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Linear Ordinary Differential Equations with Gevrey Coefficients

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We consider the linear differential operator

$$(1)\qquad P(x, \frac{d}{dx}) = a_m(x)\frac{d^m}{dx^m} + a_{m-1}(x)\frac{d^{m-1}}{dx^{m-1}} + \dots + a_0(x)$$

and the equation

$$(2)\qquad P(x, \frac{d}{dx})u(x) = f(x)$$

on an open interval Ω in R^n.

We assume that the singular points, i.e. the zeros of $a_m(x)$, are isolated and of finite order. Then the irregularity σ of a singular point x is defined by

$$(3)\qquad \sigma = \max\{1, \max_{0\leqq i<m} \frac{\operatorname{ord}_x a_m - \operatorname{ord}_x a_i}{m-i}\}.$$

Let $*$ denote one of $\emptyset$, (s) and $\{s\}$ for an $s > 1$ and assume that the following irregularity condition is satisfied at every singular point x in Ω :

$\sigma = 1$ if $* = \emptyset$;

$\sigma \leqq s/(s-1)$ if $* = (s)$;

$\sigma < s/(s-1)$ if $* = \{s\}$.

Furthermore we assume that the coefficients $a_i(x)$ are in the space $E^*(\Omega)$ of ultradifferentiable functions of class $*$. Then we have the following theorems:

Theorem A. For any ultradistribution $f \in D^{*\prime}(\Omega)$ of class $*$ equation (2) has a solution $u \in D^{*\prime}(\Omega)$.

Theorem B. The homogeneous equation

$$P(x, \frac{d}{dx}) u(x) = 0 \tag{4}$$

has

$$m + \sum_{x \in \Omega} \mathrm{ord}_x a_m(x) \tag{5}$$

linearly independent solutions $u \in D^{*\prime}(\Omega)$.

Theorem C. If $f \in D^{*\prime}(\Omega)$, then any solution $u_1 \in D^{*\prime}(\Omega_1)$ of (2) on an open subinterval Ω_1 of Ω can be continued to a solution $u \in D^{*\prime}(\Omega)$ on Ω.

Here $E(\Omega)$ and $D'(\Omega)$ are Schwartz' spaces of differentiable functions and distributions respectively. $E^{(s)}(\Omega)$ (resp. $E^{\{s\}}(\Omega)$) is the space of all functions $a \in E(\Omega)$ such that for every $K \subset\subset \Omega$ and $h > 0$ there is a C (resp. there are h and C) satisfying $\sup_{x \in K} |a^{(p)}(x)| \leq C h^p p!^s$. They are called Gevrey classes of functions. $D^*(\Omega) = \{\varphi \in E^*(\Omega);$ supp φ is compact$\}$ has a natural locally convex topology and its dual $D^{*\prime}(\Omega) = \{f : D^*(\Omega) \to C$ continuous and linear$\}$ is by definition the space of ultradistributions of class $*$ on Ω [2].

When the coefficients a_i are all real analytic these theorems have been proved by the auther [3] in 1973. We employed the theory of linear ordinary differential equations in the complex domain, in particular the index formulas [1] and the characterization of ultradistributions of class * by the growth order of their defining functions as hyperfunctions [2].

Similarly to that case the theorems are derived from the existence theorem in the non-singular case and the index formulas of $P(x, d/dx)$ acting in various spaces of ultradifferentiable functions as sketched below.

1. Non-singular case.

We assume that a_m never vanishes in Ω. Then the proof is reduced to the case where $a_m = 1$ by the following.

Lemma 1 (Rudin [6]). If $a \in E^*(\Omega)$ never vanishes on Ω, then the inverse $1/a$ belongs to $E^*(\Omega)$.

For such an operator we have the following existence theorem of Cauchy type.

Lemma 2. Let $x_0 \in \Omega$. For each $f \in E^*(\Omega)$ and $c_0, \dots, c_{m-1} \in C$ there is a unique solution $u \in E^*(\Omega)$ of

$$\begin{cases} Pu = f, \\ u^{(j)}(x_0) = c_j, \quad j = 0, \dots, m-1. \end{cases}$$

This is proved by Picard's method of successive approximation or by Cauchy's method of majorants [5].

Applying the lemma to P and its formal dual P', we obtain the topological exact sequences

$$(4) \qquad 0 \longrightarrow C^m \longrightarrow E^*(\Omega) \xrightarrow{P} E^*(\Omega) \longrightarrow 0,$$

$$(5) \qquad 0 \longrightarrow D^*(\Omega) \xrightarrow{P'} D^*(\Omega) \longrightarrow C^m \longrightarrow 0.$$

The dual of (5) is the exact sequence

$$(6) \qquad 0 \longrightarrow C^m \longrightarrow D^{*\prime}(\Omega) \xrightarrow{P} D^{*\prime}(\Omega) \longrightarrow 0.$$

This proves Theorems A and B. In view of Theorem A we need to prove Theorem C only for homogeneous solutions. Comparing (4) with (6), we find that every homogeneous solution in $D^{*\prime}(\Omega)$ is actually in $E^*(\Omega)$. Hence extendability follows from Lemma 2.

2. The case where 0 is a unique singular point.

We note that P and its formal dual P' have the same singular points and the same irregularity at each singular point.

The following is a key lemma.

Lemma 3. Let I be a compact interval containing 0 and let d and p be non-negative integers. If 0 is a zero of $\psi \in C^{d+p}(I)$ of order $\geq d$, then the function φ defined by

$$(7) \qquad \varphi(x) = \begin{cases} \psi(x)/x^d, & x \neq 0, \\ \psi^{(d)}(0)/d!, & x = 0 \end{cases}$$

is in $C^p(I)$ and we have

$$(8) \qquad \sup_{x\in I} |\varphi^{(p)}(x)| \leq \frac{p!}{(d+p)!} \sup_{x\in I} |\psi^{(d+p)}(x)|.$$

Hence it follows that the original equation is divisible by an invertible function so that we may assume that

$$(9)\qquad P(x, \frac{d}{dx}) = x^d \frac{d^m}{dx^m} + b_{m-1}(x)\, x^{d_{m-1}} \frac{d^{m-1}}{dx^{m-1}} + \dots + b_0(x)\, x^{d_0},$$

where $d = \mathrm{ord}_0\, a_m$, d_i are integers satisfying

$$(10)\qquad d - \sigma(m - i) \leqq d_i < d$$

and b_i are functions in $E^*(\Omega)$.

As we will see later Theorem A follows from the following uniquness theorem for the formal dual P'.

Lemma 4. Suppose that the irregularity condition is fulfilled. If $u \in E^*(\Omega)$ satisfies $Pu = 0$ and $u^{(p)}(0) = 0$ for all p, then u is identically equal to 0.

Let $\Omega_\pm = \{x \in \Omega;\ \pm x \geqq 0\}$. To prove Theorems A, B and C we make use of the topological exact sequence

$$(11)\qquad 0 \longrightarrow D^*_{\Omega_-}(\Omega) \longrightarrow D^*(\Omega) \xrightarrow{\ \rho\ } D^*(\Omega_+) \longrightarrow 0$$

and its dual

$$(12)\qquad 0 \longrightarrow D^{*\prime}_{\Omega_+}(\Omega) \longrightarrow D^{*\prime}(\Omega) \longrightarrow \tilde{D}^{*\prime}(\Omega_-) \longrightarrow 0,$$

where $D^*_{\Omega_-}(\Omega) = \{\varphi \in D^*(\Omega);\ \mathrm{supp}\, \varphi \subset \Omega_-\}$, $D^{*\prime}_{\Omega_+}(\Omega) = \{f \in D^{*\prime}(\Omega);\ \mathrm{supp}\, f \subset \Omega_+\}$ and $\tilde{D}^{*\prime}(\Omega_-)$ is the space of all ultradistributions on $\Omega_- \setminus \{0\}$ which are extendable across 0. ρ is the restriction mapping to Ω_+. The topological exactness of (11) is the Whitney type extension theorem with bounds due to

Ritt, Carleson and Komatsu [4].

We claim that all rows of the following diagram are topologically exact.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D^*_{\Omega_-}(\Omega) & \xrightarrow{P'} & D^*_{\Omega_-}(\Omega) & \longrightarrow & C^m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D^*(\Omega) & \xrightarrow{P'} & D^*(\Omega) & \longrightarrow & C^{d+m} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D^*(\Omega_+) & \xrightarrow{P'} & D^*(\Omega_+) & \longrightarrow & C^d & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{13}
$$

The exactness on the left follows from Lemma 4 for every row. The exactness of the middle and lower rows is proved by the stability of indices. If $P' = x^d(d/dx)^m$, it is easy to see that the cokernels of P' in $D^*(\Omega)$ and in $D^*(\Omega_+)$ are C^{d+m} and C^d respectively. Since we have good stability theorems only for operators in Banach spaces, we approximate the spaces by

$$D^*(\Omega) = \varinjlim_{K \Subset \Omega} \varinjlim_{M_p} D^{M_p}{}_K, \qquad D^*(\Omega_+) = \varinjlim_{K \Subset \Omega} \varinjlim_{M_p} D^{M_p}{}_K(\Omega_+),$$

where $D^{M_p}{}_K$ (resp. $D^{M_p}{}_K(\Omega_+)$) is the space of all functions $\varphi \in D(\Omega)$ (resp. $D(\Omega_+)$) such that $\operatorname{supp} \varphi \subset K$ and $\sup |\varphi^{(p)}(x)| / M_p \longrightarrow 0$ as $p \longrightarrow \infty$. M_p ranges over the arbitrary sequences of positive numbers such that the multiplications by b_i are continuous if $* = \emptyset$, the sequences of the form $h_1 h_2 \dots h_p p!^s$, where $h_p > 0$ tends to 0 as $p \to \infty$, if $* = (s)$ and $h^p p!^s$ for large constants $h > 0$ if $* = \{s\}$.

In $D^{M_p}_K$ (resp. $D^{M_p}_K(\Omega_+)$) the operator $x^d (d/dx)^m$ is a closed linear operator with dense domain and index $-d-m$ (resp. $-d$). It follows from (10) and Lemma 3 that $x^{d_i}(d/dx)^i$ are compact relative to $x^d (d/dx)^m$. Hence the middle and lower rows of (13) are exact as the inductive limits of exact sequences of the same form. Then the exactness of the upper row is proved by the exactness of the columns. The topological exactness follows from the open mapping theorem.

The dual dagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^d & \longrightarrow & D^{*\prime}_{\Omega_+}(\Omega) & \xrightarrow{P} & D^{*\prime}_{\Omega_+}(\Omega) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^{d+m} & \to & D^{*\prime}(\Omega) & \xrightarrow{P} & D^{*\prime}(\Omega) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^m & \longrightarrow & \tilde{D}^{*\prime}(\Omega_-) & \xrightarrow{P} & \tilde{D}^{*\prime}(\Omega_-) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{14}$$

proves Theorems A, B and C. The exactness of the middle row is exactly Theorems A and B. Compare the lower row with the exact sequence

$$0 \to C^m \longrightarrow D^{*\prime}(\Omega_- \setminus \{0\}) \xrightarrow{P} D^{*\prime}(\Omega_- \setminus \{0\}) \to 0$$

in the non-singular case. Then we find that any homogeneous solution $u_1 \in D^{*\prime}(\Omega_- \setminus \{0\})$ is an extendable ultradistribution. Let $u_2 \in D^{*\prime}(\Omega)$ be an extension. Since $Pu_2 \in D^{*\prime}_{\Omega_+}(\Omega)$, we can find a solution $u_3 \in D^{*\prime}_{\Omega_+}(\Omega)$ of $Pu_3 = Pu_2$ by the exactness of the upper row of (14). Then $u = u_2 - u_3 \in D^{*\prime}(\Omega)$ is an extension of u as a homogeneous solution. Similarly every homogeneous solution $u_1 \in D^{*\prime}(\Omega_+ \setminus \{0\})$ on $\Omega_+ \setminus \{0\}$ can be extended to a homogeneous solution $u \in D^{*\prime}(\Omega)$ on Ω.

3. General case.

We arrange the singular points x_i in Ω so that

$$\ldots < x_{-i} < \ldots < x_{-1} < x_0 < x_1 < x_2 < \ldots < x_j < \ldots .$$

Theorems A and C are immediate consequences of those theorems on intervals (x_{j-1}, x_{j+1}). To prove Thorem B we extend m linearly independent homogeneous solutions in $D^{*\prime}((x_{-1}, x_0))$ to homogeneous solutions in $D^{*\prime}(\Omega)$ by Theorem C. For each $j \geqq 0$ (resp. $j < 0$) there are $\operatorname{ord}_{x_j} a_m$ linearly independent homogeneous solutions in $D^{*\prime}_{[x_j, x_{j+1})}((x_{j-1}, x_{j+1}))$ (resp. $D^{*\prime}_{(x_{j-1}, x_j]}((x_{j-1}, x_{j+1}))$) by the upper row of (14). We extend them to homogeneous solutions on Ω with support in $\{x \in \Omega;\ x \geqq x_j\}$ (resp. $\{x \in \Omega;\ x \leqq x_j\}$). Then it is easy to see that these homogeneous solutions on Ω form a basis of the homogeneous solutions in $D^{*\prime}(\Omega)$.

4. Example

In case the coefficients are real analytic it is proved in [3] that for each singular point x_0 of irregularity σ there is a homogeneous solution which belongs to $\mathcal{D}^{(\sigma/(\sigma-1))}{}'(\Omega_0)$ but does not to $\mathcal{D}^{\{\sigma/(\sigma-1)\}}{}'(\Omega_0)$ on a neighborhood Ω_0 of x_0. Hence the irregularity condition is necessary for Theorem B to hold. To demonstrate the same for Theorems A and C we give an example.

The operator

$$P(x, \frac{d}{dx}) = x^2 \frac{d}{dx} - 1$$

has the singular point 0 of irregularity 2. On each interval which does not contain 0 every homogeneous solution is written const $e^{-1/x}$. The solution $e^{-1/x}$ on $(-\infty, 0)$ cannot be extended across 0 as an ultradistribution of class $\{2\}$. The famous infinitely differentiable function

$$\varphi(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0\ , & x \leqq 0 \end{cases}$$

is an ultradifferentiable homogeneous solution of class $\{2\}$. Therefore Lemma 4 does not hold without the irregularity condition.

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