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The Fixed Point Property for Continua with Finitely Many Arc Components

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1. Introduction. A space X has the *fixed point property* (FPP) if each map $f: X \longrightarrow X$ leaves some point fixed —— that is, there is a point $x \in X$ such that $f(x) = x$. Let X be a space approximated from within by subsets with FPP. Then it is natural to ask when X has FPP.

In [5] Young showed that if X is an arcwise connected Hausdorff space such that every monotone increasing sequence of arcs is contained in an arc, then X has FPP. While Borsuk [1] proved that every dendroid has FPP. Afterward Ward [3] generalized both of these results as follows: Let X be a chained acyclic Hausdorff space. Suppose that there is a point $e \in X$ such that for every ray R with initial point e the set $\overline{\bigcap_x \{R - [e, x)\}}$ has FPP. Then X has FPP.

In Section 2 of this article we show a sufficient condition that a continuum, approximated from within by Peano continua with FPP, has FPP. As a consequence we have that for every positive integer n the Cartesian product of n Warsaw circles is a T^n-like continuum with FPP and the n-fold suspension of Warsaw circle is an S^n-like continuum with FPP, where T^n, S^n mean an n-dimensional torus, an n-sphere, respectively. Here refer that Dyer [2] proved that the Cartesian product of n chainable continua has FPP.

In Section 3 we consider FPP for continua with two or more arc components.

2. Continua approximated from within by Peano continua with FPP.

DEFINITION. A *continuum* is a compact connected metric space and a *Peano continuum* is a locally connected continuum. A *map* is a continuous function. Let (M, d) be a metric space and ε a positive number. A map $f:M \longrightarrow M$ is said to be ε-*near to the identity map* or simply to be ε-*near* if $d(x, f(x)) < \varepsilon$ for every $x \in M$.

THEOREM 1. *Let X be a continuum for which there exists a sequence $C_1 \subset C_2 \subset \cdots$ of Peano continua such that $X = \bigcup_i C_i$ and every C_i has FPP. If the following (1) and (2) hold, then X has FPP.*

(1) For every $\varepsilon > 0$, there exist a C_i and a function $f:X \longrightarrow X$ such that for each $s > i$ the restriction $f|C_s$ is an ε-near map of C_s to C_i.

(2) There exists a closed subset A, which may be empty, of $\bigcap_j \overline{X - C_j}$, such that every f in (1) is continuous on A, $f(A) \subset C_1$ and every point $x \in \bigcap_j \overline{X - C_j} - A$ has a neighborhood U whose component containing x lies in a C_l.

PROOF (SKETCH). Let $g:X \longrightarrow X$ be an arbitrary map. For every C_i and every $\delta > 0$, there exists a C_s with $g(C_i) - N_\delta(A) \subset C_s$, where $N_\delta(A)$ is a δ-neighborhood of A in X.

Then for every C_i, we have $f \circ g(C_i) \subset C_i$, and there exists an $x \in X$ with $g(x) = x$.

REMARK. An n-sphere S^n $(n \geqq 1)$ satisfies condition (1) in Theorem 1 but has no FPP.

THEOREM 2. *Suppose that for each* k $(1 \leqq k \leqq n)$ X_k *and* $C_{k1} \subset C_{k2} \subset \ldots$ *satisfy the conditions in Theorem 1. If every* $C_{1i} \times C_{2i} \times \ldots \times C_{ni}$ $(i = 1, 2, \ldots)$ *has FPP, then so does* $X_1 \times X_2 \times \ldots \times X_n$.

To prove this it is sufficient to show that $X = X_1 \times \ldots \times X_n$ and $C_i = C_{1i} \times \ldots \times C_{ni}$ $(i = 1, 2, \ldots)$ satisfy the conditions in Theorem 1.

DEFINITION. Let X and Y be compact metric spaces. Then X is said to be *Y-like* if for every $\varepsilon > 0$ there is a map f of X onto Y such that for every $y \in Y$ the diameter of $f^{-1}(y)$ is less than ε.

COROLLARY 1. *The Cartesian products of* n *Warsaw circles is a* T^n*-like continuum with FPP for* $1 \leqq n \leqq \omega$.

COROLLARY 2. *The Cartesian product of the above* T^n*-like continuum and an m-cell* $(1 \leqq m \leqq \omega)$ *has FPP.*

For a set P, the symbols $P^{\#}$ and P^{*} denote the cone over P and the suspension of P, respectively.

THEOREM 3. *Assume that* X *and* $C_1 \subset C_2 \subset \cdots$ *satisfy the conditions in Theorem 1. If every* $C_i^{\#}$ (C_i^*) $(i = 1, 2, \ldots)$ *has FPP, then so does* $X^{\#}$ (X^*).

To prove this, it is sufficient to show that $X^{\#}$ (X^*) and $C_1^{\#} \subset C_2^{\#} \subset \cdots$ $(C_1^* \subset C_2^* \subset \cdots)$ satisfy the conditions of Theorem 1. In this case the vertex of $X^{\#}$ and the suspension points of X^* correspond to the set A in Theorem 1.

COROLLARY 3. *For every positive integer* n, *the n-fold suspension of Warsaw circle is an* S^n*-like continuum with FPP. Also the Cartesian product of this continuum and an m-cell* $(1 \leqq m \leqq \omega)$ *has FPP.*

REMARK. Recently Watanabe [4] obtained fixed point theorems for cones over certain general spaces.

3. Continua with finitely many arc components.

DEFINITION. A finite set T is called an *ordered tree* if (1) T is the set of vertices of a one-dimensional polyhedron containing no simple closed curve, and (2) T is a partially ordered set such that a pair p, q of points is the vertices of an edge if and only if one of them covers the other. (By "*p covers q*" in a partially ordered set $\{P, \geq\}$, it is meant that $p > q$, but that $p > x > q$ for no $x \in P$.) A function $f: P \longrightarrow Q$ between partially ordered sets is *isotone* if $f(x) \geq$

$f(y)$ whenever $x \geq y$. A partially ordered set P has the *fixed point property* (FPP) if every isotone function $f:P \longrightarrow P$ leaves an element of P fixed, i. e., there exists an $x \in P$ with $f(x) = x$.

LEMMA 1. *Let P be a partially ordered finite set. If there exists a maximum or minimum element in P, then P has FPP.*

The case where P has a minimum element follows from Knaster-Tarski's theorem.

LEMMA 2. *Every ordered tree has FPP.*

This follows from induction on the number of elements of T. Let g be a collection of mutually exclusive subsets G_λ of a topological space X such that $\bigcup_\lambda G_\lambda = X$. Then we define a binary relation $\leq$ on g as follows: For $G_\lambda, G_\mu \in g$, $G_\lambda \leq G_\mu$ if and only if there exists a finite sequence $G_1, G_2, \ldots, G_k$ of elements of g such that $G_1 = G_\lambda$, $G_k = G_\mu$ and $\overline{G}_i \cap G_{i+1} \neq \phi$ $(1 \leq i \leq k)$. The relation $\leq$ is not necessarily a partial order.

LEMMA 3. *Let G_1, G_2 be arc components of a space X, and let $f:X \longrightarrow X$ be a continuous map. If $f(G_1) \cap G_2 \neq \phi$, then $f(G_1) \subset G_2$.*

Let g be the collection of arc components of a topological space X. If $f: X \longrightarrow X$ is continuous, then for every $G_\lambda \in g$ there exists a $G_\mu \in g$ with $f(G_\lambda) \subset G_\mu$. Thus we define a function $f^*: g \longrightarrow g$ by $f^*(G_\lambda) = G_\mu$. From Lemma 3 we have

LEMMA 4. *If* $G_1 \leq G_k$, *then* $f^*(G_1) \leq f^*(G_k)$.

THEOREM 4. *Let* X *be a continuum with finitely many arc components* $G_1, G_2, \ldots, G_n$ *such that each* $\overline{G}_i$ *has FPP. If* $g = \{G_1, G_2, \ldots, G_n; \leq\}$ *is an ordered tree or a partially ordered set with a maximum or minimum element, then* X *has FPP.*

This follows from Lemmas 4, 2 and 1.

COROLLARY 4. *Let* X *be the continuum in Theorem 4. Let* Y *be an arcwise connected continuum such that each* $\overline{G}_i \times Y$ *has FPP. Then* $X \times Y$ *has also FPP.*

THEOREM 5. *Let* X *be a continuum with finitely many arc components* $G_1, G_2, \ldots, G_n$ *satisfying the following conditions:*

(1) For every i *there exists a monotone increasing sequence* $C_{i1} \subset C_{i2} \subset \ldots$ *of subsets of* G_i *such that* $G_i = \bigcup_j C_{ij}$ *has FPP.*

(2) $\bigcap_j \overline{G_i - C_{ij}} = \overline{G}_i - G_i$ $(1 \leqq i \leqq n)$.

(3) $g = \{G_1, G_2, \ldots, G_n; \leq\}$ *is an ordered tree each of whose elements is covered by at most one element.*

Then X *has FPP.*

PROOF(SKETCH). Let $f: X \longrightarrow X$ be a map. Then by Lemma 2 there exists an s with $f(G_s) \subset G_s$. If G_s is the maximum element of g, then $G_s = C_{sj}$ for some j, and hence f leaves a point of C_{sj} fixed. Suppose that G_s is not the maximum element. If $f(C_{sj}) \subset C_{sj}$ for some j, then there exists a fixed point of f in C_{sj}. If for every j, $f(C_{sj})$ is not contained in C_{sj}, then there exist a point $x_0 \in X$ and G_t such that $x_0 \cup f(x_0) \subset \overline{G}_s - G_s \subset G_t$. Therefore we have $f(G_t) \subset G_t$. Continuing this process, we can find a fixed point of f.

COROLLARY 5. *Let X be the continuum in Theorem 5. Let Y be an arcwise connected continuum such that $C_{ij} \times Y$ $(1 \leqq i \leqq n,\ j = 1, 2, \ldots)$ have FPP. Then $X \times Y$ has FPP.*

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