数理解析研究所講究録
第 533 巻 1984 年 252-255

Logarithms of pseudodifferential operators

By Takashi AOKI (青木貴史)

Department of Mathematics, Kinki University

In this note, we clarify the relation between operators and their exponentials. As an application, a sufficient condition for invertibility of pseudodifferential operators of infinite order is obtained.

A pseudodifferential operator is defined by a symbol $P(x, \xi)$. The operator defined by $P(x, \xi)$ is denoted by $:P(x, \xi):$. The composite operator of two operators $:P(x, \xi):$ and $:Q(x, \xi):$ is written in the following form:

$$:P(x, \xi)::Q(x, \xi): = :\exp(\partial_\xi \cdot \partial_y)P(x, \xi)Q(y, \eta)\Big|_{\substack{y=x \\ \eta=\xi}}:.$$

1. Composition formula.

Let us consider a symbol of the form $\exp\{p(x, \xi)\}$. In analytic category, such a symbol makes sense if $p(x, \xi)$ is of order at most 1-0, i.e., if $\lim_{|\xi|\to\infty} p(x, \xi)/|\xi| = 0$. More generally, a formal sum of symbols $\exp\{\sum_{j\geq 0} p_j(x, \xi)\}$ makes sense if

$\exists d > 0,\ 1 > \exists A > 0;\ \forall h > 0,\ \exists H > 0$ such that

$$|p_j(x, \xi)| \leq A^j(h|\xi| + H),\ j \geq 0,\ |\xi| \geq (j+1)d,$$

(then $\sum_j p_j(x, \xi)$ is said to be a formal symbol of order 1-0).
Let us recall the composition formula for operators with exponential symbols:

Theorem 1. Let $p(x,\xi)$ and $q(x,\xi)$ be symbols of order 1-0. Let $\{w_j\}$ and $\{r_j\}$ be sequences of symbols defined by

(1.1) $$w_0(x,y,\xi,\eta) = p(x,\xi) + q(y,\eta),$$

(1.2) $$w_{j+1} = \frac{1}{j+1}\left(\partial_\xi \cdot \partial_y w_j + \sum_{k=0}^{j} \partial_\xi w_k \cdot \partial_y w_{j-k}\right),\ j \geq 0,$$

(1.3) $$r_j(x,\xi) = w_j(x,x,\xi,\xi),\ j \geq 0.$$

Then $\sum_j r_j(x,\xi)$ is a formal symbol of order 1-0 satisfying

(1.4) $$:\exp\{p(x,\xi)\}::\exp\{q(x,\xi)\}: = :\exp\{\sum_j r_j(x,\xi)\}:.$$

In the preceding theorem, we can replace p and q by formal symbols $\sum_j p_j$ and $\sum_j q_j$ of order 1-0, respectively.

2. Exponential of operators and operators with exponential symbols.

Let $\sum p_j(x,\xi)$ and $\sum q_j(x,\xi)$ be formal symbols of order 1-0. We give an answer to the following problem: Under what conditions does the equality

$$\exp:\sum_j p_j(x,\xi): = :\exp\{\sum_j q_j(x,\xi)\}:$$

hold ?

First part of the answer is

Theorem 2. Let $\sum_j p_j(x,\xi)$ be a formal symbol of order 1-0. Let $\{\psi^{(j)}_{\ell,k}(x,y,\xi,\eta)\}$ and $\{q^{(j)}_k(x,\xi)\}$ be sequences of symbols defined by

(2.1) $\psi^{(o)}_{\ell,o}(x,y,\xi,\eta) = p_\ell(x,\xi)$, $\ell=0,1,2,\cdots$,

(2.2) $\psi^{(j)}_{\ell,o}(x,y,\xi,\eta) = 0$, $j=1,2,\cdots$, $\ell=0,1,2,\cdots$,

(2.3) $q^{(j+1)}_k(x,\xi) = \frac{1}{j+1}\sum_{\ell=o}^{k}\psi^{(j)}_{\ell,k-\ell}(x,x,\xi,\xi)$,

(2.4)
$$\psi^{(j)}_{\ell,k+1}(x,y,\xi,\eta) = \frac{1}{k+1}\{\partial_\xi\cdot\partial_y\psi^{(j)}_{\ell,k}(x,y,\xi,\eta) + \sum_{\nu=o}^{\ell}\sum_{\mu=o}^{j-1}\partial_\xi\psi^{(\mu)}_{\nu,k}(x,y,\xi,\eta)\cdot\partial_y q^{(j-\mu)}_{\ell-\mu}(y,\eta)\}.$$

Set $q_k(s,x,\xi) = \sum_{j=1}^{k+1} s^j q^{(j)}_k(x,\xi)$ $(s\in\mathbf{C})$. Then, for each s, the formal series $\sum q_k(s,x,\xi)$ is a formal symbol of order 1-0 such that

(2.5) $\exp\{s:\sum_j p_j(x,\xi):\} = :\exp\{\sum_k q_k(s,x,\xi)\}:$

holds.

<u>Example 3.</u> $\exp(sx\sqrt{D}) = :\exp(sx\sqrt{\xi} + \frac{s^2x}{4}):$ $(n=1;\ D = :\xi: = \partial/\partial x,\ x=x_1)$.

Conversely, we have

<u>Theorem 4.</u> Let $\sum_j q_j(x,\xi)$ be a formal symbol of order 1-0. Let $\{\psi^{(j)}_{\ell,k}(x,y,\xi,\eta)\}$ be a sequence of symbols defined by

(2.6) $\psi^{(o)}_{o,o}(x,y,\xi,\eta) = q_o(x,\xi)$,

(2.7) $\psi^{(j)}_{\ell,o}(x,y,\xi,\eta) = 0$, $j=1,2,\cdots$, $\ell=0,1,2,\cdots$, (2.8) $\psi^{(j)}_{\ell,k+1}(x,y,\xi,\eta) = \frac{1}{k+1}\{\partial_\xi\cdot\partial_y\, \psi^{(j)}_{\ell,k}(x,y,\xi,\eta)$

$+ \sum_{\nu=0}^{\ell}\sum_{\mu=0}^{j-1}\sum_{i=0}^{\ell-\nu} \frac{1}{j-\mu}\,\partial_\xi\psi^{(\mu)}_{\nu,k}(x,y,\xi,\eta)\cdot\partial_y\,\psi^{(j-\mu-1)}_{i,\ell-\nu-i}(y,y,\eta,\eta)\},$

(2.9) $\psi^{(o)}_{k,o}(x,y,\xi,\eta) = q_k(x,\xi) - \sum_{j=o}^{k}\sum_{\ell=o}^{k-1}\frac{1}{j+1}\psi^{(j)}_{\ell,k-\ell}(x,x,\xi,\xi).$

Set $p_k(x,\xi) = \psi^{(o)}_{k,o}(x,x,\xi,\xi)$. Then $\sum p_k(x,\xi)$ is a formal symbol of order 1-0 such that

(2.10) $:\exp\{\sum_j q_j(x,\xi)\}: = \exp:\sum_k p_k(x,\xi):$

holds.

3. Invertibility for pseudodifferential operators of infinite order.

It is well known that a pseudodifferential operator of finite order is invertible if its symbol is invertible as a symbol. The same is true for infinite order case.

Theorem 5. Let $P(x,\xi)$ be a symbol. Suppose that $1/P(x,\xi)$ is also a symbol, i.e., for each $h>0$, there is a constant $C_h>0$ such that

$$C_h^{-1}\exp(-h|\xi|) \le |P(x,\xi)| \le C_h\exp(h|\xi|).$$

Then $:P(x,\xi):$ is invertible.