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Characteristic classes of surface bundles

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In this paper we define characteristic classes of surface bundles, namely smooth fibre bundles whose fibres are a closed orientable surface Σ_g of genus $g \geq 2$ and announce some non-triviality results for them. As a consequence we obtain lower bounds for the Betti numbers of the mapping class group $M(g)$ of Σ_g.

It is known [EE] that the connected component of the identity of $\mathrm{Diff}_+\Sigma_g$, the group of orientation preserving diffeomorphisms of Σ_g, is contractible. Therefore $\mathrm{BDiff}_+\Sigma_g$ is a $K(M(g),1)$. Now let ξ be the tangent bundle along the fibres of an oriented surface bundle and let $e(\xi)$ be its Euler class. If we apply the Gysin homomorphism to $e^{i+1}(\xi)$, we obtain an integral cohomology class of the base space of degree $2i$. By the naturality, this defines certain cohomology classes

$$e_i \in H^{2i}(M(g):\mathbb{Z}) \quad (i=1,2,\ldots).$$

$M(g)$ acts on $H^1(\Sigma_g;\mathbb{Z})$ preserving the symplectic form given by the cup product and so we obtain a homomorphism $M(g) \longrightarrow Sp(2g;\mathbb{Z})$, where $Sp(2g;\mathbb{Z})$ is the group of all $2g\times 2g$ symplectic matrices with integral entries. This induces a homomorphism $M(g) \longrightarrow Sp(2g;\mathbb{R})$. Since $Sp(2g;\mathbb{R})$ has $U(g)$ as a maximal compact subgroup, we have a g-dimensional complex vector bundle η on $K(M(g),1)$. Let $c_i(\eta) \in H^{2i}(M(g);\mathbb{Z})$ be its i-th Chern class. From the argument of Atiyah in [A] and the fact that η is

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flat as a real vector bundle, we can conclude

$$e_{2i-1} = \frac{(-1)^i 2i}{B_i} s_{2i-1}(c(\eta)) \qquad (i=1,2,\ldots \text{ and}$$

$$s_{2i}(c(\eta)) = 0 \qquad \text{coefficients are in } \mathbb{Q})$$

where $s_i(c(\eta))$ stands for the characteristic class of η corresponding to the formal sum $\sum_j t_j^{\,i}$ and B_i is the i-th Bernoulli number. These two relations induce those among monomials of e_{2i-1}'s and the quotient $\mathbb{Q}[e_1,e_3,\ldots]/(\text{relations})$ is naturally isomorphic to the relative Lie algebra cohomology $H^*(sp(2g;\mathbb{R}),u(g))$ which in turn is additively isomorphic to $H^*(S^2\times S^4\times\ldots\times S^{2g};\mathbb{Q})$ (see [BH]). It is known that $M(g)$ acts properly discontinuously on the Teichmüller space $T(g) \cong \mathbb{R}^{6g-6}$ with non-compact quotient $\mathcal{M}_g$, the moduli space for Riemann surfaces of genus g. Hence $\mathrm{vcd}(M(g)) \leq 6g-7$. Thus any monomial of e_i's of degree $\geq 6g-6$ vanishes. To sum up we have a homomorphism

$$\phi\colon \mathbb{Q}[e_1,e_2,\ldots]/(\text{above relations}) \longrightarrow H^*(M(g);\mathbb{Q}),$$

here we use the letter e_i for both symbolical and actual meanings. Since $\mathrm{vcd}(M(g))$ is conjectured to be $3g-3$ ([Hv]), ϕ will surely have a still large kernel. Our main results are

THEOREM 1. For any $k \in \mathbb{N}$, there exists a natural number $g(k)$ such that the elements $e_1,\ldots,e_k$ are all non-trivial in $H^*(M(g);\mathbb{Q})$ if $g \geq g(k)$.

COROLLARY 2. The natural surjective homomorphism $\mathrm{Diff}_+\Sigma_g \longrightarrow M(g)$ does not have a right inverse if $g \geq g(3)$ (we can take 86 for $g(3)$). In fact the induced homomorphism $H_{2i}(\mathrm{Diff}_+\Sigma_g) \longrightarrow H_{2i}(M(g))$ is not surjective for $i=3,4,\ldots,k$ if $g \geq g(k)$ (here we consider $\mathrm{Diff}_+\Sigma_g$ as a discrete group).

This result should be compared with the recent affirmative solution of the Nielsen realization problem by Kerckhoff [Ke].

THEOREM 3. For any $k \in \mathbb{N}$, there exists a natural number $g'(k)$ such that the 2i-th Betti number $b_{2i}(M(g))$ of $M(g)$, which is equal to $b_{2i}(\mathcal{M}_g)$, is at least i for all $i=1,\ldots,k$ if $g \geq g'(k)$.

We mention that Harer [Ha] has proved that $b_2(M(g)) = 1$ for $g \geq 5$. Since "half" of our characteristic classes come from $S_p(2g;\mathbb{Z})$, we also have informations on the homomorphism $H_{2i}(M(g)) \longrightarrow H_{2i}(S_p(2g;\mathbb{Z}))$. We omit the precise statement.

Sketch of Proofs. The proofs of the above results are given by constructing sufficiently many surface bundles with non-trivial characteristic classes. Roughly speaking we apply the method of Atiyah [A] (see also [Ko]) iteratively. To be more precise, let $\pi: E \longrightarrow X$ be a surface bundle with fibre Σ_g. We assume that X is an iterated surface bundle. Given $(n,n') \in \mathbb{N}\times\mathbb{N}$, an "(n,n')-construction on $\pi: E \longrightarrow X$" is described by the following diagram of surface bundles:

$$\begin{array}{ccccccccc}
F & \longrightarrow & E_2^* & \longrightarrow & {}'E_1^* & \longrightarrow & E_1^* & \longrightarrow & E^* \\
\downarrow{\scriptstyle \Sigma''} & & \downarrow{\scriptstyle \Sigma'} & & \downarrow{\scriptstyle \Sigma'} & & \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \Sigma} \\
E_2 & = & E_2 & \longrightarrow & E_1 & = & E_1 & \longrightarrow & E \\
 & & & & & & & & \downarrow{\scriptstyle \Sigma} \\
 & & & & & & & & X \; .
\end{array}$$

Here $E^* \longrightarrow E$ is the pull back bundle $\pi^*(E)$. E^* contains a cross-section D as the diagonal. $E_1 \longrightarrow E$ is a covering map which kills first the action of $\pi_1(E)$ on $H^1(\text{fibre};\mathbb{Z}/nn')$ and then kills $H^1(\ ;\mathbb{Z}/nn')$. $E_1^* \longrightarrow E^*$ is the pull back by this map. $'E_1^* \longrightarrow E_1^*$ is a fibre-wise nn'-fold covering map. $E_2 \longrightarrow E_1$ is a covering map which satisfies the condition: the homology class of the inverse image D' of D under the map $E_2^* \longrightarrow E^*$ is divisible by n. The assumption that X is an iterated surface bundle guarantees the existence of such covering. Finally $F \longrightarrow E_2^*$ is an n-fold cyclic ramified covering ramified along D'. $F \longrightarrow E_2$ is a surface bundle with fibre Σ'' whose genus is $n^2n'g - \frac{1}{2}n(n+1)n'+1$. The $(2,1)$-construction on the trivial surface bundle $\Sigma_g \longrightarrow pt.$ is nothing but Atiyah's method in [A]. Theorem 1 is proved by calculating e_k of surface bundles which are defined by applying (n_j,n_j')-constructions on $\Sigma_g \longrightarrow pt.$ successively $(j=1,\ldots,k)$. It turns out that e_k of such a surface bundle is $(g-1)$ times a non-trivial polynomial of n_j,n_j''s. Since such surface bundles admit multi-valued cross-sections, once the statement of Theorem 1 is proved for one g_0, it holds for all $g \geq g_0$. Corollary 2 follows from Theorem 1 and the Bott vanishing theorem [B]. We can also compute other characteristic classes than e_k. It turns out that $e_{i_1}^{d_1}\ldots e_{i_s}^{d_s}$ $(\sum_j i_jd_j = k)$ is a linear combination of $(g-1)$, $(g-1)^2,\ldots,$ $(g-1)^{d_1+\cdots+d_s}$ with coefficients in polynomials of n_j,n_j''s. Theorem 3 follows from this.

It is very likely that these examples of surface bundles are enough to prove the injectivity of ϕ in small degrees. However necessary computations for that are extremely complicated. Also it seems to be interesting to test the surjectivity of ϕ by examining these examples because it is by no means clear that characteristic numbers of a surface bundle which is obtained by applying an (n,n')-construction on another surface bundle depend only on those of the latter. The details together with these points will appear elsewhere.

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以上は1983年7月現在のまとめである。その後の進展も含めた詳細な結果は現在論文を準備中。とくに定理3に関しては、準同型 ϕ は低い次数で injective であることが証明された。