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TOPOLOGICAL TRANSITIVITY AND C*-ALGEBRAS

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ABSTRACT

We show the relation between the transitive topological dynamics and the C*-algebras generated by a family of shift operators on a Hilbert space. We classify the C*-algebras corresponding to subgroups of the unit circle and that associated with Bernoulli shifts.

1. Introduction.

Let H be the direct sum $\sum_{n\in Z}\oplus K_n$ of a family of countable copies of a separable Hilbert space K, where Z is the set of all integers. The author has been interested in the structure of a family of shift operators on H. A bounded linear operator A is said to be a (weighted) shift operator if A translates the subspace K_n into K_{n+1} for each n in Z. In [4], the author analyzed the structure of invariant subspaces for a family of shift operators on H. This work is originated by Beurling's paper [2], which was taken over by McAsey, Muhly and Saitô [9] in the context of the theory of non-self-adjoint crossed product.

In addition to the work concerning invariant subspace, the author and Takemoto studied the structure of C*-algebras generated by a family of shift operators of multiplicity one [6]. These C*-algebras contain some important examples such as irrational rotation C*-algebras ([10],

[11]) in the theory of C*-algebras. Moreover they are interested in the classification of these C*-algebras under the stable isomorphism [7] (two C*-algebras A and B are said to be stable isomorphic if $A \otimes K(H)$ and $B \otimes K(H)$ are *-isomorphic, where $K(H)$ is the set of all compact operators on a Hilbert space H).

We here take notice that each C*-algebra associated with a family of shift operators corresponds to a transitive topological dynamical system. Hence we are interested in the transitivity in the theory of topological dynamics. In this note, we discuss the relation between some typical transitive dynamical systems and corresponding C*-algebras.

2. Transitive Dynamical Systems and Families of Shift Operators.

Let $\Sigma = (X, \sigma)$ be a compact dynamical system, that is, X is a (Hausdorff) compact space and σ is a homeomorphism of X . We denote by C(X) the set of all continuous functions on X . Then C(X) becomes a commutative C*-algebra under the usual addition, scalar multiplication, multiplication and *-operation ($f^*(x) = \overline{f(x)}$ for f in C(X)). We denote by α the *-automorphism of C(X) induced by σ, that is, $\alpha(f)(x) = f(\sigma(x))$. Throughout this note, we assume that σ is topologically transitive, that is, there exists a point x in X such that the orbit $\{\sigma^n(x) : n \in Z\}$ of x is dense in X (cf. [14, §5.4]), and say that Σ is a transitive (topological) dynamical system. Under this assumption, C(X) is embedded into the commutative C*-algebra $\ell^\infty(Z)$ of all bounded sequences on Z . For f in C(X) and n in Z , let

$$\pi_1(f)(n) = f(\sigma^n(x)) .$$

Since σ is topologically transitive, π_1 becomes a *-isomorphism of $C(X)$ onto the C*-subalgebra $\pi_1(C(X))$ of $\ell^\infty(Z)$. Let β be the *-automorphism of $\ell^\infty(Z)$ defined by

$$\beta((a_n)_{n \in Z}) = (a_{n-1})_{n \in Z} , \quad a = (a_n)_{n \in Z} \in \ell^\infty(Z) .$$

Then it follows that $\pi_1(\alpha(f)) = \beta(\pi_1(f))$. We here note that $\ell^\infty(Z)$ is *-isomorphic with $C(\beta Z)$, where βZ is the Stone-Čech's compactification of Z . Next we consider a representation of $\ell^\infty(Z)$ on a separable Hilbert space H with a basis $\{e_n\}_{n \in Z}$. For $a = (a_n)_{n \in Z}$ in $\ell^\infty(Z)$, we define a bounded linear operator $\pi_2(a)$ on H by

$$\pi_2(a)e_n = a_n e_n , \quad n \in Z .$$

Then π_2 is a *-isomorphism of $\ell^\infty(Z)$ onto a maximal commutative C*-algebra $\pi_2(\ell^\infty(Z))$ in the non-commutative C*-algebra $B(H)$ of all bounded linear operators on H . Let S be the usual shift, that is, $Se_n = e_{n+1}$ for each n in Z . Then it follows that

$$\pi_2(\beta(a)) = S^*\pi_2(a)S , \quad a = (a_n)_{n \in Z} \in \ell^\infty(Z) .$$

For a dynamical system $\Sigma = (X, \sigma)$, let

$$\mathcal{S} = \{\pi(f)S: f \in C(X)\}$$

and

$$W(\mathcal{S}) = \{\pi(f): f \in C(X)\} ,$$

where $\pi = \pi_1 \cdot \pi_2$. Since, for $a = (a_n)_{n \in Z}$ in $\ell^\infty(Z)$, the operator $\pi(a)S$ ($\pi(a)Se_n = a_{n+1}e_{n+1}$, $n \in Z$) becomes a shift operator on H , $\mathcal{S}$ is a family of shift operators with diagonal part $W(\mathcal{S})$ such that $S^*W(\mathcal{S})S = W(\mathcal{S})$.

Conversely, let $\mathcal{S}$ be a family of shift operators with diagonal part $W(\mathcal{S}) = \{W \in B(H): U = WS \in \mathcal{S}\}$ satifying the following condition:

$$(*) \begin{cases} (1)\ W(\mathcal{S}) \text{ is a C*-algebra,} \\ (2)\ S^*W(\mathcal{S})S = W(\mathcal{S}). \end{cases}$$

Then $W(\mathcal{S})$ is a C*-subalgebra of $\pi_2(\ell^\infty(Z))$ and the map $\alpha_S: W \longrightarrow S^*WS$ is a *-automorphism of $W(\mathcal{S})$. By the Gelfand transform, $W(\mathcal{S})$ is *-isomorphic with $C(X_{\mathcal{S}})$ for some compact space $X_{\mathcal{S}}$ and the map α_S induces a homeomorphism σ_S of $X_{\mathcal{S}}$. In this case, $X_{\mathcal{S}}$ consists of all multiplicative linear functionals on the commutative C*-algebra $W(\mathcal{S})$. Let A be a C*-subalgebra of $\ell^\infty(Z)$. For each n in Z, the linear functional $\phi(n)$ on A defined by

$$\phi(n)(a) = a_n, \qquad a = (a_n)_{n \in Z} \in A,$$

is multiplicative on A. Since $W(\mathcal{S})$ is *-isomorphic with some C*-subalgebra A of $\ell^\infty(Z)$, the set Z of all integers is canonically embedded onto the dense subset $\phi(Z)$ of X and it follows that $\sigma_S(\phi(n)) = \phi(n + 1)$. Therefore the transitive dynamical system $\Sigma_{\mathcal{S}} = (X_{\mathcal{S}}, \sigma_S)$ corresponds to $\mathcal{S}$.

Our purpose is to study the structure of C*-algebras generated by a family of shift operators. Thus we investigated the relation between the structure of those C*-algebras and the properties of transitive dynamical systems. Let $\mathcal{S}$ be a family of shift operators on H such that $W(\mathcal{S})$ satisfies the condition (*) and $\Sigma = (X, \sigma)$ be the transitive topological dynamical system corresponding to $\mathcal{S}$. We denote by $C^*(\Sigma)$ the C*-algebra generated by $\pi(C(X))$ and $\{S\}$. The following holds naturally.

Theoren 2.1. (1) σ is minimal if and only if $C^*(\Sigma)$ is simple.
(2) There exists a σ-invariant measure on X if and only if there exists a tracial state (cf. the discussions before Theorem 2.2) of $C^*(\Sigma)$.
(3) $\phi(Z)$ is open if and only if $C^*(\Sigma)$ contains all compact operators.

Now we consider the class of monothetic compact abelian group X . Namely, there exists a homomorphism ϕ of Z onto a dense subgroup $\phi(Z)$ of X . Defining a homeomorphism σ by $\sigma(x) = x + \phi(1)$ $(x \in X)$, $\Sigma = (X, \sigma)$ becomes a transitive topological dynamical system. Moreover it is known that X is the dual group of a subgroup G of the unit circle T_d with discrete topology [13, Theorem 2.3.3]. Hence we denote by Σ_G this dynamical system (X, σ_G) corresponding to G .

Before stating our theorem, we discuss the generalized dimension of a projection in a C*-algebra. For a subspace M of H , let P_M be the projection of H onto M . The dimension of the subspace M is equal to the number

$$Tr(P_M) = \sum_{n \in Z} (P_M e_n, e_n) ,$$

where $(\cdot,\cdot)$ is inner product in H . For a general C*-algebra A in $B(H)$, a linear functional τ on A is said to be a tracial state if $\tau(1) = 1$, $\tau(A) > 0$ for every positive operator A in A and $\tau(AB) = \tau(BA)$ for A and B in A . For a projection P in A , the positive number $\tau(P)$ is regarded as a generalized dimension of $M = PH$ with respect to the system (A, τ). When A has a unique trace τ , we denote by $D(A)$ the set of all dimensions of projections in A , i.e.,

$$D(A) = \{\tau(P): P \text{ is a projection in } A\} .$$

Using this set, we can classify the C*-algebras $C^*(\Sigma_G)$ associated with subgroups G of T_d .

Theorem 2.2. ([10],[11],[6]) (1) $C^*(\Sigma_G)$ is a simple C*-algebra with the unique tracial state.
(2) $D(C^*(\Sigma_G)) = \{t \in [0, 1]: \exp(2\pi i t) \in G\}$.

The following are some interesting examples of $C^*(\Sigma_G)$.

Example 2.3. Let $G(\theta) = \{\exp(2\pi i n\theta): n \in Z\}$, where θ $(0 < \theta < 1/2)$ is an irrational number. Then X is the unit circle and $\sigma(\exp(2\pi i t)) = \exp(2\pi i(t + \theta))$ and $\phi(n) = \exp(2\pi i n\theta)$. Hence $C^*(\Sigma_{G(\theta)})$ is an irrational rotation C*-algebra and by the theorem mentioned above it follows that $D(C^*(\Sigma_{G(\theta)})) = (Z + Z\theta)\cap[0, 1]$. Hence $C^*(\Sigma_{G(\theta_1)})$ is *-isomorphic with $C^*(\Sigma_{G(\theta_2)})$ if and only if $\theta_1 = \theta_2$ ([10]).

Example 2.4. Let J be a set of irrational numbers such that $J\cup\{1\}$ are linearly independent over ratinal numbers. Let $G(J) = \{\exp(2\pi i(n_1\theta_1 + n_2\theta_2 + \cdots + n_k\theta_k)): n_i \in Z,\ \theta_i \in J\}$. Then $X = \prod_{\theta\in J} T_\theta$, where T_θ is the unit circle, $\sigma((\exp(2\pi i t_\theta))_{\theta\in J}) = (\exp(2\pi i t_\theta + \theta))_{\theta\in J}$, $\phi(n) = (\exp(2\pi i n\theta))_{\theta\in J}$ and $D(C^*(\Sigma_{G(J)}) =$

$\{t \in [0, 1]: t = n_0 + n_1\theta_1 + \cdots + n_k\theta_k, n_i \in Z, \theta_i \in J\}$.

Example 2.5. Let $G(Q) = \{\exp(2\pi it): t \in Q\}$ (resp. $G(p) = \{\exp(2\pi ik/p^n): k = 0, 1, \cdots, p^n - 1, n \in Z\}$. Then X is the profinite group $\hat{Z}$ (resp. the group of p-adic integers Z_p), $\sigma(x) = x + \phi(1)$ and $D(C^*(\Sigma_{G(Q)})) = Q \cap [0, 1]$ (resp. $D(C^*(\Sigma_{G(p)})) = \{k/p^n: k = 0, \cdots, p^n, n \in Z\}$).

Example 2.6. Let $G = T_d$. Then X is the Bohr-compactification of Z , $\sigma(x) = x + \phi(1)$ and $D(C^*(\Sigma_G)) = [0, 1]$. The commutative C*-algebra C(X) is *-isomorphic with the C*-algebra of all almost periodic sequences in $\ell^\infty(Z)$.

Remark 2.7. Let Σ be a transitive dynamical system (X, σ) such that X is homeomorphic with the unit circle T . According to the classical theorem established by Poincaré, every topologically transitive homeomorphism of T is topologically conjugate to a irrational rotation of T (cf. [5]). Hence $C^*(\Sigma)$ is *-isomorphic with $C^*(\Sigma_{G(\theta)})$ for some irrational number θ $(0 < \theta < 1/2)$.

Now we consider the relation between the minimality of σ and the uniqueness of tracial state τ . Let $X = T^2 = [0, 1) \times [0, 1)$ and $\sigma(s, t) = (s + \theta, t + f(s))$, where f is a continuous map of T into itself. By Furstenberg [3], σ becomes a non-uniquely (i.e., there exist at least two σ-invariant measures on X) ergodic homeomorphism for a suitable θ and f . The C*-algebra $C^*(\Sigma)$ associated with this dynamical system $\Sigma = (T^2, \sigma_{\theta,f})$ has at least two tracial states, but is simple because every ergodic homeomorphism is minimal. By Theorem 2.2, $C^*(\Sigma)$ is not *-isomorphic with the C*-algebra $C^*(\Sigma_G)$ for any subgroup G of T_d. In the case where $f(s) = s$, Anzai [1] showed that σ is a minimal uniquely ergodic homeomorphism of X. In this case $C^*(\Sigma)$ is a simple C*-algebra which has a unique tracial state. It is unknown whether this C*-algebra is *-isomorphic with $C^*(\Sigma_G)$ for a subgroup G of T_d. However, recently, Watatani and Ichihara showed that the K_0-group for $C^*(\Sigma)$ is group isomorphic with $Z \oplus Z \oplus Z$. Thus $C^*(\Sigma)$ is not *-isomorphic with $C^*(\Sigma_{G(J)})$ for any $J = \{\theta_1, \theta_2\}$ such that $\{1, \theta_1, \theta_2\}$ are linearly independent over rational numbers.

3. Finite-Dimensional Representations of $C^*(\Sigma)$.

In the case where σ is not minimal, $C^*(\Sigma)$ is not simple, so that we study the class of irreducible representations of $C^*(\Sigma)$. First we consider Bernoulli shifts. Let $X_k = \prod_{n \in Z} \{0, \cdots, k-1\}$, $\sigma_k((x_n)_{n \in Z}) = (x_{n-1})_{n \in Z}$ and $\Sigma_k = (X_k, \sigma_k)$. The C*-algebra $C^*(\Sigma_k)$ is the uniform closure in $B(H)$ of the following operators of the form: $\sum_{\text{finite}} \pi(f_n) S^n$, where f_n is a function in $C(X_k)$. For a fixed element x for σ_k and θ in $[0, 1)$, we define a linear functional $\pi_{x,\theta}$ on the above dense *-algebra as follows;

$$\pi_{x,\theta}\Big(\sum_{\text{finite}} \pi(f_n) S^n\Big) = \sum_{\text{finite}} f_n(x) \exp(2\pi i n \theta) .$$

The map $\pi_{x,\theta}$ is extended to the C*-algebra $C^*(\Sigma)$ as a multiplicative linear functional on $C^*(\Sigma_k)$ is regarded as a compact subspace of the dual space of $C^*(\Sigma_k)$ with σ-weak topology, and is homeomorphic with the topological direct sum of k-copies of the unit circle because the set of fixed points for σ_k consists of k-points. Hence we obtain a complete classification of the C^*-algebras associated with Bernoulli shifts.

Theorem 3.1. $C^*(\Sigma_k)$ is *-isomorphic with $C^*(\Sigma_j)$ if and only if $k = j$.

For a C^*-algebra A , we denote by $\mathrm{Irr}_n(A)/\sim$ the space of unitary equivalence classes of n-dimensional irreducible representations of A . Then $\mathrm{Irr}_1(C^*(\Sigma_k))/\sim$ is homeomorphic with $\underbrace{T \oplus \cdots \oplus T}_{k}$. This result is generalized to the case of n-dimensional representations as follows.

Theorem 3.2. ([7, Theorem A]) $\mathrm{Irr}_n(C^*(\Sigma))/\sim$ is homeomorphic with $(X_n/\approx) \times T$, where X_n is the set of points x in X with $\sigma^n(x) = x$ and $\sigma^k(x) \neq x$ for $1 \leqq k \leqq n - 1$ and where $\approx$ is the orbit equivalence relation.

The above theorem derives an interesting result for Markov chains. Let $M = (a_{i,j})_{i,j=0}^k$ be a $k \times k$ matrix with $a_{i,j} \in \{0, 1\}$ and $X_M = \{(x_n)_{n} \ Z: a_{x_n, x_{n-1}} = 1\}$ and σ_M the restriction of σ_k to X_M . Let $\Sigma_M = (X_M, \sigma_M)$. Then, for each $n \geqq 1$, $(X_M)_n$ is a finite set and by Theorem 3.2 the cardinal nimber of $(X_M)^n = \{x \in X_M: \sigma^n(x) = x\}$

are determined by the C*-algebra $C^*(\Sigma_M)$. Under the condition that M is irreducible, the topological entoropy $h(\sigma_M)$ is determined by the cardinal numbers $N_n(\sigma_M)$ of $(X_M)^n$, i.e.,

$$h(\sigma_M) = \lim (1/n)\log N_n(\sigma_M)$$

(cf. [14, Theorem 8.17]). Hence we have the following theorem.

Theorem 3.3. Let M and N be irreducible matrices. If $C^*(\Sigma_M)$ and $C^*(\Sigma_M)$ are *-isomorphic, then $h(\sigma_M) = h(\sigma_N)$.

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