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Paris-Harrington Theory and reflection Principles

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Introduction

In their paper [PH], Paris and Harrington showed that in Peano Arithmetic PA Harrington principle (H) is equivalent to the uniform reflection principle RFN_{Σ_1}. Since uniform reflection principles RFN_{Σ_p} $(p = 1,2,3,\dots)$ make a hierarchy over PA ([Sm],*4,1[a]), it is natural to ask for a hierarchy of extensions of (H) which corresponds to the hierarchy of reflection principles.

In order to prove the unprovability of (H) in PA, Paris and Harrington considered a theory T, showing that (H) implies Con(T) and Con(T) implies Con(PA) in PA. If one see the proof precisely, then we can find that (H) is equivalent to $Mod(T)_\omega$ and $Mod(T)_\omega$ implies Con(T), where $Mod(T)_\omega$ means that every finite subset of T has a model on ω. We consider theories T_n $(n \in \omega)$ and T_∞, which are extensions of $T = T_0$. However all the sentences $Mod(T_n)_\omega$ $(n \in \omega)$ and $Mod(T_\infty)$ become equivalent to (H). In addition, all of $Con(T_n)$ $(n \in \omega)$ are equivalent to Con(PA). By considering T_n $(n \in \omega)$ we cannot produce any hierarchy corresponding to RFN_{Σ_p} $(p = 1,2,3,\dots)$.

Next we extend Harrington principle directly and define sentences (H_p) $(p = 1,2,3,\dots)$ where (H_1) is (H). Then this hierarchy is the one just seeked for, since (H_p) is exactly equivalent to RFN_{Σ_p} for every $p = 1,2,3,\dots$. So the problem is solved in one sense. However, since the principles (H_p) are rather complicated in the view point of arithmetical and combinatorial formula, so it is desirable to find more simple hierarchies.

§1. Equiconsistency of PA and Theory T_n and equivalency of $\mathrm{Mod}(T_n)_\omega$ to (H)

1.1 Definitions and notations

(1) PA ; Peano Arithmetic with μ-symbol.

(2) (H) ; Harrington Principle i.e.

$$\forall e \forall r \forall k \, \exists M (M \xrightarrow[*]{} (k)^e_r)$$

(3) Theory T_n $(n \in \omega)$, T_∞

T_0 is the theory of T in [PH].

T_n $(n \geqq 1)$ is as follows;

Language of T_n ; 0, 1, +, $\cdot$, < and constants c_i $(i \in \omega)$.

Axioms of T_n;

(i) Defining equations for 0, 1, +, $\cdot$, < and the mathematical induction axioms for Σ_n-formulas.

(ii) $c_i^2 < c_{i+1}$.

(iii) For any $i < \mathbb{k}, \mathbb{k}'$ and Σ_n-formula $\phi(\mathbf{y},\mathbf{z})$ (where $\mathbb{k}$, $\mathbb{k}'$ and $\mathbf{z}$ have the same length),

$$\forall \mathbf{y} < c_i [\phi(\mathbf{y}, c(\mathbb{k})) \leftrightarrow \phi(\mathbf{y}, c(\mathbb{k}'))].$$

T_∞ is obtained from T_n by changing the Σ_n-formulas into unrestricted ones.

Remark) $T_0 \subset T_n \subset T_{n+1} \subset T_\infty$.

1.2 Equiconsistency and conservativity.

<u>Proposition 1.</u> $PA \vdash \mathrm{Con}(T_0) \to \mathrm{Con}(PA)$ (2.2 in [PH]).

<u>Proposition 2.</u> $PA \vdash \mathrm{Con}(PA) \to \mathrm{Con}(T_\infty)$.

So T_n $(n \in \omega)$, T_∞, and PA are provably equiconsistent in PA.

Proposition 3. T_∞ is a conservative extension of PA.

Lemma. Let $\phi_j(y,z) \equiv \phi_j(y_1,\ldots,y_m,z_1,z_2,\ldots,z_n)$ $(j = 1,\ldots,l)$ be a finite set of formulas in PA. For any number $k > n$, there is a sequence of terms $\tilde{c}_0,\ldots,\tilde{c}_{k-1}$ in PA which satisfies

(i) $PA \vdash \tilde{c}_i^2 < \tilde{c}_{i+1}$ for $0 \leq i < k-1$ and

(ii) $PA \vdash \forall y < c_i[\phi_j(y,\tilde{c}(k)) \leftrightarrow \phi_j(y,\tilde{c}(k'))]$ $j = 1,\ldots,l$ for $i < k$, $k' < k$.

From this Lemma, we can derive Proposition 2 and 3 immediately. In [PH], both Proposition 2 for T_0 and the Lemma are not mentioned explicitly. But Dr. Uesu pointed out to me that the proofs of 2.10 and 2.11 of [PH] can be regarded the proof of the above Lemma. For, (H) is not provable in PA, but for each number e, the formula $\forall k \forall r \exists M\ (M \underset{*}{\rightarrow} (k)_r^e)$ is provable in PA. In this proof the fact that "ϕ_j is limited" is never used, so the Lemma holds for any formulas ϕ_j.

1.3 Models of finite subsets of T_n on ω.

Let $\mathrm{Mod}(T_n)_\omega$ be the formula expressing "Every finite set of axioms of T_n has a model on ω".

Proposition 4. For all $n \in \omega$, $PA \vdash (H) \leftrightarrow \mathrm{Mod}(T_n)_\omega$

Proof) The Proposition 2.11 in [PH] leads to one direction, $PA \vdash (H) \to \mathrm{Mod}(T_n)_\omega$, by constructing a model. The axioms in (i) of T_n are also satisfied in this model, because the truth-definition for Σ_n-formulas can be constructed in PA itself. On the other hand, Paris and Harrington proved that $PA \vdash \mathrm{Mod}(T_0)_\omega \to RFN_{\Sigma_1}$ and $PA \vdash RFN_{\Sigma_1} \leftrightarrow (H)$, so $PA \vdash \mathrm{Mod}(T_n)_\omega \to (H)$.

$\mathrm{Con}(T_n)$ and $\mathrm{Mod}(T_n)_\omega$ give no hierarchy corresponding to RFN_{Σ_n}.

§2. Extensions of Harrington principle and Reflection principles

2.1 Harrington Principles and Reflection principles

In this section we suppose that PA and T have all the symbols of primitive recursive functions and their defining equations as axioms.

A Π_p-sentence ϕ of PA can be written as

$$\phi := \forall x_0 Qx_1 \ldots Qx_{p-1} A(x_0, x_1, \ldots, x_{p-1}) \tag{1}$$

where Q'_s are $\exists$ or $\forall$ alternately, and A is a quantifier free formula. Then define

$$\phi^*(z_0, z_1, \ldots, z_{p-1}) := \forall x_0 < z_0 Qx_1 < z_1 \ldots Qx_{p-1} < z_{p-1} A(x_0, x_1, \ldots, x_{p-1}).$$

<u>Definition 1.</u> $M \xrightarrow[*]{\phi} (k)^e_r$

For $k, e, r, M \in \omega$ and a Π_p-sentence ϕ, $M \xrightarrow[*]{\phi} (k)^e_r$ is the following formula:

For every partition $P : [M]^e \to r$ there is a subset $Y \subseteq M$ ($Y = \{y_0, y_1, \ldots, y_{q-1}\}$, $y_0 < y_1 < \ldots < y_{q-1}$) such that

(i) Y is homogeneous for P,

(ii) $\text{card}(Y) \geqq k$,

(iii) Y is relatively large, i.e. $\text{card}(Y) \geqq \min(Y)$, and

(iv) $\phi^*(y_0, y_1, \ldots, y_{p-1})$ holds.

Note that $M \xrightarrow[*]{\phi} (k)^e_r$ is a primitive recursive formula.

<u>Definition 2.</u> (H_p)

For $p = 1, 2, \ldots$, (H_p) means the following sentence:

For all k, e, r and for all true Π_p-sentence ϕ, there exists an $M \in \omega$ such that $M \xrightarrow[*]{\phi} (k)^e_r$.

If $p = 1$, since the condition (iv) holds trivially, (H_1) concides with the Harrington principle (H). Clearly (H_{p+1}) implies (H_p).

Proposition 1.

(H_p) is a Π_{p+1}-sentence of PA.

Proof).

For every Π_p-sentence ϕ of the form (1) there is a number f such that ϕ is equivalent to the formula

$$\tau(f) := \forall x_0 Q x_1 \dots Q x_{p-1} T'_{p-1}(f, x_0, \dots, x_{p-1})$$

where T'_{p-1} is Kleene's T_{p-1} if Qx_{p-1} is $\exists x_{p-1}$ and is $\neg T_{p-1}$ if Qx_{p-1} is $\forall x_{p-1}$.

So (H_p) is expressed as

$$\forall k \forall e \forall r \forall f [\forall x_0 Q x_1 \dots Q x_{p-1} T'_{p-1}(f, x_0, \dots, x_{p-1}) \rightarrow \exists M (M \xrightarrow[*]{\tau(f)} (k)^e_r)],$$

which is a Π_{p+1}-sentence.

Definition 3.

For every true Π_p-sentence ϕ of the form (1) we define a finite sequence of arithmetical functions

$$f_1(x_0),\ f_3(x_0, x_2),\ f_5(x_0, x_2, x_4) \dots$$

by the following way:

$$f_1(x_0) = \mu x_1 Q x_2 \dots Q x_{p-1} A(x_0, x_1, \dots, x_{p-1})$$

$$f_3(x_0, x_2) = \mu x_3 Q x_4 \dots Q x_{p-1} A(x_0, f_1(x_0), x_2, \dots, x_{p-1})$$

$$f_5(x_0,x_2,x_4) = \mu x_5 Q x_6 \ldots Q x_{p-1} A(x_0,f_1(x_0),x_2,f_3(x_0,x_2),x_4,x_5,\ldots,x_{p-1})$$

. . .

We call $(f_1,f_3,\ldots,f_s)$ the function sequence of ϕ.

Proposition 2.

Let ϕ be a true Π_p-sentence and $(f_1 f_3 f_5 \ldots)$ be its function sequence. Put the functions f_1^*, f_3^*, f_5^* ... as following:

$$f_1^*(y_0) = \max\{f_1(x_0),\ x_0 < y_0\}$$

$$f_3^*(y_0,y_2) = \max\{f_3(x_0,x_2);\ x_0 < y_0,\ x_2 < y_2\}$$

$$f_5^*(y_0,y_2,y_4) = \max\{f_5(x_0,x_2,x_4);\ x_0 < y_0,\ x_2 < y_2,\ x_4 < y_4\}$$

. . . .

Then the condition (iv) of Definition 1 is equivalent to:

(iv)' $f_1^*(y_0) < y_1$, $f_3^*(y_0,y_2) < y_3$, $f_5^*(y_0,y_2,y_4) < y_5$, ...

(The proof is obvious.)

Proposition 3.

(H_p) is equivalent to the sentence obtained from the definition of (H_p) by replacing the condition (iv) of $M \xrightarrow[*]{\phi} (k)_r^e$ with the following condition:

(iv)" For all $i_0, i_1, \ldots, i_{p-1} \in \omega$

$$i_0 < i_1 < \ldots < i_{p-1} < q-1 \rightarrow \phi^*(x_{i_0},x_{i_1},\ldots,x_{i_{p-1}}).$$

Proof).

Use 2.9 in [PH].

2.2 Truth of (H_p)

Proposition 4.

(i) (H_p) is true.

(ii) For each e and each true Π_p-sentence ϕ

$$PA \vdash \forall k \forall r \exists M(M \xrightarrow[*]{\phi} (k)^e_r).$$

(iii) $PA \vdash RFN_{\Sigma_p} \to (H_p)$.

Proof) (Cf. 2.1 and 3.1 in [PH])

(i) Suppose H_p were false, construct the tree of counter examples <P,M>, take an infinite path by König lemma, and put a homogeneous infinite set by infinite Ramsey theorem. Then we can find its finite subset that satisfies the conditions (i)-(iv) for Y in Definition 1.

(ii), (iii) Formalize the above proof.

2.4 Relation to reflection principles

Proposition 5. (2.4 in [PH])

For every model $\mathcal{A}$ of T there is a model $\mathcal{J}$ of PA such that for all prenex formula $\theta(\mathbf{y})$ in PA and for all $i < \mathbf{k}$ and $\mathbf{a} < c_i$.

$$\mathcal{J} \models \theta(\mathbf{a}) \quad \text{iff} \quad \mathcal{A} \models \theta^*(\mathbf{a}, c(\mathbf{k})).$$

Proposition 6.

In PA + (H_p) it is proved that for all true Π_p-sentence ϕ and finite subset S of T, S + $\{\phi^*(c_0, \ldots, c_{p-1})\}$ has a model on ω.

Proof)

Similar to 2.11 om [PH].

Proposition 7.

$$PA + (H_p) \vdash RFN_{\Sigma_p} .$$

Proof)

In PA, suppose (H_p) and let ϕ be a true Π_p-sentence. By Proposition 6 and Compactiness theorem $T + \{\phi^*(c_0, \ldots, c_{p-1})\}$ has a model. Then by Proposition 5 $PA + \{\phi\}$ is consistent.

Formalizing the above discussion, we can obtain

$$PA + (H_p) \vdash Tr_p(\ulcorner\phi\urcorner) \rightarrow \neg Pr_{pA}(\ulcorner\neg\phi\urcorner). \tag{2}$$

where Tr_p is the partial truth-definition of order p (Cf. [Sm]). Let $\phi(a)$ be a Π_p-formula whose only free variable is a. Since for the sentences $\phi(\bar{n})$ for all numeral $\bar{n}$ (2) holds, we have

$$PA + (H_p) \vdash Tr_p(\ulcorner\phi(\dot{a})\urcorner) \rightarrow \neg Pr_{pA}(\ulcorner\neg\phi(\dot{a})\urcorner).$$

And $\quad PA \vdash Tr_p(\ulcorner\phi(\dot{a})\urcorner) \leftrightarrow \phi(a) \qquad$ (5.21 in [Sc]),

so for all Σ_p-formula $\psi(a)$,

$$PA + (H_p) \vdash Pr_{pA}(\ulcorner\psi(\dot{a})\urcorner) \rightarrow \psi(a).$$

Combining this proposition and Proposition 4 (iii), we have the following theorem.

Theorem.

$$PA \vdash (H_p) \leftrightarrow RFN_{\Sigma_p} \qquad (p = 1, 2, 3, \ldots).$$

References

[PH] J. Paris and L. Harrington, A mathematical incompleteness in Peano Arithmetic (Handbook of Mathematical Logic, D.8.), 1977.

[Sm] C. Smorynski, The incompleteness theorem (ibid. D.1.).

[Sc] H. Schwichtenberg, Proof theory: Some applications of cut elimination (ibid. D.2.).