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SOME PROPERTIES OF ONE-STEP RECURRENT TERMS IN LAMBDA CALCULUS

SACHIO HIROKAWA
Department of Computer Science
Shizuoka University
Hamamatsu 432, Japan

ABSTRACT

One-step recurrent term (3) is a term whose one-step reductums are reducible to itself. The following properties of one-step recurrent terms are shown.

(a) Every leftmost reduction of one-step recurrent term is a part of some cycle which returns to that term.

(b) Leftmost reductum of one-step recurrent term is one-step recurrent.

(c) If a reduction cycle can be left by contracting a redex at a point, then the redex occurs at any point in the cycle.

(d) If a term is one-step recurrent, then either it is a nf or has no nf.

These properties are proved by analyzing the behavior of the residuals through the reductions. A fundamental lemma, which the author calls "decomposition lemma", plays a central role.

1. INTRODUCTION

A one-step recurrent term is a term whose one-step reductums are all reducible to itself. The notion was introduced in(3) referring to some conjecture (4). It is a weakened notion of the recurrent term given in (1,2,4,5) or of the minimal form stated in (1,2). A recurrent term is a term which is reducible to itself after any reduction. Roughly, its reduction graph contains only cycles returning to itself. It is obvious that every recurrent term is one-step recurrent. The converse, which is equivalent to the above conjecture, is still open, and it is our motivation of this paper.

To study the problem more closely, this paper examines some properties of one-step recurrent terms. The proof of the results will be carried out by analyzing the behavior of the residuals in the reduction process.

We state some definitions which are needed in the later arguments. The general notions and terminology are referred to (1).

We write $M \rightarrow N$ when term M is reducible to term N by one-step reduction, i.e., by contracting some β-redex Δ in M. Sometimes we write $M \xrightarrow[\Delta]{} N$ when we want to emphasize the redex being contracted. The symbol "$\twoheadrightarrow$" is used for the transitive closure of $\rightarrow$. When Δ is the leftmost redex or the head redex, we write $M \xrightarrow{l} N$ or $M \xrightarrow{h} N$, respectively. Given a set $\mathcal{F}$ of redex occurrences in M, $M \xrightarrow{\mathcal{F}\text{-}cpl} N$ represents a complete development of $(M, \mathcal{F})$. Let $\sigma : M \twoheadrightarrow L$ be a reduction starting from M. Then $\mathcal{F}/\sigma$ represents the set of residuals of $\mathcal{F}$ in L.

Strong equivalence of reductions σ and τ is denoted by $\sigma = \tau$. In the sequel, we draw some diagrams of reduction. In these diagrams we understand any two reductions to be strongly equivalent when they start from the same point and terminate at another same point.

2. FUNDAMENTAL LEMMAS

Lemma 1 (Decomposition lemma) Let $\mathcal{F}$ be a set of redex occurrences in term M and let $\sigma : M \twoheadrightarrow N$. If N contains no residual of $\mathcal{F}$ then σ has a decomposition such that $\sigma = M \xrightarrow{\mathcal{F}\text{-}cpl}\!\!\!\!\twoheadrightarrow L \twoheadrightarrow N$.

Proof By induction on the length of σ. Suppose that σ is of the form $\sigma : M \underset{\Delta}{\xrightarrow{\sigma_0}} M' \xrightarrow{\sigma_1}\!\!\!\!\twoheadrightarrow N$ and that σ_0 contracts a redex Δ in M. Then by induction hypothesis for $M' \xrightarrow{\sigma_1}\!\!\!\!\twoheadrightarrow N$ and $\mathcal{F}/\sigma_0$, σ_1 has a decomposition such that

$$\sigma_1 = M' \xrightarrow{\mathcal{F}/\sigma_0\text{-}cpl}\!\!\!\!\twoheadrightarrow L' \twoheadrightarrow N.$$

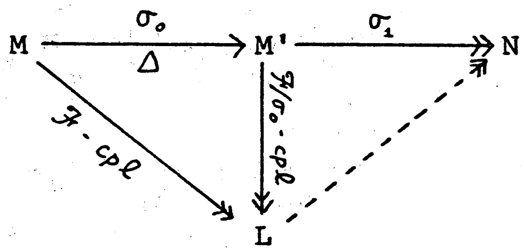

Figure 1

Case 1 Δ belongs to $\mathcal{F}$. (see Figure 1.) Then reduction σ_0 followed by a complete development of $(M, \mathcal{F}/\sigma_0)$ is a complete development of $(M, \mathcal{F})$. Therefore L=L' and $\sigma = M \xrightarrow{\mathcal{F}/\sigma_0\text{-}cpl}\!\!\!\!\twoheadrightarrow L \twoheadrightarrow N$.

Case 2 Δ does not belongs to $\mathcal{F}$. (See Figure 2.) Consider the complete developments of $(M, \mathcal{F} \cup \{\Delta\})$. Then we have the strong equivalences of reductions:

$$M \underset{\tau}{\xrightarrow{\mathcal{F}\text{-}cpl}}\!\!\!\!\twoheadrightarrow L \xrightarrow{\{\Delta\}/\tau\text{-}cpl}\!\!\!\!\twoheadrightarrow L'$$

$$= M \xrightarrow{\mathcal{F} \cup \{\Delta\}\text{-}cpl}\!\!\!\!\twoheadrightarrow L'$$

$$= M \underset{\Delta}{\xrightarrow{\sigma_0}} M' \xrightarrow{\mathcal{F}/\sigma_0\text{-}cpl}\!\!\!\!\twoheadrightarrow L'.$$

Hence $\sigma = M \xrightarrow{\mathcal{F}\text{-}cpl}\!\!\!\!\twoheadrightarrow L \twoheadrightarrow N$.

Q.E.D.

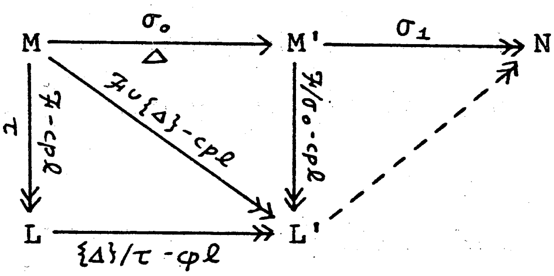

Figure 2

Lemma 2 (2) Let $\mathcal{F}$ be the set of all redexes in M and G(M) be the term obtained by a complete development of $(M, \mathcal{F})$. Then necessary and sufficient condition for M to be recurrent is that G(M) is reducible to M.

Proof If M is recurrent, then G(M) is reducible to M. So it suffices to show the sufficiency.

Then let $M=L_0 \xrightarrow[\Delta_0]{} L_1 \xrightarrow[\Delta_1]{} L_2 \xrightarrow[\Delta_2]{} L_3 \xrightarrow[\Delta_3]{} \cdots \xrightarrow[\Delta_{n-2}]{} L_{n-1} \xrightarrow[\Delta_{n-1}]{} L_n$ be an arbitrary reduction of M, where Δ_i is the redex contracted by the reduction $L_i \rightarrow L_{i+1}$. Since G(M) is reducible to M, we have a reduction cycle $M \overset{\sigma}{\twoheadrightarrow} G(M) \overset{\tau}{\twoheadrightarrow} M$.

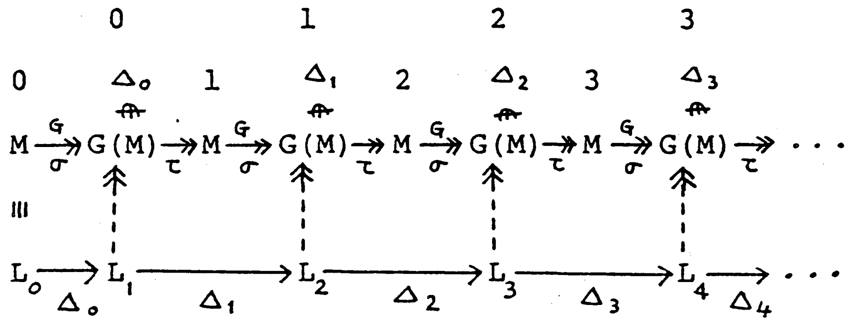

Figure 3

Thenconsider the n-times composition of $\sigma + \tau$. (See Figure 3.) The i-th σ erases all the redexes of i-th M, so i-th G(M)does not contain any residual of Δ_i in L_i. By lemma 1, L_{i+1} is reducible to i-th G(M). Hence L_i is reducible to M. Q.E.D.

Using the decomposition lemma, we can simplify the proof of the following theorem. (As for the term with only one redex in it, the theorem is an easy consequence of lemma 2.)

Theorem 3 (3) Let M be a term having two redexes. If M is one-step recurrent, then M is recurrent.

Proof Let Δ_1 be the leftmost redex in M and Δ_2 be the other one. Since M is one-step recurrent, we have two reductions such that $\sigma_1 : M \xrightarrow[\Delta_1]{\ell} M \twoheadrightarrow M$, $\sigma_2 : M \xrightarrow[\Delta_2]{} M \twoheadrightarrow M$. Consider the residual of Δ_1 in M at the end of σ_2. Since Δ_1 is leftmost, its residual must be leftmost if it exists. Hence the reduction $\sigma_2 + \sigma_1 : M \xrightarrow[\Delta_2]{} M \twoheadrightarrow M \xrightarrow{\ell} M \twoheadrightarrow M$ erases both Δ_1 and Δ_2, and by lemma 1, we have $\sigma_2 + \sigma_1 = M \xrightarrow{\{\Delta_1,\Delta_2\}\text{-}\varphi\ell} G(M) \twoheadrightarrow M$. By lemma 2, M is recurrent. Q.E.D.

<u>Lemma 4</u> Let $\sigma : M \twoheadrightarrow L$ and $\tau : L \twoheadrightarrow M$. Let $L \to N$ by contracting a redex Δ which is not residual of any redex in M. Then N is reducible to M.

<u>Proof</u> Consider the infinite reduction $\sigma + \tau + \sigma + \tau + \cdots$. (See Figure 4.) Let Δ^i be the redex Δ in L_i.

Suppose that every term $M_i (i \geqslant 1)$ contains a residual of Δ ($= \Delta^1$). Let Δ^1_i be such a residual in M. Now consider the redex Δ^2 in L_2. By the assumption, Δ^2 is not the residual of any redex in M_1, so neither of Δ^1. Since M_i $(i \geqslant 1)$ contains a residual of Δ, $M_i (i \geqslant 2)$ contains a residual of Δ^2. Continuing the similar argument, M_i has at least i redexes, i.e., the residuals of Δ^1, Δ^2, $\cdots$ and Δ^i. A contradiction. Therefore there is a term M which has no residual of Δ. By lemma 1, we have $L \twoheadrightarrow M_i = L \xrightarrow[\Delta]{} N \twoheadrightarrow M_i \equiv M$. Q.E.D.

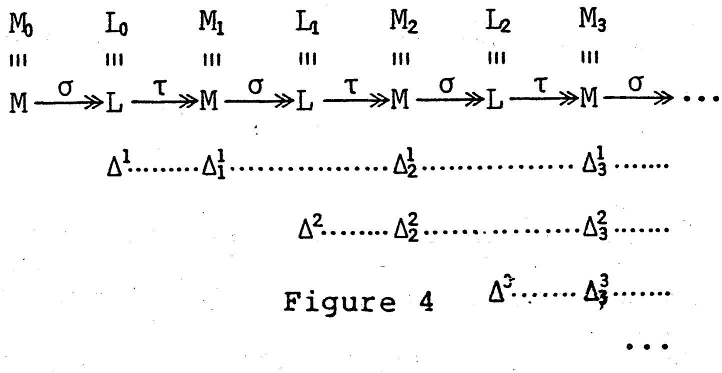

Figure 4

<u>Lemma 5</u>

Let $\tau : M \xrightarrow{\ell} N$, $\sigma : M' \twoheadrightarrow M$ and $\tau' : M' \xrightarrow{\ell} N'$. Then there is a reduction $\sigma' : N \twoheadrightarrow N'$ such that $\sigma + \tau' = \tau + \sigma'$. (See Figure 5.)

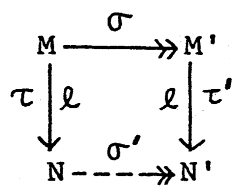

Figure 5

Proof By induction on the length n of $M \xrightarrow{\sigma}\!\!\!\!\rightarrow M'$.

Base Step: n=1. For brevity we prove only the following case such that $M=(\lambda x.P)Q$, $M'=(\lambda x.P)Q'$, $N=P[x:=Q]$ and $N'=P[x:=Q']$. If we contract Q to Q' at every occurrence of Q in $N=P[x:=Q]$, we can obtain N'. So we have $N \twoheadrightarrow N'$.(See Figure 6.)

Induction Step:Let $\sigma = M \xrightarrow{\sigma_1}\!\!\!\!\rightarrow M'' \xrightarrow{\sigma_2} M'$. By induction hypothesis for σ_1, we have the left square of Figure 6. By induction hypothesis for σ_2, we have the right square. Thus N is reducible to N'. Q.E.D.

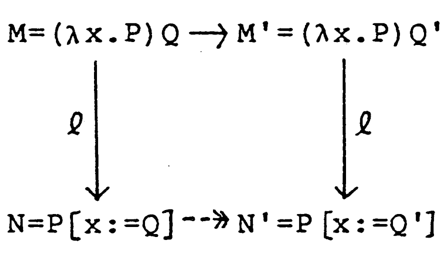

Figure 6

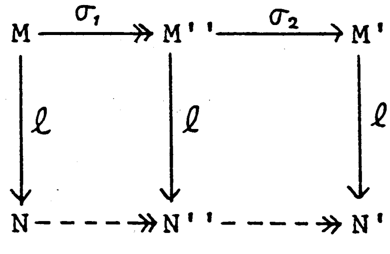

Figure 7

3. LEFTMOST REDUCTION OF ONE-STEP RECURRENT TERM

As for the actual form of terms, the lemma 13.3.2 in (1) is extended to one-step recurrent terms in the following theorem.

Theorem 6 If M is one-step recurrent, then either M is a nf or has no nf.

Proof Suppose M is not in nf. Consider the term N obtained by contracting the leftmost redex in M. Since M in one-step recurrent, N is reducible to M. Thus we have the following infinite reduction: $M \xrightarrow{\ell} N \twoheadrightarrow M \xrightarrow{\ell} N \twoheadrightarrow \cdots$, which is quasi leftmost. Since every term with infinite quasi leftmost reduction has no nf (1), M has no nf. Q.E.D.

Using the notion of quasi-head reduction we can prove:

<u>Theorem 7</u> If M is one-step recurrent, then either M is in hnf or has no hnf.

<u>Theorem 8</u> Let M be a one-step recurrent term and N be a term obtained from M by leftmost reduction . Then N is reducible to M.

<u>Proof</u> Suppose M is reducible to L by contracting the leftmost redex in M. Since M is one-step recurrent, L is reducible to M. Let n be the length of the leftmost reduction $M \overset{\ell}{\twoheadrightarrow} N$.

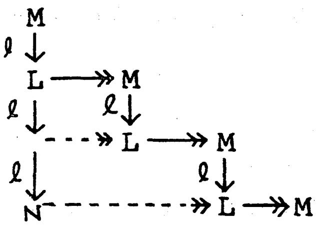

Figure 8

Then consider the reduction

$$\sigma : M \overset{\ell}{\longrightarrow} L \twoheadrightarrow M \overset{\ell}{\longrightarrow} L \twoheadrightarrow M \overset{\ell}{\longrightarrow} \cdots \twoheadrightarrow M,$$

which is n-times composition of $M \overset{\ell}{\longrightarrow} L \twoheadrightarrow M$. By lemma 5, $\sigma = M \overset{\ell}{\twoheadrightarrow} N \twoheadrightarrow M$. (See Figure 8 for the case of n=3.) Q.E.D.

<u>Theorem 9</u> Let M be a one-step recurrent term and L be the term obtained from M by leftmost reduction. If L is reduced to N by contracting a redex in L, then N is reducible to M.

<u>Proof</u> Suppose L is reduced to N by contracting a redex Δ in L .

<u>Case 1</u>: Δ is not a residual of any redex in M.

By theorem 8 we have $L \twoheadrightarrow M$, so $N \twoheadrightarrow M$ by lemma 3.

<u>Case 2</u>: Δ is a residual of some redex Δ' in M.

Since M is one-step recurrent, we have the following reductions $\sigma : M \overset{\ell}{\longrightarrow} P \twoheadrightarrow M$, $\tau : M \underset{\Delta'}{\longrightarrow} Q \twoheadrightarrow M$, where Q is obtained by contracting Δ'. Let n be the length of $M \overset{\ell}{\twoheadrightarrow} L$ and τ' be the reduction τ followed by n-times σ . By lemma 4, we have $\tau' = \tau''$: $M \overset{\ell}{\twoheadrightarrow} L \longrightarrow M$. Since τ' erases the redex Δ' , the term obtained by τ'' contains no residual of Δ . Thus $N \twoheadrightarrow M$ by lemma 1. Q.E.D.

Theorem 10 If M is obtained from a one-step recurrent term by leftmost reduction, then M is one-step recurrent.

Proof Let L be a one-step recurrent term and $L \overset{\ell}{\twoheadrightarrow} M \longrightarrow N$. Then by theorem 9 $N \twoheadrightarrow L$, so we have $N \twoheadrightarrow M$. Thus M is one-step recurrent. Q.E.D.

The following theorem is an extension of a result about recurrent terms given in (5). By theorem 10 we can prove the theorem similarly to (5).

Theorem 11 Let M be a one-step recurrent term which is not in hnf. Then M can be reduced to a one-step recurrent term of the form $\lambda x_1 \cdots x_n.(\lambda x.P)QR_1 \cdots R_\ell$, where $(\lambda x.P)Q$ is a one-step recurrent term of order zero and R_i's are one-step recurrent term.

Proof Let $M = \lambda x_1 \cdots x_m.(\lambda x.M_1)M_2 N_1 \cdots N_k$. We prove the theoremby induction on k.

Base Step: k=0. If the term $(\lambda x.M_1)M_2$ were not of order zero, then we would have a head reduction like $(\lambda x.M_1)M_2 \overset{h}{\twoheadrightarrow} y.M'$. By theorem 8, we have $\lambda y.M' \twoheadrightarrow (\lambda x.M_1)M_2$. A contradiction. Hence the order of $(\lambda x.M_1)M_2$ is zero.

Induction Step: If the term $(\lambda x.M_1)M_2$ is of order zero the theorem holds trivially. Otherwise $(\lambda x.M_1)M_2 \overset{h}{\twoheadrightarrow} y.M'$ for some M'.

Thus we have a head reduction of M such that

$$M=\lambda x_1 \cdots x_m.(\lambda x.M_1)M_2N_1\cdots N_k \xrightarrow{h} M''=\lambda x_1\cdots x_m.(\lambda y.M')N_1\cdots N_k.$$

By theorem 10, M'' is one-step recurrent. By induction hypothesis the theorem holds for M''. Hence it also holds for M. Q.E.D.

3. KLOP'S CONJECTURE

A plane is an equivalence class of terms by the cyclic equivalence, where M and N are said cyclically equivalent if and only if M N and N M. We say that a plane can be left when there is a term M in the plane and irreversible step M N for some N which does not belong to the plane. These notions are due to (4).

The original motivation of considering one-step recurrent term comes form the following theorem:

Theorem 12 (3) The following (a) and (b) are equivalent.

(a) (Klop's conjecture (4)) If a plane can be left at one point, then it can be left at any point.

(b) A term is recurrent if and only if it is one-step recurrent.

Proof Since every recurrent term is one-step recurrent, the only-if part is trivial in (b).

First we prove (a)==>(b). Let M be a one-step recurrent term. Then the plane which contains M can not be left at M. Therefore it cannot be left at any point. Hence any term reducible from M is reducible to M.

Next we prove (b)<==(a). Suppose that a plane can be left at a point M and that it can not be left at another point N. Then N is one-step recurrent. By (b), N is recurrent. Therefore M is recurrent. This contradicts to the assumption that M has an irreversible one-step reduction. Q.E.D.

<u>Theorem 13</u> If a plane can be left at one point by one-step leftmost reduction, then it can be left at any point by one-step leftmost reduction.

<u>Proof</u> Suppose that we can leave a plane α at a point M by one-step leftmost reduction $M \xrightarrow{\ell} N$. And Suppose that M' is a term in α and $M' \xrightarrow{\ell} N'$. Then there is a reduction $M \twoheadrightarrow M'$. So we have $N \twoheadrightarrow N'$ by lemma 5. Since N is not reducible to M, neither is N'. Thus the plane can be left at M' by one-step leftmost reduction. Q.E.D.

<u>Theorem 14</u> Let M be a term on a plane α which can be left at M by contacting a redex Δ in M. Let σ : $M \twoheadrightarrow N$ and τ: $L \twoheadrightarrow M$ be arbitrary reductions on α . Then
(1) N contains some residual of Δ .
(2) Δ is a residual of some redex in L. (See Figure 9.)

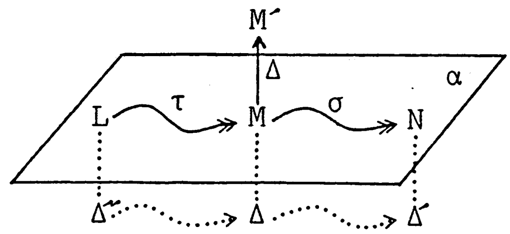

Figure 9

<u>Proof</u> Let M' be the term obtained by contracting in M.
(1) Suppose that $\{\Delta\}/_{\sigma} = \phi$. Then by lemma 1, we have a decomposition of such that $\sigma = M \xrightarrow[\Delta]{} M' \twoheadrightarrow N$. Since N belongs to α we have $N \twoheadrightarrow M$. Hence $M' \twoheadrightarrow M$. A contradictions. Therefore $\{\Delta\}/_{\sigma} = \phi$.

(2) If Δ is not the residual of any redex in L, then by lemma 4, we have $M' \twoheadrightarrow L$. Hence $M' \twoheadrightarrow M$. A contradiction. Q.E.D.

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