\mu(y)$$

$$\leqq \text{const. } \limsup_{j\to\infty} a_j^{-1} = 0,$$

and hence $\lim_{x\to 0,\, x\in R^n - E} |x|^{-m} h(|x|) \int_{B(x,|x|/2)} \log(|x|/|x-y|)d\mu(y) = 0.$

Thus the proof is complete.

Remark 1. Theorem 2' is best possible as to the size of the exceptional set. In fact, if E is a set in R^n satisfying property (ii), then one can find a nonnegative measure μ on R^n with compact support such that

$$\int |x^0 - y|^{-m} H(|x^0 - y|)d\mu(y) < \infty$$

and

$$\lim_{x\to x^0,\, x\in E} |x - x^0|^{-m} h(|x - x^0|) \int L_m(x,y)d\mu(y) = \infty.$$

Remark 2. Let μ be a nonnegative measure on R^n satisfying (1) and $\int |x^0-y|^{-m}h(|x^0-y|)d\mu(y) < \infty$. If in addition there exist M, $r_0 > 0$ such that

$$h(|x-x^0|)\mu(B(x,r)) \leqq Mr^m$$

for any $x \in B(x^0,r_0)$ and any r, $0 < r < |x-x^0|/2$, then E appeared in Theorem 2' can be taken to be an empty set and $L\mu$ is m times differentiable at x^0.

To prove this, assume that $x^0 = 0$. For the first assertion, in view of the proof of Theorem 2', it suffices to show that

$$(2) \qquad \lim_{x\to 0} |x|^{-m}h(|x|)\int_{B(x,|x|/2)} \log (|x|/|x-y|)d\mu(y) = 0.$$

For $\delta > 0$, set $\varepsilon(\delta) = \sup_{0<r\leqq\delta} r^{-m}h(r)\mu(B(0, r))$. If $0 < \delta < |x|/2$, then

$$|x|^{-m}h(|x|)\int_{B(x,|x|/2)} \log (|x|/|x-y|)d\mu(y)$$

$$= |x|^{-m}h(|x|)\int_{B(x,\delta)} \log (|x|/|x-y|)d\mu(y)$$

$$+ |x|^{-m}h(|x|)\int_{B(x,|x|/2)-B(x,\delta)} \log (|x|/|x-y|)d\mu(y)$$

$$\leqq \text{const.} \left\{(\delta/|x|)^m \log (|x|/\delta) + |x|^{-m}h(|x|)\mu(B(0,2|x|))\log (|x|/\delta)\right\}$$

$\leqq$ const. $\{(\delta/|x|)^m + \varepsilon(2|x|)\}\log(|x|/\delta)$.

Since $\lim_{x\to 0} \varepsilon(2|x|) = 0$, for x sufficiently close to 0 we can choose $\delta > 0$ so that

$$\log(|x|/\delta) = [\varepsilon(2|x|) + |x|]^{-1/2}.$$

Since $\lim_{x\to 0} (\delta/|x|) = 0$, we derive (2).

To prove the second assertion, we first note that

$$\int |x-y|^{-m+1} d\mu(y) < \infty \qquad \text{for every } x \in B(0, r_0),$$

and hence $L\mu$ is $m - 1$ times differentiable at $x \in B(0, r_0)$ and

$$(\partial/\partial x)^\lambda L\mu(x) = \int [(\partial/\partial x)^\lambda L](x - y) d\mu(y)$$

for any $x \in B(0, r_0)$ and any multi-index λ with length $m - 1$. As in the proof of Theorem 2', we can prove that

$$\lim_{x\to 0} |x|^{-1} h(|x|)\left\{u_\lambda(x) - u_\lambda(0) - \sum_{i=1}^{n} a_i x_i\right\} = 0,$$

where $x = (x_1, \ldots, x_n)$, $u_\lambda = (\partial/\partial x)^\lambda L\mu$ for a multi-index λ with length m - 1 and $a_i = \int [(\partial/\partial x_i)(\partial/\partial x)^\lambda L](-y)d\mu(y)$. This implies that $L\mu$ is m times differentiable at 0.

3. Proof of Theorem 3

We first recall the following results.

Lemma 1 (cf. [4; Lemma 1]). Let μ be a nonnegative measure on R^n such that $\lim_{r \downarrow 0} r^{\alpha-n}\mu(B(0, r)) = 0$ for some real number α. Then the following statements hold:

(i) If $\beta < 0$, then $\lim_{r\downarrow 0} r^\beta \int_{B(0,r)} |y|^{\alpha-\beta-n} d\mu(y) = 0$;

(ii) If $n - \alpha + 1 > 0$ and $\beta > 0$, then

$$\lim_{r\downarrow 0} r^\beta \int_{B(0,1)} (r + |y|)^{\alpha-\beta-n} d\mu(y) = 0.$$

Lemma 2 (cf. [4; Lemma 4]). Set $u(x) = \int_{B(x^0,1)} L(x - y)dy$. Then $u \in C^\infty(B(x^0, 1))$. Moreover, if λ is a multi-index with length n, then

$$B_\lambda \equiv [(\partial/\partial x)^\lambda u](x^0) = \int_{\partial B(0,1)} y^{\lambda'}[(\partial/\partial x)^{\lambda''} L](y)dS(y),$$

where $\lambda = \lambda' + \lambda''$ and $|\lambda'| = 1$.

Now we prove Theorem 3 by assuming that $x^0 = 0$. Let μ be a nonnegative measure on R^n satisfying (1), (a) and (b) with $x^0 = 0$, and set $\nu = \mu - a\Lambda_n$. For $x \in B(0, 1/2) - \{0\}$, we write

$$|x|^{-n}\left\{\int L_{n-1}(x,y)\,d\mu(y) - \sum_{|\lambda|=n} (\lambda!)^{-1} C_\lambda x^\lambda\right\}$$

$$= |x|^{-n}\int_{R^n - B(0,1)} L_n(x,y)\,d\mu(y)$$

$$+ |x|^{-n}\int_{B(0,1)-B(0,2|x|)} L_n(x,y)\,d\nu(y)$$

$$- |x|^{-n}\sum_{0<|\lambda|\leqq n} (\lambda!)^{-1} x^\lambda \lim_{r\downarrow 0}\int_{B(0,2|x|)-B(0,r)} [(\partial/\partial x)^\lambda L](-y)\,d\nu(y)$$

$$+ a|x|^{-n}\left\{\lim_{r\downarrow 0}\int_{B(0,1)-B(0,r)} L_n(x,y)\,dy - \sum_{|\lambda|=n} (\lambda!)^{-1} B_\lambda x^\lambda\right\}$$

$$+ |x|^{-n}\int_{B(0,2|x|)-B(x,|x|/2)} L_0(x,y)\,d\nu(y)$$

$$+ |x|^{-n}\int_{B(x,|x|/2)} L_0(x,y)\,d\nu(y)$$

$$= u_1(x) + u_2(x) - u_3(x) + au_4(x) + u_5(x) + u_6(x).$$

If $y \in R^n - B(0, 2|x|)$, then $|L_n(x,y)| \leqq \text{const.}\ |x|^{n+1}|y|^{-n-1}$ and hence

$$\lim_{x\to 0} u_1(x) = 0.$$

For simplicity, set $\tau = |\nu|$. Then $\lim_{r \downarrow 0} r^{-n}\tau(B(0, r)) = 0$ by (a), and we have

$$\limsup_{x\to 0} |u_2(x)| \leqq \text{const.} \limsup_{x\to 0} |x| \int_{B(0,1)} (|x| + |y|)^{-n-1} d\tau(y) = 0$$

because of Lemma 1, (ii).

If $0 < |\lambda| < n$, then Lemma 1, (i) yields

$$\limsup_{x\to 0} |x|^{|\lambda|-n} \int_{B(0,2|x|)} |[(\partial/\partial x)^{\lambda}L](-y)|d\tau(y)$$
$$\leqq \text{const.} \limsup_{x\to 0} |x|^{|\lambda|-n} \int_{B(0,2|x|)} |y|^{-|\lambda|} d\tau(y) = 0.$$

If $|\lambda| = n$, then $\int_{B(0,r)-B(0,s)} [(\partial/\partial x)^{\lambda}L](-y)dy = 0$ for any $r, s > 0$. Hence by the definition of A_{λ},

$$\lim_{x\to 0} \left\{\lim_{r\downarrow 0} \int_{B(0,2|x|)-B(0,r)} [(\partial/\partial x)^{\lambda}L](-y)d\nu(y)\right\} = 0.$$

Therefore $\lim_{x\to 0} u_3(x) = 0$.

Since $u(x) \equiv \int_{B(0,1)} L(x - y)dy \in C^{\infty}(B(0, 1))$ and

$$u_4(x) = |x|^{-n}\left\{u(x) - \sum_{|\lambda|\leqq n} (\lambda!)^{-1}x^{\lambda}[(\partial/\partial x)^{\lambda}u](0)\right\}$$

in view of Lemma 2, we see that $\lim_{x\to 0} u_4(x) = 0$.

As to u_5, we obtain

$$|u_5(x)| \leqq \text{const.}\ |x|^{-n} \int_{B(0,2|x|)} \log(2 + |x|/|y|)\, d\tau(y)$$

$$\leqq \text{const.}\ |x|^{1-n} \int_{B(0,2|x|)} |y|^{-1} d\tau(y),$$

which tends to zero as $x \to 0$ by Lemma 1, (i).

Finally we can show, in a way similar to the proof of Theorem 2', that $u_6(x)$ tends to zero as $x \to 0$ except for x in a set satisfying (ii) of the theorem. Thus we conclude the proof of Theorem 3.

Remark 1. If $\lim_{j\to\infty} C(E_j') = 0$, then we can find a nonnegative measure μ on R^n with compact support such that $\lim_{r\downarrow 0} r^{-n}\mu(B(0, r)) = 0$ and

$$\lim_{x\to 0, x\in E} |x|^{-n} \int L_{n-1}(x,y)\, d\mu(y) = \infty.$$

Remark 2. Let μ be a nonnegative measure on R^n satisfying (1), (a), (b) and

(c) There exist M, $r_0 > 0$ such that $\mu(B(x, r)) \leqq Mr^n$ for any $x \in B(x^0, r_0)$ and any $r \leqq r_0$.

Then the set E in Theorem 3 can be taken to be empty and, moreover, $L\mu$ is n times differentiable at x^0.

This fact can be proved in the same way as in Remark 2 in Section 2.

Remark 3. We can prove the following result similar to Theorem 2'.

Theorem 3'. Let k be as in Theorem 2', and h be a nondecreasing positive function on $(0, \infty)$ such that $\lim_{r \downarrow 0} h(r) = 0$ and

$$\int_0^1 [(r+s)H(s)]^{-1} ds \leqq \text{const. } [h(r)]^{-1} \qquad \text{for } r > 0,$$

where $H(r) = h(r)k(r)$ for $r > 0$. Let m be a nonnegative integer and μ be a nonnegative measure on R^n satisfying (1) and

$$\lim_{r \downarrow 0} r^{-m} H(r) \mu(B(x^0, r)) = 0.$$

Then there exists a set E in R^n such that

(i) $$\lim_{x \to x^0, x \in R^n - E} |x-x^0|^{-m} h(|x-x^0|) \int L_{m-1}(x,y) d\mu(y) = 0;$$

(ii) $$\lim_{j \to \infty} k(2^{-j}) C(E_j') = 0,$$

where $L_{-1}(x, y) = L(x-y)$.

Remark 4. Let $\tilde{h}$ be nonincreasing on the interval (0, 1) and

$$E = \{x \in R^n; \limsup_{r \downarrow 0} \tilde{h}(r)\mu(B(x, r)) > 0\}$$

for a nonnegative measure μ on R^n. If $\mu(E) = 0$, then $\Lambda_{\tilde{h}^{-1}}(E) = 0$, where $\Lambda_{\tilde{h}^{-1}}$ denotes the Hausdorff measure with respect to the measure function $\tilde{h}^{-1}$.

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