数理解析研究所講究録
第 521 巻 1984 年 47-62

On a problem in combinatorics.

小樽商大*) 兼岩龍二 (Ryuji Kaneiwa)

1. Introduction.

PROBLEM. There are n boys and n girls. They are sitting at table, the boys looking at the girls, and the girls the boys. Each member of each group chooses one of the other group. A boy and a girl that choose each other, we call them a couple. i) What is probability of that the number of couples is $k(0 \leqslant k \leqslant n)$? ii) How behave the probability distribution of the random variable k, for increasing n?

This problem was proposed by Prof. Hisatsugu*) in statistics. More generally, we define the (fixed points) problem of $A_1 \times \cdots \times A_\lambda$ by the problem of counting the number

$$H_{A_1\times\cdots\times A_\lambda}(k) = \#\{(f_1,\ldots,f_\lambda) \in A_1\times\cdots\times A_\lambda;\ \varphi(f_\lambda\circ\cdots\circ f_1) = k\},$$

where

$$A_1,\ldots,A_\lambda \subset N^N = \{f;\ f \text{ is a map from } N \text{ into } N\},$$

$$\varphi(f) = \#\{x \in N;\ f(x) = x\} \qquad (\text{for } f \in N^N)$$

and $N = \{1,\ldots,n\}$. Let $H_\lambda(n,k) = H_{(N^N)^\lambda}(k)$. It is plain that

$$H_\lambda(n,n) = (n!)^{\lambda-1}, \qquad \sum_{k=0}^{n} H_\lambda(n,k) = n^{\lambda n}.$$

Our probability asked at i) is $H_2(n,k)/n^{2n}$. We have explicit formula

$$H_\lambda(n,k) = \frac{n^{\lambda n}}{k!}\sum_{j=0}^{n-k}\frac{(-1)^j}{j!}\left\{\frac{(n)_{j+k}}{n^{j+k}}\right\}^\lambda, \tag{1.1}$$

*) Otaru University of Commerce, Otaru, Hokkaido o47 Japan.

where $(x)_k = x(x-1)\ldots(x-k+1)$, $(x)_0 = x^0 = 1$. (For some values of $H_\lambda(n,k)$, see the table in the appendix.) Especially in the case of $\lambda = 1$, we have

$$H_1(n,k) = \binom{n}{k}(n-1)^{n-k}.$$

We consider only the problem of $(N^N)^\lambda$ herein, however, we would think that is not vacant to compare with the problem of $(N!)^\lambda$, where $N!$ is symmetric group on N. We get following

TABLE I.

problem of	$(N^N)^\lambda$	$(N!)^\lambda$ *)	Poisson d. with pm. 1	def.
number	$H_\lambda(n,k)$	$(n!)^{\lambda-1}D(n,k)$		
probability	$\frac{H_\lambda(n,k)}{n^{\lambda n}} = \frac{1}{k!}\sum_{j=0}^{n-k}\frac{(-1)^j}{j!}\left\{\frac{(n)_{j+k}}{n^{j+k}}\right\}^\lambda$	$\frac{D(n,k)}{n!} = \frac{1}{k!}\sum_{j=0}^{n-k}\frac{(-1)^j}{j!}$	$\frac{1}{e\cdot k!}$	$p(k)$
j-th moment	$\sum_{m=0}^{j} S(j,m)\left\{\frac{(n)_m}{n^m}\right\}^\lambda$	$\sum_{m=0}^{n} S(j,m)$ **)	$\hat{B}_j$ †	$\sum k^j p$
characteristic function	$\sum_{m=0}^{n}\frac{(e^{it}-1)^m}{m!}\left\{\frac{(n)_m}{n^m}\right\}^\lambda$	$\sum_{m=0}^{n}\frac{(e^{it}-1)^m}{m!}$	$e^{e^{it}-1}$	$\sum e^{ikt}p$

*) The problem of $(N!)^\lambda$ is reduced to the probrem of $N!$, that is well known as 'problème des rencontres' (see [1]).

**) $S(j,m)$ is a Stirling number of the second kind, that is defined by $x^j = \sum_{m=0}^{j} S(j,m)(x)_m = \sum_{m=0}^{\infty} S(j,m)(x)_m$.

† $\hat{B}_j$ is the j-th Bell number, that is defined by $\hat{B}_j = \sum_{m=0}^{j} S(j,m)$.

Our purpose is to get an asymptotic expression of $H_\lambda(n,k)$ for increasing n. We firstly obtain an infinite sum form

(1.2) $$H_\lambda(n,k) = \frac{n^{\lambda n}}{e\,k!}\sum_{m=0}^{\infty} c_m(\lambda,n)\, d_m(k),$$

where

$$c_m(\lambda,n) = \frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\left\{\frac{(n)_j}{n^j}\right\}^\lambda,$$

$$d_m(k) = \sum_{t=0}^{m}(-1)^{m-t}\binom{m}{t}(k)_t.$$

When we set

(1.3) $$c_m(\lambda,n) = \sum_{\ell=0}^{\lambda m}\alpha_\lambda(\ell,m)\, n^{-\ell},$$

we have

(1.4) $\alpha_\lambda(0,0) = 1$,

(1.5) $$c_m(\lambda,n) = \sum_{\ell=[\frac{m+1}{2}]}^{\lambda(m-1)}\alpha_\lambda(\ell,m)\, n^{-\ell} \qquad (m>0)$$

(for some values of $\alpha_\lambda(\ell,m)$, see the appendix). Observing (1.2) and (1.5), we may conjecture that

(1.6) $$\frac{H_\lambda(n,k)}{n^{\lambda n}} = \frac{1}{e\cdot k!}\sum_{\ell=0}^{L-1} n^{-\ell}\sum_{\frac{\ell}{\lambda}\leq m\leq 2\ell}\alpha_\lambda(\ell,m)\, d_m(k) + O_{\lambda,L}(n^{-L}).$$

We get following

THEOREM. For $\lambda = 1,2$, (1.6) is correct.

COROLLARY. If $\lambda = 1$ or 2, $H_\lambda(n,k)/n^{\lambda n}$ converges to $1/e\cdot k!$, uniformly for $k = 0,1,2,\ldots$, as n tends to infinity.

EXAMPLE. By the theorem, we have the asymptotic expansion

$$H_2(n,0) \approx \frac{n^{2n}}{e}\left(1-\frac{1}{n}-\frac{2}{3n^2}-\frac{1}{2n^3}+\cdots\right).$$

If $n = 10$, we have

$$H_2(10,0) = 32845\ 29863\ 68548\ 28800,$$

$$\frac{n^{2n}}{e}\left(1-\frac{1}{n}-\frac{2}{3n^2}-\frac{1}{2n^3}\right)\Big|_{n=10} = 0.32845\ 50277\times 10^{20}.$$

2. Proof of (1.1).

$$\sum_{k=j}^{n}\binom{k}{j}H_\lambda(n,k) = \sum_{k=j}^{n}\binom{k}{j}\sum_{\substack{(f_1,\cdots,f_\lambda)\in(N^N)^\lambda \\ \varphi(f_\lambda\circ\cdots\circ f_1)=k}} 1$$

$$= \sum_{k=j}^{n}\ \sum_{\substack{(f_1,\cdots,f_\lambda)\in(N^N)^\lambda \\ \varphi(f_\lambda\circ\cdots\circ f_1)=k}}\ \sum_{\substack{A_1\subset\{x\in N;\ f_\lambda\circ\cdots\circ f_1(x)=x\} \\ \# A_1=j}} 1$$

$$= \sum_{\substack{A_1\subset N \\ \# A_1=j}}\ \sum_{\substack{(f_1,\cdots,f_\lambda)\in(N^N)^\lambda \\ \{x\in N;\ f_\lambda\circ\cdots\circ f_1(x)=x\}\supset A_1}} 1$$

$$= \sum_{\substack{A_1\subset N \\ \# A_1=j}}\cdots\sum_{\substack{A_\lambda\subset N \\ \# A_\lambda=j}}\ \sum_{\substack{(f_1,\cdots,f_\lambda)\in(N^N)^\lambda \\ f_i(A_i)=A_{i+1}\ (i=1,\cdots,\lambda-1) \\ (f_\lambda|A_\lambda)\circ\cdots\circ(f_1|A_1)=I_{A_1}}} 1$$

$$= \left\{\binom{n}{j}\right\}^\lambda (j!)^{\lambda-1}(n^{n-j})^\lambda$$

$$(1.2) \qquad = \frac{n^{\lambda n}}{j!}\left\{\frac{(n)_j}{n^j}\right\}^\lambda,$$

where $f_i|A_i$ is a restriction of f_i to A_i and I_{A_1} is the identity map on A_1. Thus we have

$$\begin{pmatrix} \binom{0}{0} & \cdots & \binom{n}{0} \\ & \ddots & \vdots \\ O & & \binom{n}{n} \end{pmatrix}\begin{pmatrix} H_\lambda(n,0) \\ \vdots \\ H_\lambda(n,n) \end{pmatrix} = \begin{pmatrix} \frac{n^{\lambda n}}{0!}\left\{\frac{(n)_0}{n^0}\right\}^\lambda \\ \vdots \\ \frac{n^{\lambda n}}{n!}\left\{\frac{(n)_n}{n^n}\right\}^\lambda \end{pmatrix}.$$

Since

$$\begin{pmatrix} \binom{0}{0} & \cdots & \binom{n}{0} \\ & \ddots & \vdots \\ O & & \binom{n}{n} \end{pmatrix}^{-1} = \begin{pmatrix} \binom{0}{0} & -\binom{1}{0} & \binom{2}{1} & \cdots & (-1)^{n-0}\binom{n}{0} \\ & \binom{1}{1} & -\binom{2}{1} & & \vdots \\ & & \binom{2}{2} & \ddots & \vdots \\ O & & & & (-1)^{n-n}\binom{n}{n} \end{pmatrix},$$

we have

$$H_\lambda(n,k) = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} \frac{n^{\lambda n}}{j!} \left\{ \frac{(n)_j}{n^j} \right\}^\lambda$$

$$= \frac{n^{\lambda n}}{k!} \sum_{j=k}^{n} \frac{(-1)^{j-k}}{(j-k)!} \left\{ \frac{(n)_j}{n^j} \right\}^\lambda = \frac{n^{\lambda n}}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \left\{ \frac{(n)_{j+k}}{n^{j+k}} \right\}^\lambda .$$

q.e.d.

3. Moments.

By (2.1), we have

$$\sum_{k=j}^{n} (k)_j \frac{H_\lambda(n,k)}{n^{\lambda n}} = \left\{ \frac{(n)_j}{n^j} \right\}^\lambda \qquad (0 \leqslant j \leqslant n).$$

Since $(k)_j = 0$ $(k < j)$, the right hand side may be replaced by $\sum_{k=0}^{n}$ instead of $\sum_{k=j}^{n}$. Let it be so. Then the both sides vanish if $j > n$. Hence we have, for all non-negative integer j,

$$\sum_{k=0}^{n} (k)_j \frac{H_\lambda(n,k)}{n^{\lambda n}} = \left\{ \frac{(n)_j}{n^j} \right\}^\lambda$$

Thus, for $j = 0,1,2,\ldots$, we have the j-th moment

(3.1) $$\sum_{k=0}^{n} k^j \frac{H_\lambda(n,k)}{n^{\lambda n}} = \sum_{k=0}^{n} \frac{H_\lambda(n,k)}{n^{\lambda n}} \sum_{m=0}^{j} S(j,m)(k)_m$$

$$= \sum_{m=0}^{j} S(j,m) \sum_{k=0}^{n} (k)_m \frac{H_\lambda(n,k)}{n^{\lambda n}} = \sum_{m=0}^{j} S(j,m) \left\{ \frac{(n)_m}{n^m} \right\}^\lambda .$$

4. Characteristic function.

We consider the function $\varphi_{\lambda,n}(t)$ that is called characteristic function of the probability distribution

$$\Phi_{\lambda,n}(I) = \sum_{k\in I} \frac{H_\lambda(n,k)}{n^{\lambda n}} \qquad (I \subset N).$$

We have, by (2.1),

$$\varphi_{\lambda,n}(t) \overset{\text{def.}}{=} \sum_{k=0}^{n} e^{ikt} \frac{H_\lambda(n,k)}{n^{\lambda n}} = \sum_{k=0}^{n} \frac{H_\lambda(n,k)}{n^{\lambda n}} \sum_{m=0}^{k} \binom{k}{m} (e^{it}-1)^m \tag{4.1}$$

$$= \sum_{m=0}^{n} (e^{it}-1)^m \sum_{k=m}^{n} \binom{k}{m} \frac{H_\lambda(n,k)}{n^{\lambda n}} = \sum_{m=0}^{n} \frac{(e^{it}-1)^m}{m!} \left\{ \frac{(n)_m}{n^m} \right\}^\lambda .$$

(3.1) could be deduced also from (4.1), by using the formula

$$\frac{(e^x-1)^m}{m!} = \sum_{j=m}^{\infty} S(j,m) \frac{x^j}{j!} \qquad \text{(see [1])}.$$

Since $\varphi_{\lambda,n}(t)/\exp(e^{it}-1)$ is an entire function of $e^{it}-1$, we may put

$$e^{-z} \varphi_{\lambda,n}(t) = \sum_{n=0}^{\infty} c_m(\lambda,n)\, z^m \qquad (z = e^{it}-1). \tag{4.2}$$

Then, by (4.1), we have

$$c_m(\lambda,n) = \sum_{j=0}^{\min(m,n)} \frac{1}{j!} \left\{ \frac{(n)_j}{n^j} \right\}^\lambda (-1)^{m-j} \frac{1}{(m-j)!}$$

$$= \frac{(-1)^m}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left\{ \frac{(n)_j}{n^j} \right\}^\lambda , \tag{4.3}$$

$$|c_m(\lambda,n)| \leqslant \frac{2^m}{m!} . \tag{4.4}$$

5. Proof of (1.2).

In this section, we shall show (1.2) with error term, that is the following

LEMMA 1.

$$\frac{H_\lambda(n,k)}{n^{\lambda n}} = \frac{1}{e\,k!}\sum_{m=0}^{M-1} c_m(\lambda,n)\, d_m(k) + E_M \quad \left(|E_M| \leqslant \frac{e^4 4^M}{M!} \right),$$

where

$$(4.5) \qquad d_m(k) = \sum_{t=0}^{m} (-1)^{m-t}\binom{m}{t}(k)_t .$$

COROLLARY. (1.2) is valid.

PROOF. By Cauchy's theorem with (4.1), (4.2),

$$\frac{H_\lambda(n,k)}{n^{\lambda n}} = \operatorname*{Res}_{z=-1} \frac{\varphi_{\lambda,n}(t)}{(z+1)^{k+1}} = \sum_{m=0}^{\infty} c_m(\lambda,n) \operatorname*{Res}_{z=-1} \frac{e^z z^m}{(z+1)^{k+1}}$$

$$= \frac{1}{e}\sum_{m=0}^{M-1} c_m(\lambda,n) \operatorname*{Res}_{z=0} \frac{e^z (z-1)^m}{z^{k+1}} + \frac{1}{2\pi i}\int_{|z|=\rho} \frac{e^{z-1}}{z^{k+1}} \sum_{m=M}^{\infty} c_m(\lambda,n)(z-1)^m\, dz .$$

We have

$$\operatorname*{Res}_{z=0} \frac{e^z (z-1)^m}{z^{k+1}} = \frac{d_m(k)}{k!},$$

where $d_m(k)$ is defined by (4.5). On the other hand, by Taylor's theorem with (4.4),

$$\left| \sum_{m=M}^{\infty} c_m(n) x^m \right| \leqslant \sum_{m=M}^{\infty} \frac{2^m}{m!}|x|^m \leqslant \frac{|2x|^M}{M!} e^{2|x|} .$$

Therefore,

$$|E_M| \leqslant \frac{1}{2\pi e}\int_{|z|=\rho} \frac{e^{\operatorname{Re} z}}{\rho^{k+1}}\, e^{2|z-1|}\, \frac{(2|z-1|)^M}{M!}\, |dz|$$

$$\leqslant \frac{2^M}{M!}\rho^{-k} e^{3\rho+1} (\rho+1)^M .$$

If we take $\rho = 1$, we have $|E_M| \leqslant e^4 \cdot 4^M / M!$. q.e.d.

6. On $s_\lambda(j,\ell)$.

Let

$$(6.1)\qquad \{(x)_j\}^\lambda = \sum_{\ell=0}^{\lambda j} s_\lambda(j,\ell)\,x^\ell = \sum_{\ell=-\infty}^{\infty} s_\lambda(j,\ell)\,x^\ell$$

Especially, $s(j,\ell) = s_1(j,\ell)$ is a Stirling number of the first kind. By the definition (6.1) of $s_\lambda(j,\ell)$, we have recurence formulae

$$s_\lambda(j+1,\ell) = \sum_{g=0}^{\lambda} (-1)^{\lambda-g}\binom{\lambda}{g} j^{\lambda-g} s_\lambda(j,\ell-g) \qquad (j\geqslant 0),$$

$$(6.2)\qquad s_\lambda(j+1,\lambda(j+1)-\ell) = \sum_{g=0}^{\lambda} (-1)^{g}\binom{\lambda}{g} j^{g} s_\lambda(j,\lambda j-(\ell-g)) \qquad (j\geqslant 0).$$

The latter implies that $s_\lambda(j,\lambda j-\ell)$ is a polinomial of degree 2ℓ in $\mathbb{Q}[j]$, for $\ell \geqslant 0$. So we may define $\alpha_\lambda(\ell,m)$ by

$$(6.3)\qquad s_\lambda(j,\lambda j-\ell) = \sum_{m=0}^{2\ell} \alpha_\lambda(\ell,m)\,(j)_m$$

We define $\alpha_\lambda(\ell,m)=0$ outside $\ell\geqslant 0$, $0\leqslant m\leqslant 2\ell$. (6.2) leads us to

$$(6.4)\qquad (m+1)\alpha_\lambda(\ell,m+1) = \sum_{g=1}^{\lambda}(-1)^g\binom{\lambda}{g}\sum_{u=0}^{g}\alpha_\lambda(\ell-g,m-u)\,\eta_{g,u}(m-u),$$

where

$$\eta_{g,u}(m) = \sum_{t=u}^{g}\binom{g}{t} m^{g-t} S(t,u) \in \mathbb{Z}[m].$$

We now show the following

LEMMA 2. i) $\alpha_\lambda(0,0)=1$. ii) If $\ell>0$, then $\alpha_\lambda(\ell,0)=0$. iii) For $m>0$, if $\ell>\lambda(m-1)$ then $\alpha_\lambda(\ell,m)=0$.

PROOF. i) It is clear from (6.1) and (6.3). ii) If $\ell>0$, by the definition (6.1) of $s_\lambda(j,\ell)$, we have $\alpha_\lambda(0,-\ell)=0$. When we substitute $j=0$ in the both hand sides of (6.3), we obtain $\alpha_\lambda(\ell,0)=0$. iii) Induction on m. If $m=1$ and $\ell>0$, we have, by (6.4),

$$\alpha_\lambda(\ell,1) = \sum_{q=1}^{\lambda}(-1)^q\binom{\lambda}{q}\alpha_\lambda(\ell-q,0)\,\eta_{q,0}(0).$$

Since $\eta_{q,0}(0) = S(q,0) = 0$ $(q > 0)$, we have $\alpha_\lambda(\ell,1) = 0$. Suppose $\alpha_\lambda(\ell,\mu) = 0$, for $\ell > \lambda(\mu-1)$ and $\mu = 1,\ldots,m$. Let $\ell > \lambda m$. If $1 \leqslant q \leqslant \lambda$ and $0 \leqslant u \leqslant q$, we have

$$\ell - q > \lambda m - q \geqslant \lambda(m-u) - q \geqslant \lambda(m-u) - \lambda$$

and $\alpha_\lambda(\ell-q,m-u) = 0$. By (6.4) we get $\alpha_\lambda(\ell,m+1) = 0$. q.e.d.

COROLLARY. If $\ell > 0$, then

$$\text{(6.5)}\qquad s_\lambda(j,\lambda j-\ell) = \sum_{\frac{\ell}{\lambda}+1\leqslant m\leqslant 2\ell}\alpha_\lambda(\ell,m)\,(j)_m\,.$$

7. On $c_m(\lambda,n)$.

We have

$$\left\{\frac{(n)_j}{n^j}\right\}^\lambda = \frac{1}{n^{\lambda j}}\sum_{j=0}^{m}s_\lambda(j,\ell)\,n^\ell = \sum_{\ell=0}^{\lambda j}s_\lambda(j,\lambda j-\ell)\,n^{-\ell}.$$

By (4.3),

$$c_m(\lambda,n) = \frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\sum_{\ell=0}^{\lambda j}s_\lambda(j,\lambda j-\ell)\,n^{-\ell}$$

$$= \frac{(-1)^m}{m!}\sum_{\ell=0}^{\lambda m}n^{-\ell}\sum_{\frac{\ell}{\lambda}\leqslant j\leqslant m}(-1)^j\binom{m}{j}s_\lambda(j,\lambda j-\ell).$$

If $\ell = 0$, we have

$$\sum_{\frac{\ell}{\lambda}\leqslant j\leqslant m}(-1)^j\binom{m}{j}s_\lambda(j,\lambda j-\ell) = \sum_{j=0}^{m}(-1)^j\binom{m}{j} = \delta_{m,0}\,.$$

If $\ell > 0$,

$$\sum_{\frac{\ell}{\lambda}\leqslant j\leqslant m}(-1)^j\binom{m}{j}s_\lambda(j,\lambda j-\ell) = \sum_{\frac{\ell}{\lambda}\leqslant j\leqslant m}(-1)^j\binom{m}{j}\sum_{\frac{\ell}{\lambda}+1\leqslant t\leqslant 2\ell}\alpha_\lambda(\ell,t)\,(j)_t$$

$$= \sum_{\frac{\ell}{\lambda}+1\leqslant t\leqslant 2\ell}\alpha_\lambda(\ell,t)\sum_{\frac{\ell}{\lambda}\leqslant j\leqslant m}(-1)^j\binom{m}{j}(j)_t\,,$$

since $(j)_t = 0$ (if $m < t$, $j \leq m$),

$$= \sum_{\substack{\frac{\ell}{\lambda}+1 \leq t \leq 2\ell \\ t \leq m}} \alpha_\lambda(\ell,t) \sum_{\frac{\ell}{\lambda} \leq j \leq m} (-1)^j \binom{m}{j} (j)_t ,$$

here the parameter t runs in the range $t > \ell/\lambda$, so that,

$$= \sum_{\substack{\frac{\ell}{\lambda}+1 \leq t \leq 2\ell \\ t \leq m}} \alpha_\lambda(\ell,t)\, t! \sum_{j=t}^{m} (-1)^j \binom{m}{j}\binom{j}{t}$$

$$= \sum_{\substack{\frac{\ell}{\lambda}+1 \leq t \leq 2\ell \\ t \leq m}} \alpha_\lambda(\ell,t)\, t!\, (-1)^t\, \delta_{m,t}$$

$$= (-1)^m\, m!\, \alpha_\lambda(\ell,m).$$

We have

$$c_m(\lambda,n) = \sum_{\ell=0}^{\lambda m} n^{-\ell} \alpha_\lambda(\ell,m).$$

Thus (1.3)-(1.5) are consistent with (6.1)-(6.5).

8. Inequalities.

We provide several inequalities for proving the theorem.

LEMMA 3.

(8.1) $$|d_m(k)| \leq 2^m k! ,$$

(8.2) $$|c_m(1,n)| \leq n^{-[\frac{m+1}{2}]} ,$$

(8.3) $$|c_m(2,n)| \leq n^{-[\frac{m+1}{2}]} (m+1)! .$$

PROOF. (8.1) is clear from (4.5). To prove (8.2) and (8.3), we introduce functions

$$f_{\lambda,m}(x) = \sum_{\ell=0}^{\lambda m} \alpha_\lambda(\ell,m)\, x^\ell$$

of x. Then we have $c_m(\lambda,n) = f_{\lambda,m}(1/n)$. Since

$$|\alpha_\lambda(\ell,m)| = (-1)^\ell \alpha_\lambda(\ell,m) \qquad \text{(see (6.4))},$$

if $m > 0$, we have

$$|c_m(\lambda,n)| \leqslant \sum_{\ell=[\frac{m+1}{2}]}^{\lambda(m-1)} |\alpha_\lambda(\ell,m)|\, n^{-\ell} \leqslant n^{-[\frac{m+1}{2}]} f_{\lambda,m}(-1).$$

Let $\beta_{\lambda,m} = f_{\lambda,m}(-1)$. By (6.4), we have

$$(m+1) f_{\lambda,m+1}(x) = \sum_{q=1}^{\lambda} (-x)^q \binom{\lambda}{q} \sum_{u=0}^{q} \eta_{q,u}(m-u) f_{\lambda,m-u}(x),$$

(8.4) $$m\,\beta_{1,m} = \beta_{1,m-2} + (m-1)\,\beta_{1,m-1}\,,$$

(8.5) $$m\,\beta_{2,m} = \beta_{2,m-3} + (2m-1)\beta_{2,m-2} + (m^2-1)\beta_{2,m-1}\,.$$

It is enough to show $\beta_{1,m} \leqslant 1$ and $\beta_{2,m} \leqslant (m+1)!$. It follows from (8.4) that

$$\beta_{1,m} = \sum_{j=0}^{m} \frac{(-1)^m}{m!} \leqslant 1\,.$$

We now prove $\beta_{2,m} \leqslant (m+1)!$ by induction. $\beta_{2,0} = 1 \leqslant 1!$, $\beta_{2,1} = 0 \leqslant 2!$, $\beta_{2,2} = 3/2 \leqslant 3!$. Suppose $m \geqslant 3$ and $\beta_{2,\mu} \leqslant (\mu+1)!$ for $\mu < m$. Thus, by (8.5), we have

$$\begin{aligned}
\beta_{2,m} &\leqslant \frac{1}{m}\left\{ (m-2)! + (2m-1)(m-1)! + (m-1)(m+1)! \right\} \\
&= (m+1)! \left\{ \frac{1}{(m+1)m^2(m-1)} + \frac{2-\frac{1}{m}}{(m+1)m} + \frac{m-1}{m} \right\} \\
&\leqslant (m+1)! \left\{ \frac{1}{4\cdot 3\cdot 2\,m} + \frac{2}{m^2} + 1 - \frac{1}{m} \right\} \\
&= (m+1)! \left\{ 1 - \frac{1}{m}\left(\frac{23}{24} - \frac{2}{m}\right)\right\} \leqslant (m+1)!.
\end{aligned}$$

q.e.d.

9. Proof of the theorem.

We devide the series (1.2) into three parts as

$$\frac{H_\lambda(n,k)}{n^{\lambda n}} = \frac{1}{e\,k!}\sum_{m=0}^{2\lambda L} c_m(\lambda,n)\,d_m(k) + \frac{1}{e\,k!}\sum_{m=2\lambda L+1}^{[x]} c_m(\lambda,n)\,d_m(k) + E_{[x]+1}.$$

By Lemma 1, we have

$$(9.1)\qquad |E_{[x]+1}| \leqslant e^4\,\frac{4^{x+1}}{\Gamma(x+1)}.$$

By (1.4),(1.5) and (8.1), we obtain

$$(9.2)\qquad \frac{1}{e\,k!}\sum_{m=0}^{2\lambda L} c_m(\lambda,n)\,d_m(k) = \frac{1}{e\,k!}\sum_{\ell=0}^{L-1} n^{-\ell}\sum_{\frac{\ell}{\lambda}\leqslant m\leqslant 2\ell} \alpha_\lambda(\ell,m)\,d_m(k) + O_{\lambda,L}(n^{-L}).$$

By Lemma 3, for $\lambda = 1,2$, we have

$$(9.3)\qquad \frac{1}{e\,k!}\left|\sum_{m=2\lambda L+1}^{[x]} c_m(\lambda,n)\,d_m(k)\right| \leqslant e^{-1}\,n^{-(\lambda L+1)}\sum_{m=0}^{[x]} 2^m A_\lambda(m),$$

where $A_\lambda(m) = 1$ if $\lambda = 1$, $= (m+1)!$ if $\lambda = 2$.

Suppose $\lambda = 1$. Then

$$(9.4)\qquad e^{-1}\,n^{-(L+1)}\sum_{m=0}^{[x]} 2^m \leqslant e^{-1}\,2^{x+1}\,n^{-(L+1)}.$$

For a sufficiently large n, there exists $x = x(n) > 0$ such that

$$(9.5)\qquad n^{-\mu} = \frac{2^x}{\Gamma(x+1)},$$

where $\mu = L + 1$. If we set $x = x(n)$ as (9.5), by Stirling's formula,

$$n^{\mu} = 2^{-x}\,\Gamma(x+1) = 2^{-x}\sqrt{2\pi}\,x^{x+\frac12}e^{-x}(1+o(1)) \qquad (n\to\infty),$$

$$\mu\log n = \tfrac12\log 2\pi + (x+\tfrac12)\log x - x(1+\log 2) + o(1),$$

$$\mu\log n \sim x\log x,$$

$$(9.6)\qquad \log\mu + \log\log n = \log x + \log\log x + o(1).$$

Hence we get

$$(9.7)\qquad \log\log n \sim \log x.$$

Let $x = y(n)\log n$. Then by (9.6),

$$\log y = \log x - \log\log n = \log\mu - \log\log x + o(1),$$

$$y \sim \frac{\mu}{\log x}.$$

By (9.7), we have

$$x \sim \mu\frac{\log n}{\log\log n}. \tag{9.8}$$

(9.3)-(9.5) and (9.8) lead us to

$$\frac{1}{ek!}\sum_{m=2L+1}^{[x]} c_m(1,n)\,d_m(k) = O(n^{-\mu}2^{x})$$

$$= O(n^{-\mu}2^{\mu\frac{\log n}{\log\log n}(1+o(1))}) = O(n^{-\mu+o(1)}) = O_L(n^{-L}).$$

Similarly by (9.1),(9.5),(9.8), we have

$$E_{[x]+1} = O(n^{-\mu}2^{x}) = O_L(n^{-L}).$$

In the case of $\lambda = 1$, we get the theorem.

Suppose $\lambda = 2$. We have

$$\sum_{m=0}^{[x]} 2^m (m+1)! \leqslant 2^{[x]}[x+1]!\sum_{m=0}^{[x]} 2^{-m} \leqslant 2^{x+1}\Gamma(x+2).$$

Let set

$$n^{-\mu} = \frac{2^x}{(x+1)\Gamma^2(x+1)} \qquad (\mu = 2L+1),$$

for large n. On a similar way, we shall get

$$x \sim \frac{\mu}{2}\frac{\log n}{\log\log n}.$$

Thus we obtain

$$\frac{1}{e\,k!}\sum_{m=4L+1}^{[x]} c_m(2,n)\,d_m(k) = O(n^{-\mu}2^{x}\Gamma(x+2))$$

$$= O(n^{-\frac{\mu}{2}}2^{\frac{3}{2}x}\sqrt{x}) = O_\mu(n^{-\mu/2+o(1)}\sqrt{\frac{\log n}{\log\log n}}) = O_L(n^{-L}),$$

$$E_{[x]+1} = O(\frac{4^x}{\Gamma(x+1)}) = O(n^{-\mu/2}2^{\frac{3}{2}x}\sqrt{x}) = O_L(n^{-L}).$$

q.e.d.

Reference

[1] Riordan, An introduction to combinatorial analysis, Wiley, 1958.

[2] R.Kaneiwa, On the multiplicative partition function, Tsukuba J. of Math. vol. 7, No. 2 , 1983 , p.355-365.

APPENDIX

Table of $H_\lambda(n,k)$.

$H_1(n,k)$

n \ k	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	1	2	1						
3	8	12	6	1					
4	81	108	54	12	1				
5	1024	1280	640	160	20	1			
6	15625	18750	9375	2500	375	30	1		
7	279936	326592	163296	45360	7560	756	42	1	
8	5764801	6588344	3294172	941192	168070	19208	1372	56	1

$H_2(n,k)$

n \ k	0	1	2	3	4	5	6
0	1						
1	0	1					
2	2	12	2				
3	156	423	144	6			
4	16920	33185	13968	1440	24		
5	2764880	4581225	2088800	316200	14400	120	
6	650696400	973830816	460350000	85521600	6231600	151200	720

$H_3(n,k)$

n \ k	0	1	2	3	4	5
0	1					
1	0	1				
2	4	56	4			
3	2880	13959	2808	36		
4	3392064	10139392	3100032	145152	576	
5	7258985600	16544150125	6178856000	526644000	8928000	14400

Table of $\alpha_\lambda(\ell, m)$.

$\alpha_1(\ell, m)$

$\ell \backslash m$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	0	$-\frac{1}{2}$								
2	0	0	0	$\frac{1}{3}$	$\frac{1}{8}$						
3	0	0	0	0	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{1}{48}$				
4	0	0	0	0	0	$\frac{1}{5}$	$\frac{13}{72}$	$\frac{1}{24}$	$\frac{1}{384}$		
5	0	0	0	0	0	0	$-\frac{1}{6}$	$-\frac{11}{60}$	$-\frac{17}{288}$	$-\frac{1}{144}$	$-\frac{1}{3840}$

$\alpha_2(\ell, m)$

$\ell \backslash m$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	0	-1								
2	0	0	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{1}{2}$						
3	0	0	0	-2	-4	$-\frac{5}{3}$	$-\frac{1}{6}$				
4	0	0	0	$\frac{2}{3}$	$\frac{59}{8}$	$\frac{337}{30}$	$\frac{185}{36}$	$\frac{5}{6}$	$\frac{1}{24}$		
5	0	0	0	0	$-\frac{11}{2}$	$-\frac{83}{3}$	$-\frac{827}{24}$	$-\frac{241}{15}$	$-\frac{29}{9}$	$-\frac{5}{18}$	$-\frac{1}{120}$