数理解析研究所講究録
第 533 巻 1984 年 24

On the work of J. ECALLE

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I. Let D be a small disc $\{|x|<\varepsilon\}$ in $\mathbb{C}$; write $D^* = D - \{0\}$, $\tilde{D}^*$ = the universal covering of D^* with some fixed base-point $a \in D^*$.

Put $\tilde{\mathcal{O}} = \mathcal{O}(\tilde{D}^*)$, the space of holomorphic functions on $\tilde{D}^*$, and $\tilde{\mathcal{C}} = \tilde{\mathcal{O}}/\mathcal{O}(D)$; one has two well-known morphisms $\tilde{\mathcal{O}} \underset{\text{var}}{\overset{\text{can}}{\rightleftarrows}} \tilde{\mathcal{C}}$, where "can" is the quotient map, and "var" is defined by $\text{var} \circ \text{can} = T - \text{id}$; here T is the action of the monodromy on $\tilde{\mathcal{O}}$. The space $\tilde{\mathcal{C}}$ can be considered as a space of microfunctions at $0 \in \mathbb{R}$; on $\tilde{\mathcal{C}}_-$, the convolution product $(f,g) \mapsto f * g$ is well defined, with all the usual properties.

Now, let Ω be a discrete subgroup of $\mathbb{C}$, for instance $\Omega = \mathbb{Z}$; we suppose $D \cap \Omega = \{0\}$

Definition 1 We denote by $\mathcal{C}(\Omega)$ the set of the $f \in \tilde{\mathcal{C}}$ such that $\text{var} f$ has an analytic continuation to the whole space $\widetilde{\mathbb{C}-\Omega}$; here, $\widetilde{\mathbb{C}-\Omega}$ denotes the universal covering of $\mathbb{C}-\Omega$ with the same base-point $a \in D^*$ as before

Theorem 2 (Ecalle). $\mathcal{C}(\Omega)$ is a convolution subalgebra of $\tilde{\mathcal{C}}$

This convolution algebra is the basic object of Ecalle's theory. A important result is the description of the singularities of $f * g$ ($f, g \in \mathcal{C}(\Omega)$) in terms of the singularities of f and g; this is done with the introduction of "alien derivations"; for more precise statements, see (E 1).

II. Let G be the group of germs of analytic automorphisms $(\mathbb{C},0)\circlearrowleft$, and let H be the subgroup of germs tangent to identity. One of the works of Ecalle is the classification of conjugacy classes in H; two methods are given to describe that classification; for the first one, I refer to [E2] or [M]; I will describe briefly the second one.

For $h \in \mathcal{H}_1$, write $h(z) = z + a_2 z^2 + \cdots$; using the fact that h can be formally embedded in a one-parameter group, it is easy to prove that the formal conjugacy class of h is determined by two invariants

i) The pair (n, a_n), where $n = \inf\{m \mid a_m \neq 0\}$

ii) The coefficient of $\frac{1}{z}$ in $\frac{1}{h(z)-z}$ (this depends only on $a_n, \ldots, a_{2n-1}$)

Then one has to find the analytic invariants corresponding to a given formal class. For simplicity, I will consider only the formal class defined by $a_2 = a_3 = 1$. Then, we have $h(z) = z + z^2 + z^3 + a_4 z^4 + \cdots$; write $h_0(z) = \frac{z}{1-z}$. After the change of variable $z = \frac{1}{\zeta}$, $g = \frac{1}{h}$, one has $g(\zeta) = \zeta - 1 + b_2 \zeta^{-2} + b_3 \zeta^{-3} + \cdots$, and $g_0(\zeta) = \zeta - 1$; one sees easily that there exist one and only one formal power series $\varphi(\zeta) = \zeta + c_1 \zeta^{-1} + c_2 \zeta^{-2} + \cdots$, which satisfies $\varphi \circ g = g_0 \circ \varphi$ ...

Let Φ be the Fourier-Borel transform of φ, i.e. the formal microfunction $\Phi = \delta' + \sum c_k \frac{x^{k-1}}{(k-1)!} Y$ (Y = the Heaviside function). One can prove that one has in fact $\Phi \in \tilde{\mathcal{E}}$; a much stronger result is the following

<u>Theorem</u> 3 (Ecalle, [E 2]) <u>If</u> Ω <u>denotes the set</u> $2\pi i \mathbb{Z}$, <u>one has</u>: $\Phi \in \mathcal{C}(\Omega)$

Ecalle proves more precise results ; consider for instance the analytic continuation of Φ at the half-plane $\mathrm{Re}\, x > 0$, and denote by Φ_n the microfunction that this continuation determines at the point $2\pi i n$; then one has, for some $c_n \in \mathbb{C}$ and f_n holomorphic near $2\pi i n$: $\Phi_n = c_n \delta(x - 2\pi i n) + f_n Y(x - 2\pi i n)$. Ecalle gives some functional equation for : Φ_n (and other singularities over $2\pi i n$ of Φ), in terms of alien derivations, and proves also that the c_n give a complete list of analytic invariants.

References

[E 1], [E 2] J. Ecalle, Théorie des fonctions résurgentes, ~~...~~ vol 1 and 2, Publications Mathématiques de l'Université d'Orsay, (1981-82)

[M] B. Malgrange, Travaux d'Ecalle et de Martinet-Ramis sur les systèmes dynamiques,