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Infinitesimal Deformations of Cusp Singularities

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Introduction. The purpose of this article is to compute infinitesimal deformations T^1 of cusp singularities of two dimension. Let T be a cusp singularity, C the exceptional set of the minimal resolution of T, r the number of irreducible components of C. Then C is a (reduced) cycle of r rational curves. Our main consequence is that dim T^1 is equal to $r-C^2$ if $C^2 \leq -5$. This has been conjectured by Behnke [1]. After completing this work, I was informed that Behnke [2] solved this in a manner slightly different from ours.

§1 Definitions

(1.1) Let M be a complete module in a real quadratic field K, $U^+(M)$ the group of all totally positive units keeping M invariant by multiplication, V an infinite cyclic subgroup of $U^+(M)$. We define a subgroup G(M,V) of $SL(2,\mathbb{R})$ by

$$G(M,V) = \left\{ \begin{pmatrix} v & m \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{R});\ v\in V,\ m\in M \right\}.$$

We define an action of G(M,V) on the product $\mathbb{H}\times\mathbb{H}$ of two upper half planes by

$$\begin{pmatrix} v & m \\ 0 & 1 \end{pmatrix} : (z_1,z_2) \to (vz_1+m, v'z_2+m')$$

where v' and m' denote the conjugates of v and m respectively. The action of G(M,V) on $\mathbb{H}\times\mathbb{H}$ is free and properly discontinuous. We have a nonsingular surface X'(M,V) as quotient. This X'(M,V) is partially compactified by adding a point ∞ into a normal complex space X(M,V). Let $f : Y(M,V) \to X(M,V)$ be the minimal resolution of X(M,V), C the exceptional set of f, $\pi : \mathcal{D} \to Y(M,V)$

the universal covering of $Y(M,V)$, $C = \pi^{-1}(C)$. For brevity we denote $X(M,V)$ and $Y(M,V)$ by X and Y respectively. The space X has a unique isolated singularity at ∞, which we call a cusp singularity. The exceptional set C is a (reduced) cycle of rational curves.

(1.2) Let M^* be the dual of M, i.e. by definition $M^* = \{x \in K;\ tr(xy) \in \mathbb{Z}$ for any $y \in M\}$. Define a mapping i of K into $\mathbb{R}^2$ by $i(x) = (x,x')$, $x \in K$. Let $(M^*)^+ = \{x \in M^*;\ x>0,\ x'>0\}$, and let $\Sigma^+(M^*)$ be the convex closure of $i((M^*)^+)$, $\partial\Sigma^+(M^*)$ be the boundary of $\Sigma^+(M^*)$. Then we number lattice points lying on $\partial\Sigma^+(M^*)$ in a consecutive order. Namely we let $i^{-1}(\Sigma^+(M^*) \cap i(M^*)) = \{B_j;\ j \in \mathbb{Z}\}$ with $B_j < B_k$ for $j>k$. The group V acts on M^*, $\Sigma^+(M^*)$ and $\partial\Sigma^+(M^*)$. Let v be a generator of V with $0<v<1$. Then there exists s such that $vB_k = B_{k+s}$ for any k. We know that $s = -C^2$ by [3]. Moreover there are positive integers b_k (≥ 2) ($k \in \mathbb{Z}$) such that $b_{k+s} = b_k$ and $b_kB_k = B_{k-1}+B_{k+1}$ for any $k \in \mathbb{Z}$.

§2. Theorem.

Theorem <u>Let</u> T <u>be a cusp singularity with</u> $s \geq 5$. <u>Then the space</u> $\mathbb{T}^1$ <u>of infinitesimal deformations of</u> T <u>is, as a subspace of</u> $H^1(V, H^0(\mathcal{D}, \Theta_{\mathcal{D}}(nC)))$ <u>for</u> n <u>large enough, generated by</u>

$$\delta_{i,j} := \theta(-iB_j)\delta_j, \quad 0 \leq j \leq s-1, \quad 1 \leq i \leq b_j - 1$$

<u>where</u> $\delta_j = B_j'\partial_1 - B_j\partial_2$. <u>In particular</u> $\dim \mathbb{T}^1 = s+r$.

References.

[1] Behnke, K.: Infinitesimal Deformations of Cusp Singularities, Math. Annalen, 265, 407-422 (1983)

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[3] Nakamura, I.: Inoue-Hirzebruch Surfaces and a Duality of Hyperbolic Unimodular Singularities, Math. Annalen, 252, 221-235 (1980)