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Homology theories on strong shape

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0. In this note we shall introduce new homology theories on the category CPHTOP. One is the strong homology theory, which is defined by Lisica and Mardešić [5], and is the generalization of Steenrod-Sitnikov homology theory to arbitrary inverse systems. The other is the coherent singular homology theory [2], which is an extension of the usual singular homology theory. Both theories can be applied to strong shape theory, and are useful algebraic invariants in the theory.

Throughout this note spaces mean topological spaces, and maps mean continuous functions. ANR is an absolute neighborhood retract for metric spaces.

Notations: For each $n \geq 0$, let Δ^n be the standard n-simplex, i.e.,

$$\Delta^n = \{(t_0, \ldots, t_n) \in R^{n+1} \mid t_i \geq 0 \text{ for every } i, \sum_{i=1}^{n} t_i = 1\}.$$

For $n > 0$ and $0 \leq j \leq n$, let $\partial_j^n : \Delta^{n-1} \longrightarrow \Delta^n$ be the j-th face operator. For $n \geq 0$ and $0 \leq j \leq n$, let $\sigma_j^n : \Delta^{n+1} \longrightarrow \Delta^n$ be the j-th degenerate operator.

1. In this note we consider only inverse systems of spaces and maps $\underline{X} = (X_a, p_{aa'}, A)$ over cofinite directed sets.

A <u>coherent map</u> $f: \underline{X} \longrightarrow \underline{Y} = (Y_b, q_{bb'}, B)$ consists of an increasing function $\phi : B \longrightarrow A$ and of maps $f_{\underline{b}} : \Delta^n \times X_{\phi(b_n)} \longrightarrow Y_{b_0}$, $\underline{b} = (b_0,\dots,b_n) \in B^n$, $n \geq 0$, which satisfy

$$f_{\underline{b}}(\partial_j^n(t),x) = \begin{cases} q_{b_0 b_1} f_{\underline{b}_0}(t,x) & \text{if } j = 0, \\ f_{\underline{b}_j}(t,x) & \text{if } 0 < j < n, \\ f_{\underline{b}_n}(t, p_{\phi(b_{n-1})\phi(b_n)}(x)) & \text{if } j = n, \end{cases}$$

where $x \in X_{\phi(b_n)}$, $t \in \Delta^{n-1}$, $n > 0$,

$$f_{\underline{b}}(\sigma_j^n(t),x) = f_{\underline{b}^j}(t,x), \text{ for } 0 \leq j \leq n, \text{ where } x \in X_{\phi(b_n)},$$

$t \in \Delta^{n+1}$, $n \geq 0$,

here B^n, $n \geq 0$, denotes the set of all increasing sequences $\underline{b} = (b_0,\dots,b_n)$ in B, and $\underline{b}_j = (b_0,\dots,b_{j-1},b_{j+1},\dots,b_n)$ and $\underline{b}^j = (b_0,\dots,b_j,b_j,\dots,b_n)$ for $\underline{b} = (b_0,\dots,b_n) \in B^n$ and $0 \leq j \leq n$.

A <u>coherent homotopy from</u> f <u>to</u> $f' = (\phi', f'_{\underline{b}})$ is a coherent map $F = (\Phi, F_{\underline{b}}): \underline{X} \times I = (X_a \times I, p_{aa'} \times 1, A) \longrightarrow \underline{Y}$ such that

$\Phi \geq \phi, \phi'$, and

$F_{\underline{b}}(t,x,0) = f_{\underline{b}}(t, p_{\phi(b_n)\Phi(b_n)}(x))$,

$F_{\underline{b}}(t,x,1) = f'_{\underline{b}}(t, p_{\phi'(b_n)\Phi(b_n)}(x))$, where $x \in X_{\Phi(b_n)}$, $t \in \Delta^n$,

which is written by $F: f \simeq f'$.

Next we define the <u>composition</u> gf of f and $g = (\psi, g_{\underline{c}}): \underline{Y} \longrightarrow \underline{Z} = (Z_c, r_{cc'}, C)$. In the case f is a system map $\underline{f} =$

(ϕ, f_b),

$$(gf)_{\underline{c}}(t,x) = g_{\underline{c}}(f_{\psi(c_n)}(x)), \text{ where } \underline{c} = (c_0,\dots,c_n) \in C^n,$$

$$n \geq 0,\ x \in X_{\phi\psi(c_n)}, \text{ and } t \in \Delta^n.$$

Hence if $\underline{X}$ and $\underline{Y}$ are rudumentary systems (X) and (Y), respectively, and f is a map from from X to Y, then $(gf)_{\underline{c}}(t,x) = g_{\underline{c}}(t, f(x))$ for $x \in X$, $t \in \Delta^n$, $\underline{c} \in C^n$, $n \geq 0$.

In order to define another case, one decomposes Δ^n into subpolyhedra

$$P_i^n = \{(t_0,\dots,t_n) \in \Delta^n \mid t_0+\cdots+t_{i-1} \leq 1/2 \leq t_0+\cdots+t_i\},$$

$$0 \leq i \leq n,$$

and considers maps $\alpha_i^n \colon P_i^n \longrightarrow \Delta^{n-i}$, $\beta_i^n \colon P_i^n \longrightarrow \Delta^i$ given by

$$\alpha_i^n(t) = (\#, 2t_{i+1},\dots,2t_n), \text{ and}$$

$$\beta_i^n(t) = (2t_0,\dots,2t_{i-1},\#), \text{ where } \# = 1 - (\text{sum of remaining terms}).$$

Then

$$(gf)_{\underline{c}}(t,x) = g_{c_0,\dots,c_i}(\beta_i^n(t), f_{\psi(c_i),\dots,\psi(c_n)}(\alpha_i^n(t),x)),$$

where $\underline{c} = (c_0,\dots,c_n) \in C^n$, $n \geq 0$, $x \in X_{\phi\psi(c_n)}$, $t \in P_i^n$.

We define the <u>coherent identity map</u> $1_{\underline{X}}\colon \underline{X} \longrightarrow \underline{X}$ by putting $\phi = 1_A$, and for any $\underline{a} = (a_0,\dots,a_n) \in A^n$, $n \geq 0$,

$$1_{\underline{a}}(t,x) = p_{a_0 a_n}(x), \text{ where } x \in X_{a_n} \text{ and } t \in \Delta^n.$$

In [4] , Lisica and Mardešić showed that inverse systems of spaces and maps over cofinite directed sets and coherent homotopy classes of coherent maps construct a category, which is called <u>coherent prohomotopy category</u>, and is denoted by CPHTOP. We note that our definition of composition of coherent maps are slightly differet from the original one in [4], but by the proof of Lemma I.9.7, the coherent homotopy class of our composition coincides with the one of the original composition. Hence we have the same category CPHTOP.

2. Let $\underline{X}$ be an object of CPHTOP. For each $i \geq 0$, let $S_i(\underline{X})$ be the set of all coherent maps from Δ^i to $\underline{X}$. For each $0 \leq k \leq i$, $i \geq 0$, we define the functions $d_k = d_k^i \colon S_i(\underline{X}) \longrightarrow S_{i-1}(\underline{X})$ and $s_k = s_k^i \colon S_i(\underline{X}) \longrightarrow S_{i+1}(\underline{X})$ by formulas:

$$d_k(h) = h\partial_k^i \ , \text{ and } s_k(h) = h\sigma_k^i \quad \text{for } h \in S_i(\underline{X}).$$

Then the triple $(S_i(\underline{X}), d_k, s_k)$ is a semi-simplicial complex, which is called the <u>coherent singular complex</u> of $\underline{X}$, and is denoted by $S_c(\underline{X})$. In fact, $S_c(\underline{X})$ is a Kan complex. We note that if $\underline{X}$ is the rudimentary system (X), $S_c(\underline{X})$ is the usual singular complex $S(X)$.

Let $f\colon \underline{X} \longrightarrow \underline{Y}$ be a coherent map. For each $i \geq 0$, we define the function $S_i(f)\colon S_i(\underline{X}) \longrightarrow S_i(\underline{Y})$ by

$$S_i(f)(h) = fh \quad \text{for } h \in S_i(\underline{X}).$$

Then the collection $\{S_i(f)\}$ induces the simplicial map from $S_c(\underline{X})$ to $S_c(\underline{Y})$. Namely, each coherent map $f\colon \underline{X} \longrightarrow \underline{Y}$ induces the semi-simplicial map $S_c(f)\colon S_c(\underline{X}) \longrightarrow S_c(\underline{Y})$.

In [2], we proved the following.

<u>Theorem</u> 1. Let KAN be the category of Kan complexes and

homotopy classes of semi-simplicial maps. The correspondence S_c induces the functor from CPHTOP to KAN.

We call the functor the coherent singular complex functor.

Let H_* be the homology theory defined on KAN (see [7]). For each object $\underline{X}$ of CPHTOP, we define

$H^c_n(\underline{X};G) = H_n(S_c(\underline{X});G)$ for each $n \geq 0$ and every abelian group G. We call $H^c_n(\underline{X};G)$ the _n-dimensional coherent singular homology group of X with the coefficient group_ G. We note that if $\underline{X}$ is the rudimentary system (X), $H^c_n(\underline{X};G)$ is the usual singular homology group $H_n(X;G)$ of X.

For each coherent map $f: \underline{X} \longrightarrow \underline{Y}$, we have the homomorphism $f^c_*: H^c_n(\underline{X};G) \longrightarrow H^c_n(\underline{Y};G)$ given by $S_c(f)_*$.

By Theorem 1, we have the next result (see [2]).

Theorem 2. The correspondence H^c_* induces the functor from CPHTOP to the category Ab of abelian groups and homomorphisms.

In particular, the coherent singular homology group is an invariant on CPHTOP.

We remark that all definitions and results of §1 and §2 are extended to inverse systems of pointed spaces and pointed maps, and of pairs of spaces and maps of pairs. In the next section we shall use the extended definitions and results.

3. For each $i \geq 0$, we denote the set of all coherent homotopy classes of coherent maps from the i-sphere (S^i, s_0) to $(\underline{X,x}) = ((X_a, x_a), p_{aa'}, A)$ by

$\pi_i^c(\underline{X,x})$.

If $i \geq 1$, by the H-group structure of (S^i, s_0), $\pi_i^c(\underline{X,x})$ is a group. Moreover, if $n \geq 2$, $\pi_i^c(\underline{X,x})$ is an abelian group. We shall call $\pi_i^c(X,x)$ the <u>coherent prohomotopy group</u> of $(\underline{X,x})$. It is easily seen that $\pi_i^c(\underline{X,x})$ is an invariant on $CPHTOP_0$.

Let $|\cdot|$: KAN $\longrightarrow$ CW be the geometric realization functor, where CW is the category of CW-complexes and homotopy classes of maps. Let $(\underline{X,x})$ be an object of $CPHTOP_0$. Now we define the <u>canonical coherent map</u> $\tau_{\underline{X}}: |S_c(\underline{X})| \longrightarrow \underline{X}$ as follows:

$$\tau_{\underline{a}}(t, |h,z|) = h_{\underline{a}}(t,z), \text{ where } \underline{a} \in A^n,\ n \geq 0,\ t \in \Delta^n,$$

$$(h,z) \in S_i(\underline{X}) \times \Delta^i,\ i \geq 0.$$

If the point $|S_c(\{x_a\})|$ of $|S_c(\underline{X})|$ is considered the base point e of $|S_c(\underline{X})|$, then the coherent map $\tau_{\underline{X}}$ is the pointed coherent map from $(|S_c(\underline{X})|, e)$ to $(\underline{X,x})$. In the latter part of this note we consider $\tau_{\underline{X}}$ a pointed coherent map.

<u>Theorem</u> 3. $\tau_{\underline{X}}$ is a weak coherent homotopy equivalence. That is,

$$\tau_{\underline{X}\#}: \pi_i(|S_c(\underline{X})|, e) \cong \pi_i^c(\underline{X,x}) \quad \text{for every } i \geq 0.$$

We define the homomorphism $\Phi^{i}_{(\underline{X},\underline{x})} : \pi^{c}_{i}(\underline{X},\underline{x}) \longrightarrow H^{c}_{i}(\underline{X};Z)$ by the formula;

$$\Phi^{i}_{(\underline{X},\underline{x})}([h]) = h_{*}(1) \quad \text{for each } [h] \in \pi^{c}_{i}(\underline{X},\underline{x}),$$

which is called the i-th coherent Hurewicz homomorphism of $(\underline{X},\underline{x})$. Then by Theorem 3 and the definition of coherent singular homology groups, we have the Hurewicz isomorphism theorem in the category CPHTOP.

Theorem 4. (1) If $\pi^{c}_{k}(\underline{X},\underline{x}) = 0$ for every $0 \leq k \leq i-1$, $i \geq 2$, then $\Phi^{i}_{(\underline{X},\underline{x})} : \pi^{c}_{i}(\underline{X},\underline{x}) \cong H^{c}_{i}(X;Z)$.

(2) If $\pi^{c}_{0}(\underline{X},\underline{x}) = 0$, then $\Phi^{1}_{(\underline{X},\underline{x})}$ is an epimorphism and its kernel is the commutator subgroup of $\pi^{c}_{1}(\underline{X},\underline{x})$.

In particular, the next corollary is obtained (see [3]).

Corollary 1. Let $(\underline{X},\underline{x}) = ((X_n, x_n), p_{n,n+1})$ be an inverse sequence of pointed compact connected polyhedra. If $(\underline{X},\underline{x})$ is pointed 1-movable, then $\Phi^{1}_{(\underline{X},\underline{x})}$ is an epimorphism and its kernel is the commutator subgroup of $\pi^{c}_{1}(\underline{X},\underline{x})$.

4. Lisica and Mardešić, [5] , defined the strong homology group of inverse systems, which is an invariant in CPHTOP. For an abelian group G we associate with an object $\underline{X} = (X_a, p_{aa'}, A)$ a chain complex $C_{\#}(\underline{X};G)$, defined as follows;

a strong p-chain of $\underline{X}$, $p \geq 0$, is a function x, which assigns to every $\underline{a} \in A^{n}$ a singular (p+n)-chain $x_{\underline{a}} \in C_{p+n}(X_{a_0};G)$.

The boundary operator d: $C_{p+1}(\underline{X};G) \longrightarrow C_p(\underline{X};G)$ is defined by the formula

$$(-1)^n(dx)_{\underline{a}} = \partial(x_{\underline{a}}) - p_{a_0a_1\#}(x_{\underline{a}_0}) - \sum_{j=0}^{n} (-1)^j x_{\underline{a}_j} ,$$

where $\underline{a} = (a_0,\ldots,a_n) \in A^n$, $n \geq 0$, $x \in C_{p+1}(\underline{X};G)$ and ∂ is the boundary operator of singular chains.

Then we define

$$H_p^s(\underline{X};G) = H_p(C_\#(\underline{X};G)),$$

which is called the p-th strong homology group of $\underline{X}$ with the coefficient group G.

With a coherent map f: $\underline{X} \longrightarrow \underline{Y}$ we associate a chain map $f_\#$: $C_\#(\underline{X};G) \longrightarrow C_\#(\underline{Y};G)$, given by

$$(f_\#(x))_{\underline{b}} = \sum_{j=0}^{n} (f_{b_0,\ldots,b_j})_\#(\Delta^j \times x_{\phi(b_j),\ldots,\phi(b_n)}),$$

where $\underline{b} = (b_0,\ldots,b_n) \in B^n$, $n \geq 0$, $x \in C_p(\underline{X};G)$ and $p \geq 0$.

Then the chain map $f_\#$ induces the homomorphism f_*: $H_p^s(\underline{X};G) \longrightarrow H_p^s(\underline{Y};G)$.

We note that if $\underline{X}$ is an inverse sequence $(X_n, p_{n,n+1})$, we have the following short exact sequence;

$$0 \longrightarrow \varprojlim{}^{(1)}(H_{p+1}(X_n;G)) \longrightarrow H_p^s(\underline{X};G) \xrightarrow{\;\Xi\;} \varprojlim (H_p(X_n)) \longrightarrow 0,$$

where Ξ is induced by the formula $(\xi(x))_n = x_n$ for $x \in C_p(\underline{X};G)$. Therefore the strong homology theory is a generalization of the Steenrod-Sitnikov homology theory.

Every coherent singular i-chain $h \in S_i(\underline{X};G)$, $i \geq 0$, can be considered a strong i-chain if $\underline{X}$ by the formula

$$h(\underline{a}) = h_{\underline{a}} \quad \text{for each } \underline{a} \in A^n,\ n \geq 0.$$

Then we have the natural homomorphism $\theta^i_{\underline{X}}: H^c_i(\underline{X};G) \longrightarrow H^s_i(\underline{X};G)$, $i \geq 0$. Related to $\theta^i_{\underline{X}}$, we pose the following natural problem.

Problem. Under what conditions of $\underline{X}$ and G is the homomorphism $\theta^i_{\underline{X}}$ an isomorphism ?

In [2] , we showed that there is an inverse sequence of 1-dimensional compact polyhedra such that $\theta^1_{\underline{X}}$ is not even an epimorphism. On the other hand, by 1 and Theorem 4, we have the following partial answer of Problem.

Theorem 5. Let $(\underline{X,x}) = ((X_n,x_n), p_{n,n+1})$ be an inverse sequence of pointed compact connected polyhedra. If $\pi^c_k(\underline{X,x}) = 0$, for every $0 \leq k \leq i-1$, $i \geq 2$, then $\theta^i_{\underline{X}}: H^c_i(X;Z) \cong H^s_i(X;Z)$.

5. In [4] , Lisica and Mardešić described the strong shape theory by using ANR-resolutions in the sense of Mardešić [6]. Similarly, by using ANR-resolutions, we define the coherent singular homology groups and the strong homology groups of spaces as follows;

$H^c_*(X;G) = H^c_*(\underline{X};G)$, and $H^s_*(X;G) = H^s_*(\underline{X};G)$, where $\underline{p}: X \longrightarrow \underline{X}$ is an ANR-resolution of a space X.

It is easily seen that those homology groups are invariants in the strong shape category. All results in §2, §3 and §4 can be changed ones of spaces and strong shape category.

References

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