数理解析研究所講究録
第 533 巻 1984 年 108-117

Liouville type theorem for hyperfunctions and its applications

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§0. Introduction

In his dissertation, R.Gay proved following theorem 1

Theorem 1. Let f(z) be an entire function satisfying following conditions : for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

(1) $|f(z)| \leqq C_\varepsilon \exp(\varepsilon|z|) \qquad (z \in \mathbb{C}^n)$

Namely, f(z) is infra-exponential type.

(2) $f(m) = O(|m|^p)$ for all $m = (m_1, m_2, .., m_n) \in \mathbb{Z}^n$ and some $p \in N$.

where $\mathbb{Z}^n$ denotes the lattice points and N is the set of natural numbers.

Then f(z) is a polynomial of degree not exceeding p.

In the case of n = 1, this theorem is well known. (See Boas [2])

There is another similar type theorem due to S.Bernstein.

Theorem 2 (S.Bernstein) Suppose that entire function f(z) satisfies the following conditions: for any $\varepsilon > 0$, there exists a constant C_ε such that

(3) $|f(z)| \leqq C_\varepsilon \exp(H_K(z) + \varepsilon|z|) \qquad (z \in \mathbb{C})$

where K is a real bounded closed interval and $H_K(z) = \sup_{\zeta \in K} \mathrm{Re}(z\zeta)$.

(4) $f(x) = O(|x|^p) \qquad (x \in \mathbb{R})$.

Then f(z) is a polynomial of degree not exceeding p.

Our aim in this paper is to unify these two theorems by making use of hyperfunction theory. R.Gay gave the proof of theorem 1 with the help of L.Schwartz distribution theory while our method is based on analytic function theory.

Following theorem 3 is our main result. Our proof is inspired by the technic in Boas's book [2].

Theorem 3. Suppose that entire function f(z) satisfies the following conditions : for any $\varepsilon > 0$, there exists a constant C_ε such that

(5) $|f(z)| \leqq C_\varepsilon \exp(H_K(z)+\varepsilon|z|) \qquad (z \in \mathbb{C}^n)$

where K is a real compact convex set in $\mathbb{R}^n$, $H_K(z) = \sup_{\zeta \in K} \mathrm{Re}\langle z, \zeta\rangle$.

(6) $f(m) = O(|m|^p)$ for all $m = (m_1, m_2, .., m_n) \in \mathbb{Z}^n$.

Then f(z) is a polynomial of degree not exceeding p.

To prove theorem 3, we make use of Fourier-Laplace and Avanissian-Gay transforms of hyperfunctions with compact support. So in §1, we recall the definitions of these two transforms and their properties. Next in §2, we give the proof of theorem 3 ,and as colloraries of theorem 3 we obtain theorem 1 and 2. Finally, in §3, we will show some applications.

In what follows, K always denotes a real compact convex set and $\mathcal{B}[K]$ is the space of hyperfunctions with support contained in K.

§1. Fourier-Laplace and Avanissian-Gay transforms of hyperfunctions with compact support

For hyperfunction $T \in \mathcal{B}[K]$, we define its Fourier-Laplace transform $\tilde{T}(z)$ as follows:

$$\tilde{T}(z) = \langle T_\zeta, \exp(z\zeta)\rangle, \qquad (z \in \mathbb{C}^n)$$

where $z\zeta = z_1\zeta_1 + z_2\zeta_2 + \ldots + z_n\zeta_n$.

Following theorem 4 caracterizes Fourier-Laplace transforms of $\mathcal{B}[K]$.

Theorem 4. Suppose that T belongs to $\mathcal{B}[K]$. Then its Fourier-Laplace transform $\tilde{T}(z)$ satisfies the following estimate: for any $\varepsilon > 0$, there exists a constant C_ε such that

$$|T(z)| \leqq C_\varepsilon \exp(H_K(z)+\varepsilon|z|) \qquad (z \in \mathbb{C}^n)$$

Conversely, if an entire function $f(z)$ satisfies the above estimate, then there exists a hyperfunction $T \in \mathcal{B}[K]$ such that $f(z) = \widetilde{T}(z)$.

For the proof of theorem 4, we refer to [5].

Next we define Avanissian-Gay transforms $G_T(w)$ of $T \in \mathcal{B}[K]$ as follows:

$$G_T(w) = \langle T_\zeta , \prod_{i=1}^{n}(1-w_i\exp(\zeta_i))^{-1} \rangle.$$

Remark that in general $G_T(w)$ is defined for $w = (w_1,..,w_n)$ in $\prod_{i=1}^{n}(\mathbb{C} \setminus \exp(-K_i))$, where K_i denotes i-th projection of K.

Now we enumerate some properties of $G_T(w)$.

Proposition 5.(Avanissian-Gay [1])

(7) $G_T(w)$ is holomorphic in $\prod_{i=1}^{n}(\mathbb{C} \setminus \exp(-K_i))$.

(8) $$G_T(w) = (-1)^{\varepsilon(w_1)+\ldots+\varepsilon(w_n)} \sum_{m \in (\mathbb{N} \cup 0)^n} \widetilde{T}((-1)^{\varepsilon(w_1)}m_1,..,(-1)^{\varepsilon(w_n)}m_n)\,\theta(w_1)^{m_1}\ldots\theta(w_n)^{m_n}.$$

where $\theta(w) = w$ if $|w|<1$ and w^{-1} if $|w|>1$. $\varepsilon(w) = 0$ if $|w|<1$ and $\varepsilon(w) = 1$ if $|w|>1$.

From this development, we have that $\lim_{|w|\to\infty} G_T(w) = 0$.

(9) (Inversion formula)

$$\widetilde{T}(z) = (2\pi i)^{-n}\int_{\prod_{i=1}^{n}\Gamma_i} G_T(e^{-\zeta})\exp(z\zeta)d\zeta$$

where Γ_i is a positively oriented contour surrounding K_i.

Remark that Avanissian-Gay transform is one-to-one because of this inversion formula.

§2. Proof of Theorem 3.

First of all, we prepare the following lemma

Lemma 6. Suppose that $T \in \mathcal{B}[K]$. If its Fourier-Laplace transform $\widetilde{T}(z)$ satisfies following condition:

(10) $\limsup_{|m|\to\infty} |\widetilde{T}(m)|^{1/|m|} \leq 1.$

Then T belongs to $\mathcal{B}[\{0\}]$. In another words, $\widetilde{T}(z)$ is an entire function of infra-exponential type.

(proof) For the simplicity, we will confine ourselves to the cases of n = 1 and 2.

(i) n = 1 By proposition 5-(8), we have following expansions :

$$G_T(w) = \sum_{n=0}^{\infty} \tilde{T}(n)w^n \qquad (|w|<1),$$

$$G_T(w) = -\sum_{n=1}^{\infty} \tilde{T}(-n)w^{-n} \qquad (|w|>1).$$

From the assumption (10), it is easy to see that $G_T(w)$ is holomorphic in $|w|<1$ and $|w|>1$. From this and proposition 5-(7), $G_T(w)$ is holomorphic in $\mathbb{C}\setminus\{1\}$. By means of Inversion formula(5-(9)), we have

$$\tilde{T}(z) = (2\pi i)^{-1}\int_{\Gamma} G_T(e^{-\zeta})\exp(z\zeta)d\zeta$$

As $G_T(w)$ is holomophic in $\mathbb{C}\setminus\{1\}$, $G_T(e^{-\zeta})$ is holomorphic in $\mathbb{C}\setminus\{0\}$. Therefore we can shrink Γ towards $\{0\}$ arbitrarily. Hence $\tilde{T}(z)$ is entire function of infra-exponential type.

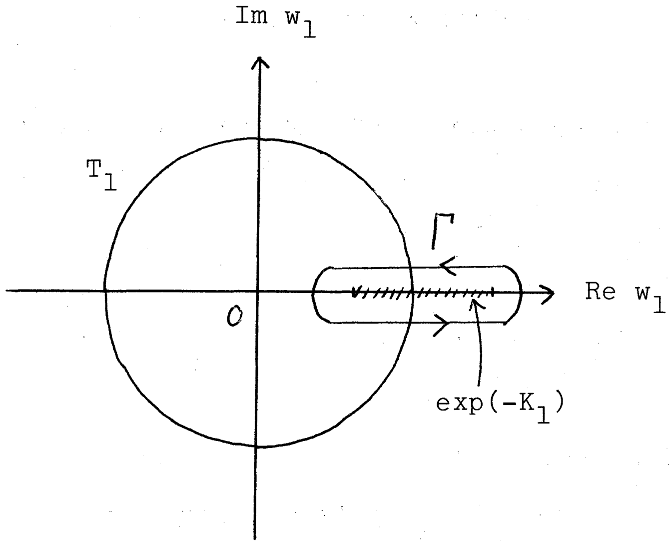
(ii) n=2 From 5-(7) and (8), it is easy to see that $G_T(w_1,w_2)$ is holomorphic in $\prod_{i=1}^{2}(\mathbb{C}\setminus\exp(-K_i))$ and $\prod_{i=1}^{2}(\mathbb{C}\setminus T_i)$, where T_i is unit circle.

To show that $G_T(w_1,w_2)$ is holomorphic in $(\mathbb{C}\setminus\{1\})\times(\mathbb{C}\setminus\{1\})$, we consider following Cauchy integral :

$$G_{\Gamma}(w_1,w_2) = -(2\pi i)^{-1}\int_{\Gamma} G_T(t, w_2)(t-w_2)^{-1}dt$$

Γ is shown in the Figure 1.

Figure 1.



Let fix w_2 in $\mathbb{C}\setminus T_2$. Then we can deform Γ towards 1, since $G_T(t, w_2)$ is holomorphic in $\prod_{i=1}^{2}(\mathbb{C}\setminus T_i)$. As $\lim_{|w|\to\infty} G_T(w_1,w_2) = 0$, $G_\Gamma(w_1,w_2)$ gives single valued analytic continuation of $G_T(w_1,w_2)$ up to $(\mathbb{C}\setminus\{1\}) \times (\mathbb{C}\setminus T_2)$. Interchanging the role of w_1 and w_2, we can obtain analytic continuation of $G_T(w_1,w_2)$ up to $(\mathbb{C}\setminus T_1) \times (\mathbb{C}\setminus\{1\})$.

By elementary set operation, we have the following identity :

(A^c denotes the complement of A.)

$$(A_1\cap B_1)^c \times B_2^c \cup B_1^c \times (A_2\cap B_2)^c \cup A_1^c \times A_2^c$$
$$= (A_1\cap B_1)^c \times (A_2\cap B_2)^c.$$

Putting $A_i = \exp(-K_i)$ and $B_i = T_i$, we see that $G_T(w_1,w_2)$ is holomorphic in $(\mathbb{C}\setminus\{1\}) \times (\mathbb{C}\setminus\{1\})$.

Like in the case of n = 1, we apply inversion fomula:

$$\widetilde{T}(z_1,z_2) = (-2\pi i)^{-2} \int_{\Gamma_1\times\Gamma_2} G_T(e^{-\zeta_1}, e^{-\zeta_2}) \exp(z_1\zeta_1+z_2\zeta_2)d\zeta_1 d\zeta_2$$

Since $G_T(e^{-\zeta_1}, e^{-\zeta_2})$ is holomorphic in $(\mathbb{C}\setminus\{0\}) \times (\mathbb{C}\setminus\{0\})$, we can shrink the contours Γ_i (i = 1, 2) towards the origin.

Hence $\widetilde{T}(z_1,z_2)$ is entire function of infra-exponential type.

This means that T belongs to $\mathcal{B}[\{0\}]$.

Now we pass to the proof of theorem 3. First we prove theorem 3 in the case of n = 1. Next we will treat the case of n = 2.

(proof of theorem 3) By means of theorem 4, there exists hyperfunction $T \in \mathcal{B}[K]$ such that $f(z) = \tilde{T}(z)$. By virtue of preceeding Lemma, f(z) is entire function of infra-exponential type.

(iii) n=1 We consider the following entire function F(z).

$$F(z) = z^{-p-1}(f(z) - \sum_{n=0}^{p} a_n z^n) \qquad (z \in \mathbb{C})$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is Maclaurin expansion of f(z).

We put $H(z) = F^2(z)$. As is easily seen, H(z) is entire function of infra-exponential type, and $H(m) = O(m^{-2})$. There exists a hyperfunction $S \in \mathcal{B}[\{0\}]$ such that $\tilde{S}(z) = H(z)$. We consider Avanissian-Gay transform $G_S(w)$ of S. $G_S(w)$ has following expansion:

$$G_S(w) = \sum_{n=0}^{\infty} \tilde{S}(n) w^n \qquad (|w| < 1)$$

$$G_S(w) = -\sum_{n=1}^{\infty} \tilde{S}(-n) w^{-n} \qquad (|w| > 1)$$

Since $\tilde{S}(m) = O(m^{-2})$, $G_S(w)$ is bounded in $\mathbb{C} \setminus 1$. By virtue of celebrated Riemann's theorem concerning with removable singularity, $G_S(w)$ is entire function. Furthermore, as $\lim_{|w| \to \infty} G_S(w) = 0$, $G_S(w)$ vanishes identically. By means of the inversion formula, $H(z) = \tilde{S}(z)$ also vanishes identically. Hence we obtain the following desire result:

$$f(z) = \sum_{n=0}^{p} a_n z^n.$$

(iv) n=2 We develop $f(z_1, z_2)$ as follows :

$$f(z_1, z_2) = \sum a_{n_1, n_2} z_1^{n_1} z_2^{n_2}$$

$$= \sum_{n_2} a_{n_2}(z_1) z_2^{n_2}$$

We put $F(z_2) = f(m_1, z_2)$ for fixed integer m_1. Since $f(z_1, z_2)$ is entire function of infra-exponential type, $F(z_2)$ has same property.

Moreover $F(z_2)$ satisfies following estimate :

$$F(m_2) = O(|m_2|^p) \qquad \text{for all } m_2 \in \mathbb{Z}.$$

Therefore by preceeding result in (iii), $F(z_2)$ is polynomial of degree not exceeding p. This means that $f(m_1,z_2) = \sum_{n_2=0}^{p} a_{n_2}(m_1)z_2^{n_2}$

On the other hand, we have

$$f(m_1,z_2) = \sum_{n_2=0}^{\infty} a_{n_2}(m_1)z_2^{n_2}.$$

Hence for $n_2 \geqq p+1$,

$$a_{n_2}(m_1) = 0 \qquad \text{for all } m_1 \in \mathbb{Z}.$$

Applying Cauchy's integral formula to $f(z_1,z_2)$, it is readily seen that $a_{n_2}(z_2)$ is entire function of infra-exponential type. From the result in (iii), we see that $a_{n_2}(z_1)$ vanishes identically for $n_2 \geqq p+1$.

Therefore

$$f(z_1,z_2) = \sum_{n_2=0}^{p} a_{n_2}(z_1)z_2^{n_2}$$

Repeating the same argument, we obtain

$$f(z_1,z_2) = \sum_{\substack{n_1=0\\ n_2=0}}^{p} a_{n_1,n_2} z_1^{n_1} z_2^{n_2}$$

By means of the assumption , $f(m_1,m_2) = O(|m|^p)$, we can conclude that $f(z_1,z_2)$ is polynomial of degree not exceeding p.

This is our desired result.

§3. Applications.

From theorem 3, immediately, we obtain following

Proposition 7. Suppose that T is a element of $\mathcal{B}$ [K]. Then followings are equivalent.

(11) T is a finite linear combination of derivatives of order not exceeding p of Dirac's delta function.

(12) $\tilde{T}(z)$ is a polynomial of degree not exceeding p.

(13) $\tilde{T}(m) = O(|m|^p)$ for all $m \in \mathbb{Z}^n$.

In next three applications, we will show how to apply theorem 1. First we treat the discriminant of Hill's equation. Following facts are well known ([4]).

(14) Discriminant $\Delta(E)$ is an entire function of order 1/2.

(15) $\lim_{E\to\infty} E^{1/2}(\Delta(E)-2\cos(\pi E^{1/2}) = 0$.

Now we put $f(E) = \Delta(E)-2\cos(\pi E^{1/2})$. Then we have following

Proposition 8. If f(E) have the following property:

$$f(-am)=O(|m|^p) \qquad (m= 1,2,3,\ldots)$$

for some positive number a, then potential V(x) in Hill's equation vanishes identically.

(proof) First we remark that f(E) is an entire function of infra-exponential type. So from the assumption, (15) and theorem 1, we can conclude that f(E) is polynomial. Furthermore, f(E) vanishes identically because of (15). Namely, $\Delta(E) = 2\cos(\pi E^{1/2})$. This means that there is no unstable intervals. Hence by virtue of Hochstadt's theorem ([4]), we can obtain desired result.

Next we pass to eigenvalue problem of trace class operator defined on a Hilbert space. Let A be a trace class oparator. According to [7], we put

$$f(z) = \det (I + zA) \qquad (z\in\mathbb{C}).$$

Then following two facts are known[7] :

(16) f(z) is entire function of infra-exponential type.

(17) $f(z) = \prod_{j=1}^{N(A)} (1+z\lambda_j(A))$

where $\lambda_j(A)$ $(j=1\ldots N(A))$ are the eigen values of A counted with algebraic multiplicity and N(A) is finite or countably infinite.

Under these preparation, we have following

Proposition 9. If $f(m) = O(|m|^p)$ for all $m\in\mathbf{Z}$ and some natural number p, then oparator A has at most p eigenvalues.

(proof) From the assumption and theorem 1, we will see that f(z) is polynomial. By virtue of (17), there exist at most p eigen values.

Remark : If the assumption holds for p=0, the we can conclude that eigenvalue of A is just 0. For the details of this we refer to exercise 156 in [7].

Finally we give one method to construct analytic functionals without unique carrier. For the definition and some special terminologies of analytic functionals, we refer [5] and [6].

Proposition 10. Suppose that entire function g(z) satisfies following conditions :

(18) g(z) is even function,

(19) there exist constants A and B such that

$$|g(z)| \leqq B\exp(A|z|) \qquad (z \in \mathbb{C})$$

(20) $g(m) = O(|m|^p)$,

(21) g(z) is not polynomial.

Put $T = FB^{-1}(g(\sqrt{z_1 z_2}))$, where FB^{-1} denotes inverse Fourier-Borel (sometime called Fourier-Laplace) transform. Then analytic functional T is carried by polydiscs $D_{a,b} = \{(\zeta_1, \zeta_2) \in \mathbb{C}^2; |\zeta_1| \leqq aA, |\zeta_2| \leqq bA$ $ab = 1, a > 0, b > 0\}$. But T is not carried by the origin.

Remark : Intersection of $D_{a,b}$ s is the origin.

(proof) From the assumption (18), $g(\sqrt{z_1 z_2})$ is entire function. By the elementary calculation and (19), $g(\sqrt{z_1 z_2})$ satisfies the following inequality : for any positive numbers a and b satisfying ab=1,

$$|g(\sqrt{z_1 z_2})| \leqq B\exp(aA|z_1| + bA|z_2|) \qquad ((z_1, z_2) \in \mathbb{C}^2)$$

This means that T is carried by the polydiscs $D_{a,b}$. Now we will show that T is not carried by the origin. If it were so, then $g(\sqrt{z_1 z_2})$ should be entire function of infra-exponential type. Therefore g(z) also have same property. Applying theorem 1 and (20), we obtain that g(z) is polynomial. This contradicts (21).

To close this final section, we give three examples of function $g(z)$.

Example 1. $g(z) = z^{2p}\cos z$. $(p = 0,1,\ldots)$

$T = FB^{-1}(\cos(\sqrt{z_1 z_2}))$ is firstly proposed by L.Hörmander in [5].

Example 2. $g(z) = z^{2p-1}\sin z$. $(p = 0,1,\ldots.)$

Example 3. $g(z) = z^{-p}J_p(z)$. ($J_p(z)$ is Bessel function of order p and $p = 0,1,\ldots$). $T = FB^{-1}(J_0(\sqrt{z_1 z_2}))$ is considered by A.Martineau with slight modification in [6].

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