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Gauss-Manin system and the flat coordinate system
(Connection with the expansion of the solutions at ∞)

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In this note, we consider the differential system of Gauss-Manin associated with a versal deformation of a simple singularity. As to such a Gauss-Manin system, it is known by K.Saito [4] that the parameter space, say S , has a canonical linear structure. A flat coordinate system is a "linear" coordinate system with respect to the linear structure of S . The purpose of this note is to give an explicit construction of the flat coordinate system by means of the expansion of the solutions near a point at infinity.

§1. Gauss-Manin system and the flat coordinate system.

We begin with a review of the notion of a flat coordinate system. Let $f(x) = f(x_1,\ldots,x_n)$ be one of the following canonical forms of simple isolated singularities:

$$(1.1)\qquad \begin{cases} A_\ell : x_1^{\ell+1} \ (n=1), & D_\ell : x_1^{\ell-1} + x_1x_2^2 \ (n=2), \\ E_6 : x_1^4 + x_2^3 , \ E_7 : x_1^3 + x_1x_2^3 , & E_8 : x_1^5 + x_2^3 \ (n=2) . \end{cases}$$

Here, the subscript ℓ or 6,7,8 stands for the Milnor number $\mu = \mu(f)$ of f at $x=0$:

$$(1.2)\qquad \mu = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / (\partial_x f) , \quad (\partial_x f) = (\partial_{x_1} f,\ldots,\partial_{x_n} f) .$$

Note that f is weighted homogeneous of degree 1 with respect to a unique weight $\rho=(\rho_1,\dots,\rho_n)$, where ρ_i are positive rational numbers. In our case, the quotient ring $\mathcal{O}_{\mathbb{C}^n,0}/(\partial_x f)$ has a $\mathbb{C}$-basis consisting of monomials. To fix the ideas, we take a set $N \subset \mathbb{N}^n$ of multi-indices as follows, so that the residue classes of $x^\nu = x_1^{\nu_1}\dots x_n^{\nu_n}$ $(\nu=(\nu_1,\dots,\nu_n)\in N)$ form a basis of the ring $\mathcal{O}_{\mathbb{C}^n,0}/(\partial_x f)$.

$$(1.3)\begin{cases} \text{(I) Case of } A_\ell, E_6, E_8 \ (f = x_1^{p_1}+\dots+x_n^{p_n}): \\ \qquad N=\{\nu\in\mathbb{N}^n;\ 0\le \nu_i\le p_i-2 \ (1\le i\le n)\}. \\ \text{(II) Case of } D_\ell, E_7 \ (f=x_1^{p_1}+x_1x_2^{p_2}): \\ \qquad N=\{\nu\in\mathbb{N}^2;\ 0\le\nu_1\le p_1-1,\ 0\le\nu_2\le p_2-2\}\cup\{(0,p_2-1)\}. \end{cases}$$

Let $S=\mathbb{C}^\mu$ be the complex affine μ-space with coordinate system $t=(t_\nu)_{\nu\in N}$, which we regard as the space of deformation parameters. We take a <u>versal</u> deformation $F=F(t,x)$ of $f=f(x)$ defined by

$$(1.4)\qquad F(t,x)=f(x)+\sum_{\nu\in N} t_\nu x^\nu .$$

The deformation F is versal in the sense that the residue classes of $\partial F/\partial t_\nu|_{t=0}$ $(\nu\in N)$ form a basis of $\mathcal{O}_{\mathbb{C}^n,0}/(\partial_x f)$. Note again that F is weighted homogeneous of degree 1 with respect to a unique weight (r,ρ) of (t,x), $r=(r_\nu)_{\nu\in N}$, and that all r_ν $(\nu\in N)$ are <u>positive</u> rational numbers as an effect of simpleness of the singularity f. We pay a special attention to the parameter t_0. Let $T=\mathbb{C}^{\mu-1}$ be the affine $(\mu-1)$-space with coordinate system $t^*=(t_\nu)_{\nu\in N^*}$, where $N^*=N\setminus\{0\}$, and $\pi: S\longrightarrow T$ the canonical projection. Then the projection π is determined by the vector field $D_0 = D_{t_0} = \partial/\partial t_0$ and the parameter t_0 gives the fiber coordinate of π.

The Gauss-Manin system associated with F, denoted by $\underline{H}_F$, is by definition the differential system on S to be satisfied by the integral

$$(1.5) \qquad u(t) = \int \delta(F(t,x))\, dx\,, \quad dx = dx_1 \wedge \ldots \wedge dx_n\,.$$

For each $\nu \in N$, define

$$(1.6) \qquad u_\nu = \int \frac{\partial F}{\partial t_\nu} \delta(F)\, dx = \int x^\nu \delta(F)\, dx \quad (u_0 = u)\,.$$

Then one can represent the Gauss-Manin system as a differential system on S including the column vector $\vec{u} = (u_\nu)_{\nu \in N}$ as the unknown.

<u>Proposition 1</u>. <u>The Gauss-Manin system</u> $\underline{H}_F$ <u>has a finite presentation</u>

$$(1.7) \qquad \begin{cases} D_{t_0} t_0 \vec{u} = (A_0(t^*) D_{t_0} + A_1(t^*))\, \vec{u}\,, \\ D_{t_\nu} \vec{u} = (B_0^\nu(t^*) D_{t_0} + B_1^\nu(t^*))\, \vec{u} \quad (\nu \in N^*)\,, \end{cases}$$

<u>where</u> $A_i,\ B_i^\nu \in M(\mu; \mathbb{C}[t^*])$ $(i = 0,1;\ \nu \in N^*)$.

Let us recall here the Gauss-Manin system $\underline{H}_{f+t_0}$. For each $\nu \in N$, define $w_\nu = \int x^\nu \delta(f + t_0)\, dx$ and consider the column vector $\vec{w} = (w_\nu)_{\nu \in N}$. Then the Gauss-Manin system $\underline{H}_{f+t_0}$ is given by

$$(1.8) \qquad D_{t_0} t_0 \vec{w} = -\Lambda \vec{w}\,, \quad -\Lambda = \mathrm{diag}(\varepsilon_\nu;\ \nu \in N)\,.$$

Note that the <u>exponents</u> ε_ν $(\nu \in N)$ of f are determined by the formulas

$$(1.9) \qquad L_{\theta_x}(x^{\nu}dx) = \varepsilon_{\nu}x^{\nu}dx \quad (\nu \in N),$$

where L_{θ_x} is the Lie derivative with respect to the Euler vector field $\theta_x = \sum_{i=1}^{n} \rho_i x_i D_{x_i}$. Comparing the differential system (1.7) with (1.8), one sees that

$$(1.10) \qquad A_0|_{t^*=0} = 0 \quad \text{and} \quad A_1|_{t^*=0} = -\Lambda,$$

since $\vec{u}$ is a deformation of $\vec{w} = \vec{u}|_{t^*=0}$.

Now we recall the notion of a flat coordinate system of S. Let $s = (s_{\nu})_{\nu \in N}$ be a coordinate system of S relative to $\pi : S \longrightarrow T$. Namely, suppose that π is realized as the projection $s = (s_0, s^*) \longmapsto s^* = (s_{\nu})_{\nu \in N^*}$ and that $D_{s_0} = D_0$. For such a coordinate system $s = (s_{\nu})_{\nu \in N}$, we define

$$(1.11) \qquad v_{\nu} = \int \frac{\partial F}{\partial s_{\nu}} \delta(F)\, dx \quad \text{for } \nu \in N.$$

Then, as to the differential system for the column vector $\vec{v} = (v_{\nu})_{\nu \in N}$, we have

Theorem 2. <u>There exists a coordinate system</u> $s = (s_{\nu})_{\nu \in N}$ <u>of</u> S <u>relative to</u> $\pi : S \longrightarrow T$ <u>such that the Gauss-Manin system</u> H_F <u>is represented in the form</u>

$$(1.12) \qquad \begin{cases} D_{s_0} s_0 \vec{v} = (A(s^*)D_{s_0} - \Lambda)\vec{v}, \\ D_{s_{\nu}} \vec{v} = B^{\nu}(s^*) D_{s_0} \vec{v} \quad (\nu \in N^*), \end{cases}$$

<u>where</u> $A, B^{\nu} \in M(\mu; \mathbb{C}[s^*])$ $(\nu \in N^*)$. <u>Moreover, such a coordinate system</u> $s = (s_{\nu})_{\nu \in N}$ <u>is determined uniquely up to linear transformation</u>.

A coordinate system $s=(s_\nu)_{\nu\in N}$ relative to π is called a <u>flat coordinate system</u> if it has the property as stated in Theorem 2. At the same time, Theorem 2 says that the space S of parameters can be endowed with a linear structure. For an intrinsic formulation of the linear structure and the flat coordinate system, we refer the reader to K.Saito [4] or S.Ishiura-M.Noumi [2] .

§2. <u>Expansion of the solutions near a point at infinity</u>.

We keep the assumptions and the notations in §1. Here we explain how one can construct the flat coordinate system of S by means of the expansion of the solutions of $\underline{H}_F$ at infinity.

As to the differential system (1.7) of Proposition 1, we define the <u>discriminant</u> $\Delta=\Delta(t)$ by

$$\Delta(t)=\det(t_0-A_0(t^*)), \tag{2.1}$$

which is a monic polynomial of degree μ in t_0 with coefficients in $\mathbb{C}[t^*]$. Then the system (1.7) defines an integrable meromorphic connection over S with logarithmic poles along the discriminant set $D=\{\Delta=0\}$. Hence, one sees that the system (1.7) has a unique fundamental system $\Phi(t)$ of many-valued holomorphic solutions on $S\setminus D$ such that

$$\Phi\in GL(\mu;\mathcal{O}(\widetilde{S\setminus D})) \quad \text{and} \quad \Phi|_{t^*=0}=t_0^{-\Lambda-1}. \tag{2.2}$$

Now we take the compactification $\overline{S}=\mathbb{P}\times T$ of $S=\mathbb{C}\times T$ in the direction of t_0-axis. Then, it is easy to see that the discriminant set D is closed in $\overline{S}$ and does not intersect with the hyperplane

$\{t_0 = \infty\}$ at infinity. Moreover, the meromorphic connection in question has regular singularities along $\{t_0 = \infty\}$ in the naive sense. So one sees that $\Phi(t)$ can be factorized in the form

$$\Phi(t) = \Psi(t)\, t_0^{-\Lambda-1} \quad \text{near} \quad (t_0, t^*) = (\infty, 0)\,, \tag{2.3}$$

where $\Psi(t) \in GL(\mu; \mathcal{O}_{\bar{S},(\infty,0)})$ and $\Psi|_{t^*=0} = 1$. (It is known that this kind of expression gives rise to a hypergeometric representation of the solutions in the case where $f = x_1^{p_1} + \ldots + x_n^{p_n}$ and $F = f + t_1 x_1 + \ldots + t_n x_n + t_0$.) We formulate this expression (2.3) in connection with the flat coordinate system.

<u>Proposistion 3</u>. (a) <u>There is a unique formal differential operator</u>

$$P(t^*, D_{t_0}) = \sum_{k=0}^{\infty} P_k(t^*) D_{t_0}^k\,, \quad P_k \in M(\mu; \mathbb{C}[t^*])\,, \tag{2.4}$$

<u>such that</u>

$$\Phi(t) = P(t^*, D_{t_0})\, t_0^{-\Lambda-1} \quad \text{near} \quad (t_0, t^*) = (\infty, 0)\,. \tag{2.5}$$

(b) <u>As to the operator</u> P <u>in (a)</u>, <u>define</u>

$$s_\nu = \delta_{0,\nu} t_0 + P_1(t^*)_{0,\nu}\,, \quad \underline{\text{for}}\ \nu \in N\,. \tag{2.6}$$

<u>Then</u>, $s = (s_\nu)_{\nu \in N}$ <u>gives a flat coordinate system of</u> S.

Remark that the 0-th row of the matrix P of operators represents the expansion of μ independent solutions of $u = \int \delta(F)\, dx$.

Now we mention how the matrix P can be determined explicitly. The first step is to solve a system of difference equations for a function $c : \mathbb{N}^n \longrightarrow \mathbb{C}$. To describe the difference system, we introduce some notations. We denote by T_i $(1 \leq i \leq n)$ the translation by one in the i-th direction. Namely, for a function $c : \mathbb{N}^n \longrightarrow \mathbb{C}$, we define a function $T_i c : \mathbb{N}^n \longrightarrow \mathbb{C}$ by

$$(2.7) \qquad (T_i c)(\alpha) = c(\alpha + 1_i) \quad \text{for } \alpha \in \mathbb{N}^n ,$$

where 1_i stands for the unit vector $(0,\ldots,\overset{i}{1},\ldots,0)$. Then the difference system to be solved is given by

$$(2.8) \qquad \alpha_i c(\alpha - 1_i) + f_i(T)c(\alpha) = 0 \quad (\alpha \in \mathbb{N}^n;\ 1 \leq i \leq n),$$

where $T = (T_1,\ldots,T_n)$ and $f_i = \partial_{x_i} f \ (1 \leq i \leq n)$. One can see that the difference system (2.8) has μ independent solutions, denoted by $c_\nu \ (\nu \in N)$, as follows.

<u>Case of</u> A_ℓ, E_6, E_8 where $f = x_1^{p_1} + \ldots + x_n^{p_n}$. We define a lattice L of $\mathbb{Z}^n$ by $L = \sum_{i=1}^{n} \mathbb{Z} p_i 1_i$. Then, for each $\nu \in N$, we define a function $c_\nu : \mathbb{N}^n \longrightarrow \mathbb{C}$ by

$$(2.9) \qquad c_\nu(\alpha) = \prod_{i=1}^{n} (-)^{k_i} \left(\frac{\nu_i + 1}{p_i} ; k_i\right), \quad k_i = \frac{\alpha_i - \nu_i}{p_i},$$

if $\alpha \in (\nu + L) \cap \mathbb{N}^n$ and by $c_\nu(\alpha) = 0$ if $\alpha \in \mathbb{N}^n \setminus (\nu + L)$.

<u>Case of</u> D_ℓ, E_7 where $f = x_1^{p_1} + x_1 x_2^{p_2}$. We define a lattice L of $\mathbb{Z}^n$ by $L = \mathbb{Z} p_1 1_1 + \mathbb{Z}(p_2 1_2 + 1_1)$. Then, for each $\nu \in N$, we define a function $c_\nu : \mathbb{N}^2 \longrightarrow \mathbb{C}$ by

$$(2.10) \qquad c_\nu(\alpha) = (-)^{k_1} \left(\frac{\nu_1 + 1}{p_1} - \frac{\nu_2 + 1}{p_1 p_2} ; k_1\right) \cdot (-)^{k_2} \left(\frac{\nu_2 + 1}{p_2} ; k_2\right)$$

with $k_1 = \dfrac{\alpha_1 - \nu_1}{p_1} - \dfrac{\alpha_2 - \nu_2}{p_1 p_2}$ and $k_2 = \dfrac{\alpha_2 - \nu_2}{p_2}$ if $\alpha \in (\nu + L) \cap \mathbb{N}^n$ and by $c_\nu(\alpha) = 0$ if $\alpha \in \mathbb{N}^2 \setminus (\nu + L)$.

In (2.9) and (2.10), the expression $(z;k)$ stands for the factorial function $\Gamma(z+k)/\Gamma(z)$. It is directly checked that the functions

c_ν $(\nu \in N)$ defined as above give μ independent solutions of (2.8). As the next step, we introduce a linear mapping $\ell : \mathbb{N}^{\mu-1} \longrightarrow \mathbb{N}^n$ by the formula

$$\exp\Big(\sum_{\nu\in N^*} t_\nu x^\nu\Big) = \sum_{\alpha^*\in\mathbb{N}^{\mu-1}} \frac{t_*^{\alpha^*}}{\alpha^*!} x^{\ell(\alpha^*)} . \tag{2.11}$$

In other words, the multi-index $\ell(\alpha^*) = (\ell_1(\alpha^*), \dots, \ell_n(\alpha^*))$ is given by $\ell_i(\alpha^*) = \sum_{\nu\in N^*} \nu_i \alpha_\nu$ for $1 \leq i \leq n$.

Theorem 4. By means of the functions $c_\nu : \mathbb{N}^n \longrightarrow \mathbb{C}$ and the linear mapping $\ell : \mathbb{N}^{\mu-1} \longrightarrow \mathbb{N}^n$ above, the operator $P(t^*, D_{t_0})$ of Proposition 3 is determined as follows:

$$P_k(t^*)_{\kappa,\nu} = \sum_{\alpha^*\in\mathbb{N}^{\mu-1}} c_\nu(\ell(\alpha^*)+\kappa) \frac{t_*^{\alpha^*}}{\alpha^*!} \tag{2.12}$$

for $\kappa, \nu \in N$ and $k \in \mathbb{N}$.

Corollary. The flat coordinate system of Proposition 3 can be represented by the formulas

$$s_\nu = \delta_{0,\nu} t_0 + \sum_{\alpha^*\in\mathbb{N}^{\mu-1}} c_\nu(\ell(\alpha^*)) \frac{t_*^{\alpha^*}}{\alpha^*!} \quad (\nu \in N) . \tag{2.13}$$

Note that $P_k(t^*)_{\kappa,\nu}$ and so s_ν are weighted homogeneous polynomials. Details of the above argument are given in M.Noumi [3] .

§3. A remark in the case of type A_ℓ .

In the case of type A, it is known by S.Ishiura-M.Noumi [1] that the flat coordinate systems of type A_ℓ $(\ell \geq 1)$ can be obtained by a method of reduction from a countable sequence of weighted homogeneous polynomials. Here we remark that this sequence of polynomials eventually appears in the expansion (2.5) of the solutions of the Gauss-Manin system.

Let $\tilde{t} = (\tilde{t}_2, \tilde{t}_3, \ldots)$ be a sequence of countably many variables and consider the formal Laurent series

$$(3.1) \qquad f(x) = x + \tilde{t}_2 x^{-1} + \tilde{t}_3 x^{-2} + \ldots .$$

Then one can find a unique Laurent series

$$(3.2) \qquad g(y) = y - \tilde{s}_2 y^{-1} - \tilde{s}_3 y^{-2} + \ldots$$

such that $g \circ f(x) = x$ and $f \circ g(y) = y$, so that one obtains a new sequence $\tilde{s} = (\tilde{s}_2, \tilde{s}_3, \ldots)$ of countably many variables. Note that $\tilde{s}_2, \tilde{s}_3, \ldots$ are weighted homogeneous polynomials in $\tilde{t}_2, \tilde{t}_3, \ldots$ and <u>vice versa</u>. Moreover, the two sequences $\tilde{t} = (\tilde{t}_2, \tilde{t}_3, \ldots)$ and $\tilde{s} = (\tilde{s}_2, \tilde{s}_3, \ldots)$ are connected to each other by the formulas

$$(3.3) \qquad \left\{ \begin{array}{ll} \tilde{s}_k = \dfrac{1}{k-1} \operatorname{Res}(\, f(x)^k \, dx) & (k = 2, 3, \ldots) \text{ and} \\ \tilde{t}_k = \dfrac{-1}{k-1} \operatorname{Res}(\, g(y)^k \, dy) & (k = 2, 3, \ldots) . \end{array} \right.$$

There is reason to call this sequence $\tilde{s} = (\tilde{s}_2, \tilde{s}_3, \ldots)$ the "flat coordinate system" associated with $\tilde{t} = (\tilde{t}_2, \tilde{t}_3, \ldots)$.

Consider the versal deformation

$$(3.4) \qquad F(t,x) = x^n + t_2 x^{n-2} + t_3 x^{n-3} + \ldots + t_n , \quad t = (t_2, \ldots, t_n)$$

of the singularity $x^n = 0$ of type A_{n-1} (in a different way of indexing from that of preceding sections). Then one can take the fractional power $F^{1/n}$ in the form (3.1), so that $\tilde{t}_2, \tilde{t}_3, \ldots$ are weighted homogeneous polynomials in $t_2, \ldots, t_n$. (Among those,

$\tilde{t}_2,\ldots,\tilde{t}_n$ are algebraically independent.) Then the variables $\tilde{s}_2,\tilde{s}_3,\ldots$ in (3.2) are determined as polynomials in $t_2,\ldots,t_n$. Now, we define a sequence $s_2,s_3,\ldots$ of polynomials in $t_2,\ldots,t_n$ as follows:

$$(3.5) \qquad s_k = n\,\tilde{s}_k \in \mathbb{C}[t] \quad \text{for } k=2,3,\ldots .$$

Then one can prove that $s_2,\ldots,s_n$ are algebraically independent and that the coordinate system $(s_2,\ldots,s_n)$ coincides with the flat coordinate system given in Corollary of Theorem 4. It is by this reduction process that the flat coordinate systems of type A_ℓ are obtained from the sequence $\tilde{s}=(\tilde{s}_2,\tilde{s}_3,\ldots)$ of "flat variables". The sequence of "flat variables" also appears in the expansion of the solutions of the Gauss-Manin system $\underline{H}_F$ at infinity.

<u>Theorem 5</u>. <u>Let</u> φ_k $(k=2,\ldots,n)$ <u>be the</u> n-1 <u>independent solutions of the Gauss-Manin system</u> $\underline{H}_F$ <u>that appear in the 0-th row of (2.5). Then we have</u>

$$(3.6) \qquad \varphi_k = \delta_{k,n}\, s^{-\frac{n-1}{n}} + \sum_{r=1}^{\infty} (-)^r \binom{\frac{k-1}{n}+r-1}{r}\, s'_{(r-1)n+k}\, s_n^{-\frac{k-1}{n}-r},$$

<u>where</u> $s'_i = s_i|_{s_n=0}$ <u>is the polynomial obtained from</u> s_i <u>by setting</u> $s_n=0$ <u>for</u> $i=2,3,\ldots$.

<u>References</u>

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