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TOPOLOGICAL ENTROPY AND THE PSEUDO-ORBIT TRACING PROPERTY

by

TAKASHI SHIMOMURA (下村尚司)

ABSTRACT

We show an inequality of the topological entropies between semiconjugate dynamical systems on compact Hausdorff spaces and apply this inequality to the bundle map on a fiber bundle whose total space, the base space and the structure group are compact Hausdorff spaces. A new method of calculating the topological entropy of a continuous map from a compact Hausdorff space to itself is given. The topologocal entropy h(f) of the expansive hmeo-morphism f with the pseudo-orbit tracing property from a compact metric space to itself satisfies the equality

$$h(f) = \limsup_{n\to\infty} (1/n)\cdot \log N_n(f) \quad ,$$

where $N_n(f)$ is the number of fixed points of f^n.

0. INTRODUCTION

Let X be a compact space. We denote by OC(X) the set of all the open coverings of X . For a continuous map $f:X \to X$ the topological entropy h(f) is defined as follows. For $\alpha \in OC(X)$ and $n \in N$, we write

$$\alpha_f^n = \{\textstyle\bigcap_{j=1}^{n-1} f^{-j}A_j \ ; \quad A_j \in \alpha \ , \ 0 \le j < n \ \} \ (\in OC(X) \ ), \tag{1}$$

For any subset $K \subset X$ and $\alpha \in OC(X)$, we write

$$N_K(\alpha) = \min \{ \ \#\beta \ ; \ \ \beta \subset \alpha \ , \ K \subset \textstyle\bigcup_{B\in\beta} B \ \} \ , \tag{2}$$

where $\#\beta$ denotes the cardinality of β .

Then the topological entropy h(f,K) of f with respect to K is defined by

$$h(f,K) = \sup_{\alpha \in OC(X)} \limsup_{n\to\infty} (1/n)\cdot \log N_K(\alpha_f^n) \quad . \tag{3}$$

Of course, $h(f,K)$ coincides with the topological entropy $h(f)$ defined by R.L.Adler, A.G.Konheim and M.H.McAndrew[1].

THEOREM 1.1. Let X be a compact space and Y a compact Hausdorff space. Let $f:X \to X$, $g:Y \to Y$ and $:X \to Y$ be a contonuous maps satisfying $\pi \circ f = g \circ \pi$ and $\pi(X) = Y$. Then the following inequality holds.

$$h(f) \leq h(g) + \sup_{y \in Y} h(f, \pi^{-1}(y)) \tag{4}$$

This has been shown by R.Bowen[3] in the case that X and Y are compact metric spaces.

THEOREM 1.2. Let $\pi:E \to X$ be a projection of a fiber bundle with the total space E and the base space X . Assume that E , X and the structure group are compact Hausdorff spaces and that $f:E \to E$ is a bundle map whose base map is $f':X \to X$, then

$$h(f) = h(f') \quad . \tag{5}$$

We say here (X,f) is a <u>cascade</u> if X is a compact Hausdorff space and $f:X \to X$ is a continuous map.

In §2 we show that the topological entropy of a cascade can be calculated by using finite closed coverings.

THEOREM 3.1. Let (X,d) be a compact metric space and $f:X \to X$ an expansive homeomorphism (Resp. a positively expansive continuous map) with the pseudo-orbit tracing property (Resp. the positive pseudo-orbit tracing property) . Then it follows that

$$h(f) = \limsup_{n\to\infty} (1/n)\cdot \log N_n(f) \tag{6}$$

where

$$N_n(f) = \#\{ x \in X \ ; \ f^n(x) = x \} \quad (n \in N) \quad . \tag{7}$$

This has been shown by R.Bowen[2] for a homeomorphism on a compact metric space with hyperbolic canonical coordinates, and K.Hiraide[5] has given this result by showing that any expansive homeomorphism with the pseudo-orbit tracing property on a compact metric space has Markov partitions of arbitrary small diameter.

1. QUOTIENTS

In this section we sketch proofs of Theorem 1.1. and Theorem 1.2. .

Sketch of a proof of Theorem 1.1.

Take $\alpha \in OC(X)$ and $n \in N$ arbitrarily. Then for each $y \in Y$ and a subset $\beta \subset \alpha_f^n$ such that $\pi^{-1}(y) \subset \bigcup_{B \in \beta} B$, there exists an open subset U_y of Y such that $y \in U_y$ and $\pi^{-1}(U_y) \subset \bigcup_{B \in \beta} B$. Because Y is a compact Hausdorff space and X is compact. Then $\gamma = \{U_y;\ y \in Y\}$ is an element of $OC(Y)$. Take $C \in \gamma_g^{ln}$ for each $l \in N$. Then we can see the following inequality.

$$N_{\pi^{-1}(C)}(\alpha_f^{ln}) \leq [\ \sup_{y \in Y} N_{\pi^{-1}(y)}(\alpha_f^n)\ ]^l \tag{7}$$

Since

$$N_X(\pi^{-1}(\gamma_g^{ln})) \leq N_Y(\gamma_g^{ln}) \tag{9}$$

where

$$\pi^{-1}(\gamma_g^{ln}) = \{\pi^{-1}(C);\ C \in \gamma_g^{ln}\}, \tag{10}$$

we see

$$N_X(\alpha_f^{ln}) \leq [\ \sup_{y \in Y} N_{\pi^{-1}(y)}(\alpha_f^n)\ ]^l \cdot N_Y(\gamma_g^{ln}). \tag{11}$$

Because $\lim_{n\to\infty} (1/n) \log N_X(\alpha_f^n)$ exists (see R.L.Adler et al.[1]), we see

$$h(f,X,\alpha) \leq \sup\ (1/n) \log N_{\pi^{-1}(y)}(\alpha_f^n) + h(g,Y,\gamma). \tag{12}$$

Since $n \in N$ and $\alpha \in OC(X)$ are arbitrary, we have the desired inequality.

Sketch of a proof of Theorem 1.2.

We have to show,

$$\sup_{x\in X} h(f,\pi^{-1}(x)) = 0. \qquad (13)$$

Take $\alpha \in OC(E)$ and $x \in X$. Assume we can find an open covering β of $\pi^{-1}(x)$ such that β refines $\{ A \cap \pi^{-1}(x) \ ; A \in \alpha_f^n \}$, i.e. for any $B \in \beta$ there exists $A \in \alpha_f^n$ such that $B \subset A$, for all $n \in N$, so that $h(f, \pi^{-1}(x), \alpha) = 0$. Since α is arbitrary, we have $h(f,\pi^{-1}(x)) = 0$. But the equicontinuity of the action of the structure group implies the existence of such an β for each $\alpha \in OC(X)$ and $x \in X$.

2. A METHOD OF CALCULATING TOPOLOGICAL ENTROPY

Let $s \in N$ be an positive integer and $A = (A_{ij})$ an (s,s)-matrix whose entries are 0 or 1. Set $S = \{1,\cdots,s\}$, then for each $n \in N$ we denote the set of all the sequences $(a_0,\cdots,a_{n-1})$ S^n of length n which satisfies $A_{a_j a_{j+1}} = 1$ for all j $(0 \leq j < n)$ by $M_n(A)$.

Let (X,f) be a cascade.

DEFINITION 2.1. The pair (α,A) of indexed finite closed covering $\alpha = \{F_1,\cdots,F_s\}$ of X and (s,s)-matrix $A = (A_{ij})$ whose entries are all 0 or 1 is said to be a <u>CM-pair</u> for (X,f), if

$$X = \bigcup_{a\in M_n(A)} \bigcap_{j=0}^{n-1} f^{-1}F_{a_j} \quad \text{where } a = (a_0,\cdots,a_{n-1}). \qquad (13)$$

Let (α,A) be a CM-pair for (X,f). For n N, a subset $P \subset M_n(A)$ is said to be <u>separated</u> if for any distinct elements p ,$p' \in P$ there exists j $(0 \leq j < n)$ such that $F_{p_j} \cap F_{p'_j} = \emptyset$ where $F_i \in \alpha$ $(0 \leq i \leq s,\ s = \#\alpha)$ and $p = (p_0,\cdots,p_{n-1})$ etc.. And for $n \in N$ and a subset $K \subset X$, a subset $P \subset M_n(A)$ is said to be <u>attached</u> to K if $K \cap \bigcap_{j=1}^{n-1} f^{-1}F_{p_j} \neq \emptyset$ for all $(p_0,\cdots,p_{n-1}) \in P$ where $F_i \in \alpha$ $(0 \leq i \leq s,\ s = \#\alpha)$.

We set,

$$S_n(f,K,(\alpha,A)) = \max\{\#P;\ P \subset M_n(A),$$
$$P \text{ is both separated and attached to } K\}, \qquad (14)$$

and

$$\overline{S}_f(K,(\alpha,A)) = \limsup_{n\to\infty} (1/n)\cdot \log S_n(f,K,(\alpha,A)). \qquad (15)$$

Then the topological entropy $h(f,K)$ with respect to K is given as follows.

PROPOSITION 2.1. Let Γ be a family of CM-pairs for (X,f). Assume that for any $\alpha_0 \in OC(X)$ there exists $(\alpha,A) \in \Gamma$ such that α refines α_0, then

$$h(f,K) = \sup \overline{S}_f(K,(\alpha,A)) \qquad ((\alpha,A) \in \Gamma) \qquad (16)$$

Proof. Let (α,A) be an arbitrary CM-pair for (X,f). For each $x \in X$, set $O(x) = \bigcap\{F^c;\ F \in \alpha,\ x \notin F\}$ where F^c is the complement of a subset $F \subset X$. Then $\beta = \{O(x);\ x \in X\} \in OC(X)$. Let $F \in \alpha$ and $x \in X$ satisfy $F \cap O(x) \neq \emptyset$. Then $x \in F$ from the definition. In particular, if $F, F' \in \alpha$ are such that $F \cap F' = \emptyset$, then for each $x \in X$, $F \cap O(x) = \emptyset$ or $F' \cap O(x) = \emptyset$. From this one sees that for any separated set $P \subset M_n(A)$ $(n \in N)$, each $\bigcap_{j=0}^{n-1} f^{-1}O(x_j) \in \beta_f^n$ $(x_j \in X,\ 0 \leq j < n)$ can intersect at most one element of $\{\bigcap_{j=0}^{n-1} f^{-j}F_{p_j};\ (p_0,\cdots,p_{n-1}) \in P\}$. This implies the following inequality,

$$N_K(\beta_f^n) \geq S_n(f,K,(\alpha,A)) \quad \text{for all } n \in N. \qquad (17)$$

And this implies that $h(f,K)$ is larger than the right hand side of the equation (16).

On the other hand, for β_0 $OC(X)$ and B β_0 set

$$U(B) = \bigcup\{B';\ B' \in \beta_0,\ B \cap B' \neq \emptyset\}. \qquad (18)$$

Fix an arbitrary $\alpha_0 \in OC(X)$, then there exists $\beta_0 \in OC(X)$ such that $\gamma = \{U(B);\ B \in \beta_0\}$, refines α_0. From the assumption there exists $(\alpha,A) \in \Gamma$ such hat α refines β_0. Fix $n \in N$ and let $P \subset M_n(A)$ be a maximal separated set attached to K. For each $x \in K$, because of the equation (13) for (α,A), there exists $a \in M_n(A)$ such that $x \in \bigcap_{j=0}^{n-1} f^{-j}F_{a_j}$ where $F_i \in \alpha$ $(1 \leq i \leq s,\ s = \#\alpha)$ and $a = (a_0,\cdots,a_{n-1})$. Then there exist $p = (p_0,\cdots,p_{n-1}) \in P$ such that $F_{a_j} \cap F_{p_j} \neq \emptyset$ for all j $(0 \leq j < n)$, so that taking B_i $(F_i \subset B_i \in \beta)$ for each i $(0 \leq i \leq S)$ we can find

$$x \in \bigcap_{j=0}^{n-1} f^{-j}F_{a_j} \subset \bigcap_{j=0}^{n-1} f^{-j}B_{a_j} \subset \bigcap_{j=0}^{n-1} f^{-j}U(B_{p_j}) \in \gamma_f^n. \tag{19}$$

This implies

$$N_K(\gamma_f^n) \leq S_n(f,X,(\alpha,A)). \tag{20}$$

Since α_0 is refined by γ, we have

$$h(f,K,\alpha_0) \leq \sup \overline{S}_f(K,(\alpha,A)) \quad ((\alpha,A) \in \Gamma). \tag{21}$$

Since $\alpha_0 \in OC(X)$ is arbitrary we are done.

3. PERIODIC POINTS

Let $f:X \to X$ be a homeomorphism on a compact metric space (X,d). Let $\delta > 0$. A sequence $\{x_i\}_{i\in Z}$ of points of X is a <u>δ-pseudo-orbit</u> (<u>δ-p.o.</u>) if $d(f(x_i),x_{i+1})) < \delta$ for all $i \in Z$. Let $\varepsilon > 0$. A point $x \in X$ <u>ε-traces</u> a δ-p.o. $\{x_i\}_{i\in Z}$ if $d(f^i(x),x_{i+1}) \leq \varepsilon$ for all $i \in Z$. A homeomorphism f from a compact metric space (X,d) to itself has the <u>pseudo-orbit tracing property</u> (<u>P.O.T.P.</u>) if for all $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-p.o. $\{x_i\}_{i\in Z}$ $(x_i \in X,\ i \in Z)$ is ε-traced by some $x \in X$ depending on the δ-p.o. $\{x_i\}_{i\in Z}$.

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A homeomorphism $f:X \to X$ on a compact metric space (X,d) is expansive if there exists $\varepsilon > 0$ such that for all distinct elements $x,y \in X$ there exists $n \in Z$ such that $d(f^n(x),f^n(y)) > \varepsilon$.

THEOREM 3.1. Let (X,d) be a metric space. And let $f:X \to X$ be an expansive homeomorphism (Resp. a positively expansive continuous map) with P.O.T.P. (Resp. positive P.O.T.P.), then it follows that

$$h(f) = \limsup_{n\to\infty} (1/n)\cdot\log N_n(f) \tag{22}$$

where

$$N_n(f) = \#\{x \in X;\ f^n(x) = x\} \qquad (n \in N). \tag{23}$$

Proof. We omit a proof.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NAGOYA UNIVERSITY, CHIKUSA-KU, NAGOYA, 464 JAPAN