数理解析研究所講究録
第 536 巻 1984 年 80-86

Liapunov Functions of a Class of Ordinary Differential Equations in $\mathbb{R}^2$

by

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§1. Introduction.

In the theory of differential equations, Liapunov function is one of the most important tools to study the stability of equiblium points and also the global behavior of the solutions. Let us consider, for example, the equations in a domain $\Omega \subset \mathbb{R}^2$,

$$\begin{cases} \dot{x} = F(x,y) \\ \dot{y} = G(x,y) \end{cases} \quad \text{in } \Omega \tag{1.1}$$

where F and G are supposed to be suitably smooth.

V(x,y) on Ω is called the (global) Liapunov function for (1.1) when $\frac{\partial V}{\partial x}F + \frac{\partial V}{\partial y}G \leqq 0$ (or $\geqq 0$) on Ω. Then, V is decreasing on every solution of (1.1), in other words, the solutions are tend to flow into the part of Ω where V is smaller. For the gradient systems and the Hamiltonian systems, we can immadiately find the Liapunov function. These systems are, however, fairly thin subsets of smooth vector fields on Ω, and for other systems, no general theory for the construction of Liapunov functions has presented. We must find a Liapunov function case by case with the concrete form of the equation. One of our aims is bringing some improvement in such a weak point in the theory of Liapunov function: to find a class of equations and to give the procedure for constructing the Liapunov function of the class. The main results for this purpose is given in §2.

§2. Main Results and Some Application.

Let Ω be a convex domain in $\mathbb{R}^2$. We are concerned with

$$\begin{cases} \dot{x} = F(x,y) \\ \dot{y} = G(x,y) \end{cases} \quad \text{in } \Omega \tag{2.1}$$

where F and G are sufficiently smooth on Ω.

Further, we assume that (2.1) has al least one equiblium point in Ω.

Proposition. If we assume

$$\frac{\partial F}{\partial y}\cdot\frac{\partial G}{\partial x} > 0 \quad \text{on } \Omega, \tag{2.2}$$

then the maps $(x,F(x,y))$ and $(G(x,y),y)$ are diffeomorphic from Ω to the images respectively. And there are unique implicit functions $p(x)$ of $F = 0$ and $q(y)$ of $G = 0$. Here, "unique" means that $y = p(x)$ for all (x,y) satisfying $F(x,y) = 0$ and $x = q(y)$ for all (x,y) satisfying $G(x,y) = 0$.

Proof. We will discuss only about $(x,F(x,y))$, for similar arguement can be applied to $(G(x,y),y)$. By the convexity of Ω, we have

$$F(x,a) - F(x,b) = \int_b^a \frac{\partial F}{\partial y}(x,s)ds,$$

Here, by (2.2), $\frac{\partial F}{\partial y} > 0$ on Ω or $\frac{\partial F}{\partial y} < 0$ on Ω, therefore $(x,F(x,y))$ is an injective map. Also, for all (x,y), the Jacobian of $(x,F(x,y))$ is $\det\begin{pmatrix} 1 & 0 \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\end{pmatrix} = \frac{\partial F}{\partial y} \neq 0$. Thus $(x,F(x,y))$ is the diffeomorphism from Ω to the image. Using the implicit function theorem, there exists $p(x)$ such that $F(x,p(x)) = 0$. Since $(x,F(x,y))$ is injective, we have the uniqueness of the implicit function $p(x)$. //

For a while, we assume that

$\Omega = (a,b) \times (c,d)$, the domain of $p(x)$ is (a,b) and the domain of $q(y)$ is (c,d). (2.3).

This is the simplest but most essential case of our study. In this case, by the proposition, p and q are unique. Hence

$$\begin{cases} F(x,y) = (y - p(x))\cdot\tilde{F}(x,y) \\ G(x,y) = (x - q(y))\cdot\tilde{G}(x,y), \end{cases}$$

where $\tilde{F}(x,y) = \int_0^1 \frac{\partial F}{\partial y}(x,ty+(1-t)p(x))dt$

$$\tilde{G}(x,y) = \int_0^1 \frac{\partial G}{\partial x}(tx+(1-t)q(y),y)dt.$$

Here let $L(x,y) = xy - \int^x p(s)ds - \int^y q(t)dt$, then we have

$$\begin{cases} F(x,y) = \frac{\partial L}{\partial x}\cdot\tilde{F} \\ G(x,y) = \frac{\partial L}{\partial y}\cdot\tilde{G} \end{cases}$$

where $\tilde{F}\cdot\tilde{G} > 0$ on Ω by (2.2). Thus we get a global Liapunov function $L(x,y)$ of (2.1).

Mere generally, let $M(x,y)$ be a positive function on Ω, then

$$M_1(x,y) = \int_{p(x)}^{y} M(x,s)ds$$

$$M_2(x,y) = \int_{q(y)}^{x} M(t,y)dt$$

can be defined on Ω and

$$\begin{cases} M_1(x,y) = (y-p(x))\tilde{M}_1(x,y) \\ M_2(x,y) = (x-q(y))\tilde{M}_2(x,y) \end{cases}$$

where $\tilde{M}_1 \cdot \tilde{M}_2 > 0$ on Ω. Therefore, let $f = \tilde{F}/\tilde{M}_1$ and $g = \tilde{G}/\tilde{M}_2$, then we have $F = fM_1$ and $G = gM_2$. Now by definition, $\frac{\partial M_1}{\partial y} = \frac{\partial M_2}{\partial x} = M$ on Ω, hence there exists $\tilde{L}(x,y)$ on Ω such that $\frac{\partial \tilde{L}}{\partial x} = M_1$ and $\frac{\partial \tilde{L}}{\partial y} = M_2$. Hence $F(x,y) = f\cdot\frac{\partial \tilde{L}}{\partial x}$ and $G(x,y) = g\cdot\frac{\partial \tilde{L}}{\partial y}$, where $f\cdot g > 0$ on Ω. Thus we obtain a global Liapunov function $\tilde{L}$ of (2.1). In such way, we can construct many global Liapunov function of (2.1) under the assumption of (2.3).

Now we return to the general cases where Ω is merely a convex domain in $\mathbb{R}^2$. If we take $\Omega' \subset \Omega$ satisfying (2.3), the discussions above is still available and we obtain the Liapunov functions on Ω'. And then, after some technical arguements, the Liapunov function on Ω' can be extended to all compact subsets of Ω. And so, we obtain the following:

Theorem. If (2.2) is satisfied, then for all compact set $K \subset \Omega$, there exist Liapunov functions of (2.1) on K given by

$$F = f\cdot\frac{\partial L}{\partial x} \quad \text{and} \quad G = g\cdot\frac{\partial L}{\partial y} \quad \text{on } K \tag{2.4}$$

where $f\cdot g > 0$ on K and L is the Liapunov function. Especially, the simplest form of L is

$$L(x,y) = xy - \int^{x} p(s)ds - \int^{y} q(t)dt.$$

The essential part of the proof has given above. We would like to omit the details of technical arguements, which is not difficult but somewhat complicated.

Corollary. Let $\Omega \subset \mathbb{R}^2$ be a convex domain. If $\frac{\partial F}{\partial y}\cdot\frac{\partial G}{\partial x} > 0$ on Ω, then
$$\begin{cases} \dot{x} = F(x,y) \\ \dot{y} = G(x,y) \end{cases} \quad \text{on } \Omega$$
have no closed orbits in Ω except the equiblium points.

Proof. If this equation has a closed orbit in Ω, then there is a compact set $K \subset \Omega$ containing the closed orbit. By the theorem, we obtain a Liapunov function L on K given by (2.4). Now, at $c \in$ the closed orbit, $\frac{dL}{dt}\big|_c = f\big|_c(\frac{\partial L}{\partial x}\big|_c)^2 + g\big|_c(\frac{\partial L}{\partial y}\big|_c)^2 = 0$. Since $f\big|_c \cdot g\big|_c > 0$, we have $\frac{\partial L}{\partial x}\big|_c = \frac{\partial L}{\partial y}\big|_c = 0$. Therefore c has turned out to be a equiblium point. //

Remark 1. If $\frac{\partial F}{\partial y} < 0$ and $\frac{\partial G}{\partial x} < 0$ on Ω, then (2.1) is the model of the competition by two species in theoretical biology. By the corollary, we can say,"the population of the species competiting another species is never periodic for time." Furthermore, by the Poincaré-Bendixson's theorem, we obtain "if a orbit is bounded on the positive direction of time, then the orbit will converge to one of the equiblium points as $t \to \infty$." This result is including that of Ch.12§6 of Hirsh-Smale[1]. They have proved the result only in the generic case without using Liapunov function.

Now, following the theorem, let us construct a Liapunov function of the equations known as the competition model of two species.

Example. We consider

$$\begin{cases} \dot{x} = (a_1 - b_1x - c_1y)x \\ \dot{y} = (a_2 - c_2x - b_2y)y \end{cases} \quad \text{on } [0,+\infty) \times [0,+\infty),$$

where a_i, b_i and c_i are positive constants for $i = 1,2$.
In the equation, x and y are the population of the species respectively. Let $\Omega = (0,+\infty) \times (0,+\infty)$, then $\frac{\partial F}{\partial y} = -c_1x < 0$ and $\frac{\partial G}{\partial x} = -c_2y < 0$ in Ω. Here (2.2) is satisfied. Next, we caluclate $p(x)$ (resp. $q(y)$) by $F(x,p(x)) = 0$ (resp. $G(q(y),y) = 0$).
Immadiately we obtain $p(x) = (b_1x - a_1)/c_1$ and $q(y) = (b_2y - a_2)/c_2$.
Therefore the simplest form of the Liapunov function in Ω is

$$L(x,y) = xy - (b_1\frac{x^2}{2} - a_1x)/c_1 - (b_2\frac{y^2}{2} - a_2y)/c_2$$

and

$$\begin{cases} \dot{x} = -c_1x(y - (b_1x - a_1)/c_1) = -c_1x\frac{\partial L}{\partial x} \\ \dot{y} = -c_2y(x - (b_2y - a_2)/c_2) = -c_2y\frac{\partial L}{\partial y} . \end{cases}$$

In the previous discussions of this section, we have not only constructed the Liapunov function but also reduced the system to the form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial y} \end{pmatrix},$$

where $f \cdot g > 0$ on Ω. Such systems can be called "quasi-gradient" systems.
Furthermore, if a differential equation on $\Omega \subset \mathbb{R}^2$ is given by

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f & h \\ -h & g \end{pmatrix}\begin{pmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial y} \end{pmatrix} \tag{2.5}$$

where $f,g \geqq 0$ in Ω, then L is a Liapunov function of (2.5). Now, let $f \equiv g \equiv 1$ and $h \equiv 0$ on Ω, then (2.5) is the gradient system. And so, let $f \equiv g \equiv 0$ and $h \equiv 1$ on Ω, then (2.5) is the Hamiltonian system. Thus, such a class of the equations as (2.5) is the natural extension including both systems and having a special Liapunov function. If we consider more general theorey about Liapunov functions, it must be important to ask that what kind of the differential equations can be reduced to the form (2.5).

Reference

[1] Hirsh-Smale, Differential equations, dynamical systems, and Linear algebra, Academic Press (1974).