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2次多項式を係数にもつLiénard方程式の周期解

Small-Amplitude Periodic Solutions of the Quadratic Liénard Equation

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SUMMARY

The quadratic Liénard equation, that is, the Lienard differential equation in which the coefficients of $\dot{x}$ and x are polynomials of degree two in x is studied with respect to its periodic solutions of small amplitude. The equation can represent a hard or a soft oscillation depending on parameter values. The existence of periodic solutions is proved and a simple formula for amplitude is given.

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1. Introduction

The periodic solutions of the Liénard equation

$$\ddot{x} + \dot{x} f(x) + g(x) = 0 \qquad (1)$$

or

$$\ddot{x} + \dot{x} f(x) + x[g(x)/x] = 0 \qquad (2)$$

have been studied for a long time. In particular, the van der Pol equation with

$$f(x) = \mu(x^2 - 1) \quad , \quad (\mu > 0) \quad ,$$
$$g(x)/x = 1 \qquad (3)$$

in (2) has been of wide interest. In a system represented by this equation , a self-oscillation develops without a trigger; in other words, the system exhibits a 'soft' oscillation. Another system represented by (2) with

$$f(x) = \mu(x^4 - \alpha x^2 + 1) \quad , \quad (\mu > 0 , \alpha > 2)$$
$$g(x)/x = 1 \qquad (4)$$

is known to exhibit a 'hard' oscillation which needs a trigger. As is seen in the above examples, it is usually assumed that $f(x) = f(-x)$ and $g(x) = -g(-x)$ or, in particular,

$$g(x) = x . \qquad (5)$$

The form of $g(x)$ seems to have attracted little attention because $g(x)$ is considered to merely modify the period of oscillation.[1] As a simple example in which $f(x) \neq f(-x)$ and the form of $g(x)$ plays an important role, we will present in this paper the equation (2) with

$$f(x) = \gamma + ax + bx^2 \quad ,$$
$$g(x)/x = 1 + px + qx^2 \quad , \qquad (6)$$

that is, the Liénard equation

$$\ddot{x} + \dot{x}(\gamma + ax + bx^2) + x(1 + px + qx^2) = 0 \quad , \qquad (7)$$

where γ, a, b, p, and q are constants. We will call (7) the quadratic Liénard equation since the coefficients of $\dot{x}$ and x are polynomials of degree two in x. This simple equation seems worth notice as a typical example that can represent both hard and soft self-oscillations. In the following, we will show that γ and $ap-b$ are decisive on self-oscillation, and give explicit formulas for the amplitude of small self-oscillation.

2. Limit Cycles

To study the periodic solutions of (7), we assume, as will be explained in the following, that

$$b>0 \quad , \tag{8}$$
$$p^2<4q \quad , \tag{9}$$
$$a>0 \quad , \tag{10}$$
$$p>0 \quad . \tag{11}$$

Inequality (8) is the condition for the dissipation to increase as $|x| \to \infty$. Inequality (9) is the condition for the stiffness $g(x)/x$ to be positive for $-\infty < x < \infty$. For $x<0$, $f(x)$ may become negative by (10) and $g(|x|)/|x| > g(x)/x$ by (11). Hence self-oscillation can be expected to exist even if $\gamma>0$ because the half period in which $x<0$ is longer than that in which $x>0$. Incidentally, it is clear that (10) and (11) can be rewritten as

$$ap>0 \tag{12}$$

by considering the substitution of x with $-x$.

Rewriting (2) with (6) as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -yf(x)-g(x) \\ &= -x-\gamma y-(px^2+axy)-(qx^3+bx^2y) \quad , \end{aligned} \tag{13}$$

we will prove the existence of limit cycles. The singularity at the origin will first be examined. Note that (13) has no finite singular point except at the origin since $g(x)$ has no real root except at $x=0$ by (9). It can be easily seen that for

$$0<|\gamma|<2 \quad , \tag{14}$$

the origin is a focus, which is stable for $\gamma>0$ and unstable for $\gamma<0$. The singularity of the origin for $\gamma=0$ can be decided by constructing[1] the following polynomial in x and y :

$$V_5(x,y)=V_4(x,y)+2aqx^2y^3/3 + (-2a^3+4aq/3)y^5/5 \quad , \tag{15}$$

where

$$V_4(x,y)=(x^2+y^2)+2(px^3-ay^3)/3 + (qx^4+a^2y^4)/2 \quad . \tag{16}$$

Let C be a positive constant. Then it is clear that if C is small enough,

$$V_4(x,y)=C \tag{17}$$

represents a closed curve which is nearly a circle centered at the origin and collapses into the origin as $C\to 0$. The same holds for V_5 and V_7 which follows.

By differentiating $V_5(x, y)$ with respect to t and making use of (13), we have

$$\frac{dV_5}{dt} = \dot{x}\frac{\partial V_5}{\partial x} + \dot{y}\frac{\partial V_5}{\partial y}$$
$$= -2ry^2[1-P_0(x,y)] + 2(ap-b)x^2y^2(1-ay+a^2y^2)$$
$$+x^2y^2P_1(x,y) \quad , \tag{18}$$

where the P(x,y)'s are polynomials in x and y which do not have constant terms and are unrelated to r. If x and y are small enough, the sign of the right-hand side of (18) is the same as that of $ap-b$ for $r=0$. Hence, if $r=0$, the origin is a stable focus for $ap-b<0$ and an unstable focus for $ap-b>0$.

The case in which

$$r=0 \quad , \quad ap-b=0 \tag{19}$$

can be dealt with by considering

$$V_7(x,y) = V_5(x,y) + [-a^2qx^2y^4 + a^2(a^2-5q/3)y^6/3]$$
$$-a^3(a^2-5q/3)x^2y^5 \quad . \tag{20}$$

By differentiation as in (18), we have

$$\frac{dV_7}{dt} = -2ry^2[1+Q_0(x,y)] + 2(ap-b)x^2y^2[1+Q_1(x,y)]$$
$$-2apq\{x^4y^2\{1+Q_2(x,y)\} + x^2y^4\{2/3+Q_3(x,y)\}\} \; , \tag{21}$$

where the Q(x,y)'s are polynomials in x and y which do not have constant terms and are unrelated to r and b. Since $ap\xi>0$ by (9) —— (11), the origin is a stable focus. To sum up, the origin of the phase plane of (13) is a focus which is

i) stable for $0<\gamma<2$,
unstable for $-2<\gamma<0$,

ii) stable for $\gamma=0$ if $ap-b\leq 0$,
unstable for $\gamma=0$ if $ap-b>0$.

As described above, V_5 and V_7 were costructed primarily for the purpose of analyzing the singularity[1] at the origin. However, they can be used as a kind of Liapounov function as will be seen in the following. We will prove that a small limit cycle exists around the origin. Since the origin is a focus, the trajectory starting from a point (Ø, y_1) near the origin on the positive y-axis surrounds the origin and again arrives at a point on the positive y-axis. We denote this point by (Ø, S(y_1)) and will use the same notation for different values of r and b. Assume first $\gamma=0$ and $a\gamma-b<0$. Then by (18)

$$S(y_1)<y_1 \quad , \qquad (22)$$

which remains true for $\gamma\neq 0$ provided $|\gamma|$ is small enough. Assume that $r<0$ and $|r|$ is small enough as mentioned above and consider a point $(\emptyset, y_0)$ such that $y_0 \ll y_1$. Then by (18)

$$S(y_0) > y_0 \quad . \tag{23}$$

Hence, by the Poincaré-Bendixon theorem,[1] there exists a stable limit cycle passing between $(\emptyset, y_0)$ and $(\emptyset, y_1)$. The uniqueness of the limit cycle can be proved by noting that the right-hand side of (18) monotonically decreases by r, while V_S and $\dot{x}$ in (13) are independent of r. However, the detail of the proof is omitted here. Similarly a unique unstable limit cycle exists for $ap-b>0$ and $r>0$ because $S(y_1)>y_1$ and $S(y_0)<y_0$. The above results can be obtained also by the bifurcation theory.[2] However, as will be shown in the following, there exists another kind of limit cycle for $ap-b>0$ which is of much mathematical interest since it is not covered by the bifurcation theory. Consider a point $(\emptyset, y_2)$ near the origin on the positive y-axis. Then, assuming $r=0$ and $b=ap$, we have by (21)

$$S(y_2) < y_2 \quad , \tag{24}$$

which remains true for $r \neq 0$ and $b \neq ap$ provided

$|\gamma|$ and the change in b are small enough. Assume next that $\gamma=0$ and $a\beta-b$ is positive but small enough as mentioned above. Then for a point(0, y_1) such that $y_1 \ll y_2$, we have by (18)

$$S(y_1) > y_1 \quad , \qquad (25)$$

which remains true for $\gamma \neq 0$ provided $|\gamma|$ is small enough. Hence there exists a stable limit cycle for $a\beta-b>0$ provided $a\beta-b$ and $|\gamma|$ are small enough. Note that a limit cycle exists even if $\gamma=0$. The above results are roughly illustrated in Fig. 1. With γ as parameter, the line for $a\beta-b<0$ corresponds to normal-type bifurcation in the bifurcation theory and the dashed line to inverted-type bifurcation. Concerning the inverted-type case it was already pointed out [2] that in some cases a stable limit cycle exists and merges for some parameter value into the inner unstable one as shown in the figure. However, as far as the authors are aware, no mathematically proven example has been reported. Therefore the quadratic Liénard equation presented in this paper seems to be a rare case in which the existence, uniqueness and merging of the stable limit cycle can be proved. On account of space consideration, however, the further proof is omitted.

3. Periodic Solutions by the Method of Perturbation

It was shown in the preceding section that the periodic solutions can exist for (13) or (7). Their approximate amplitude will be obtained by the method of perturbation on the assumption that $|\gamma|$ is small. Let

$$
\begin{aligned}
\gamma &= \epsilon^2 \gamma_2 + \epsilon^4 \gamma_4 + \cdots \\
a &= a_0 + \epsilon^2 a_2 + \cdots \\
b &= b_0 + \epsilon^2 b_2 + \cdots \\
p &= p_0 + \epsilon^2 p_2 + \cdots \\
q &= q_0 + \epsilon^2 q_2 + \cdots
\end{aligned}
\qquad (26)
$$

where ϵ is a small positive parameter, and let x and y be expanded in a Fourier series as

$$
\begin{pmatrix} x \\ y \end{pmatrix} = \sum_{\nu=-\infty}^{\infty} \begin{pmatrix} x^{(\nu)} \\ y^{(\nu)} \end{pmatrix} e^{j\nu\omega t} , \qquad (27)
$$

where ω is the fundamental angular frequency of a periodic solution. We will be concerned with describing $|x^{(1)}|$, that is, one half the amplitude of the fundamental by the time scale slower than ϵ^{-2} or ϵ^{-4}.

Assume (27) to be further expanded in powers of ϵ as follows:

$$\begin{pmatrix} x^{(\nu)} \\ y^{(\nu)} \end{pmatrix} = \sum_{n=1}^{\infty} \epsilon^n \begin{pmatrix} x_n^{(\nu)} \\ y_n^{(\nu)} \end{pmatrix} \quad , \tag{28}$$

$$\omega = \sum_{n=0}^{\infty} \epsilon^n \omega_n \quad , \tag{29}$$

$$\frac{d}{dt} = \epsilon^2 \frac{\partial}{\partial T_2} + \epsilon^4 \frac{\partial}{\partial T_4} + \cdots \quad . \tag{30}$$

Substituting the above in (13) and putting the terms $\epsilon^n e^{j\nu\omega t}$ together, we get the balance equation for (n,ν) as follows.

The balance equation for (1,1) is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -j\omega_0 & 1 \\ -1 & -j\omega_0 \end{pmatrix} \begin{pmatrix} x_1^{(1)} \\ y_1^{(1)} \end{pmatrix} \quad , \tag{31}$$

for which the solvability condition is

$$\omega_0^2 = 1 \quad . \tag{32}$$

Hence

$$\begin{pmatrix} x_1^{(1)} \\ y_1^{(1)} \end{pmatrix} = W_1 \begin{pmatrix} 1 \\ j \end{pmatrix} \quad , \tag{33}$$

where W_1 is to be determined later. The balance equation for (2,1) is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -j & 1 \\ -1 & -j \end{pmatrix} \begin{pmatrix} x_2^{(1)} \\ y_2^{(1)} \end{pmatrix} - j\omega_1 \begin{pmatrix} x_1^{(1)} \\ y_1^{(1)} \end{pmatrix} , \quad (34)$$

for which the solvability condition is, by (33),

$$\omega_1 = 0 \quad . \quad (35)$$

Hence

$$\begin{pmatrix} x_2^{(1)} \\ y_2^{(1)} \end{pmatrix} = W_2 \begin{pmatrix} 1 \\ j \end{pmatrix} , \quad (36)$$

where W_2 is generally an unknown. The solvability conditions for the balance equations for (3,1), (4,1), and (5,1) are obtained after a tedius and cumbersome calculation as

$$\frac{\partial W_1}{\partial T_2} = A W_1 , \quad (37)$$

$$\frac{\partial W_2}{\partial T_2} = A W_2 + B W_1 , \quad (38)$$

$$\frac{\partial W_1}{\partial T_4} + \frac{\partial W_3}{\partial T_2} + \frac{1}{2j}\left(C_1 W_1^2 \frac{\partial W_1^*}{\partial T_2} + C_2 |W_1|^2 \frac{\partial W_1}{\partial T_2} + \alpha \frac{\partial W_1}{\partial T_2} \right)$$
$$= A W_3 + B W_2 + (C + D) W_1 , \quad (39)$$

where W_3 is generally an unknown which defines $(x_3^{(1)}, y_3^{(1)})_T$, T denoting transposition, by

$$\begin{pmatrix} x_3^{(1)} \\ y_3^{(1)} \end{pmatrix} = W_3 \begin{pmatrix} 1 \\ j \end{pmatrix} + W_1 \begin{pmatrix} 0 \\ j\omega_2 \end{pmatrix} + \frac{\partial}{\partial T_2} W_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad (40)$$

and

$$\alpha = -\gamma_2/2 \qquad , \qquad (41)$$

and A and other parameters are as follows. Let

$$\beta = -(a_0 p_0 - b_0)/2 + j(10 p_0^2 + a_0^2 - 9 q_0)/6 \qquad , \qquad (42)$$

$$\left.\begin{aligned} \xi &= -\gamma_4/2 \\ \eta &= (a_0 p_2 + a_2 p_0 - b_2)/2 - j(20 p_0 p_2 + 2 a_0 a_2 - 9 q_2)/6 \ , \\ \zeta &= 5 b_0 q_0/3 - j[-185 p_0^4/108 + 4 b_0^2/3 + 329 p_0^2 q_0/36 \\ &\quad - a_1^4/48 - 19 a_0^2 q_0/24 + 21 q_0^2/16] \ . \end{aligned}\right\} \quad (43)$$

Then

$$\left.\begin{aligned} C_1 &= -\beta - 8 b_0/3 - j 2 a_0/3 \ , \\ C_2 &= -2\beta - 11 b_0/3 - j(5 a_0^2/3 + 16 p_0/9) \ , \end{aligned}\right\} \quad (44)$$

$$\left.\begin{aligned} A &= (\alpha - j\omega_2) - \beta |W_1|^2 \\ B &= -j\omega_3 - \beta(W_1 W_2^* + W_1^* W_2) \ , \\ C &= (\xi - j\omega_4) + \eta |W_1|^2 - \zeta |W_1|^4 \ , \\ D &= -\beta(|W_2|^2 + W_1 W_3^* + W_1^* W_3) \ . \end{aligned}\right\} \quad (45)$$

Some other results obtained together with (37)-(39) are as follows:

$$\left.\begin{aligned} &\begin{pmatrix} x_2^{(0)} \\ y_2^{(0)} \end{pmatrix} = |W_1|^2 p_0 \begin{pmatrix} -2 \\ 0 \end{pmatrix} , \quad \begin{pmatrix} x_2^{(2)} \\ y_2^{(2)} \end{pmatrix} = W_1^2 \frac{p_0 + j a_0}{3} \begin{pmatrix} 1 \\ 2j \end{pmatrix} \\ &\begin{pmatrix} x_2^{(\nu)} \\ y_2^{(\nu)} \end{pmatrix} = 0 \ , \qquad (|\nu| \geqq 3) \end{aligned}\right\} \quad (46)$$

$$\left.\begin{aligned}
&\begin{pmatrix} x_3^{(0)} \\ y_3^{(0)} \end{pmatrix} = (W_1^* W_2 + W_1 W_2^*) p_0 \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_3^{(2)} \\ y_3^{(2)} \end{pmatrix} = W_1 W_2 \frac{2(p_0 + j a_0)}{3} \begin{pmatrix} 1 \\ 2j \end{pmatrix}, \\
&\begin{pmatrix} x_3^{(3)} \\ y_3^{(3)} \end{pmatrix} = W_1^3 \left[(2p_0^2/3 - a_0^2 + q_0)/8 + j(5a_0 p_0/3 + b_0) \right] \begin{pmatrix} 1 \\ 2j \end{pmatrix}, \\
&\begin{pmatrix} x_3^{(\nu)} \\ y_3^{(\nu)} \end{pmatrix} = 0 \quad , \quad (|\nu| \geq 4) \quad .
\end{aligned}\right\} \quad (47)$$

We will determine $|W_1|$ since the fundamental and the constant components are approximately given by

$$2|x^{(1)}| \simeq 2|\epsilon x_1^{(1)}| = 2|\epsilon W_1| \quad , \tag{48}$$

$$x^{(0)} = \epsilon^2 x_2^{(0)} = -2 p_0 |\epsilon W_1|^2 \tag{49}$$

by (28), (33), and (46). We first put

$$\frac{\partial W_1}{\partial T_2} = 0 \tag{50}$$

in (37), because we are concerned with $x^{(1)}$ varying by the time scale slower than ϵ^{-2} or ϵ^{-4}. Then we get

$$A = 0 \quad , \tag{51}$$

of which the real and the imaginary parts respectively give

$$\gamma_2/2 + |W_1|^2 \mathrm{Re}\,\beta = 0 \quad , \tag{52}$$

$$\omega_2 + |W_1|^2 \mathrm{Im}\,\beta = 0 \quad . \tag{53}$$

Eq. (52) can be written by (42) as

$$-\gamma_2 + (a_0 p_0 - b_0)|W_1|^2 = 0 \quad . \tag{54}$$

This determines $|W_1| \neq 0$ for

$$\gamma_2 \cdot (ap-b) > 0 \tag{55}$$

Assume next

$$\gamma_2 = 0 \quad . \tag{56}$$

Then, in order to have $|W_1| \neq 0$, it is necessary in (54) that

$$a_0 p_0 - b_0 = 0 \quad . \tag{57}$$

Hence further assume (57) and put

$$\frac{\partial W_2}{\partial T_2} = 0 \quad , \quad \frac{\partial W_3}{\partial T_2} = 0 \quad . \tag{58}$$

Then by (38)

$$\beta = 0 \tag{59}$$

and by (39)

$$\frac{\partial W_1}{\partial T_4} = (C+D) W_1 \quad . \tag{60}$$

Eq.(59) gives by (45)

$$\omega_3 + (W_1 W_2^* + W_1^* W_2) \, \mathrm{Im}\,\beta = 0 \quad . \tag{61}$$

Eq.(60) determines $|W_1|$ by

$$\mathrm{Re}\,(C+D) = 0 \tag{62}$$

or

$$\mathrm{Re}\,C = 0 \quad , \tag{63}$$

because $\mathrm{Re}\,D = 0$ by (45), (42), and (57). Eq.(63) is rewritten by (43) as

$$-\gamma_4/2 + (1/2)(a_0 p_2 + a_2 p_0 - b_2)|W_1|^2 - (5/3)\, b_0 q_0 |W_1|^4 = 0 \quad . \tag{64}$$

Eqs.(54) and (64) can respectively be written by

(26) and (48) as

$$\gamma - (1/4)(ap-b)\rho^2 = 0 \tag{65}$$

for

$$\gamma = \epsilon^2 \gamma_2 \tag{66}$$

and

$$\gamma - (1/4)(ap-b)\rho^2 + (5/24)bq\rho^4 = 0 \tag{67}$$

for

$$\gamma = \epsilon^4 \gamma_4 \tag{68}$$

where ρ is the approximate amplitude of the fundamental and

$$\rho = 2|\epsilon W_1| \quad . \tag{69}$$

The constant component is expressed by (49) as

$$x^{(0)} = -p_0 \rho^2/2 \quad . \tag{70}$$

Since (67) has one positive root for

$$\gamma < 0 \tag{71}$$

and two positive roots for

$$0 < \gamma < 3(ap-b)^2/40bq \ , \quad ap-b > 0 \quad , \tag{72}$$

it represents Fig . 1 for small limit cycles. Note that Eq.(67) covers the applicable range of (65), because if (66) is assumed in (67), the term ρ^4 is smaller than $|\gamma|$ by (69) and serves to improving (65). Eq. (67) must be checked numerically. Of particular interest are the cases where $ap-b>0$ as can be seen by Fig. 1. Examples are given in Fig. 2.

As mentioned before, it is a very long and tiresome way to arrive at (67) by the perturbation method. However, if we are to be satisfied with a somewhat rough estimation, (67) can be obtained in a very much shorter way through V_7 in (20). Assuming $|\gamma|$ and $|ap-b|$ to be nonzero but suitably small, we rewrite (21) as

$$\frac{dV_7}{dt} = -2\gamma y^2 + 2(ap-b)x^2y^2 - 2apq(x^4y^2 + 2x^2y^4/3) + (\text{smaller terms}) \quad . \tag{73}$$

Let

$$x = R\cos\theta \quad , \quad y = R\sin\theta \quad . \tag{74}$$

Then (73) becomes

$$\frac{dV_7}{dt} \simeq -2\gamma R^2 \sin^2\theta + 2(ap-b)R^4\cos^2\theta\,\sin^2\theta - 2apqR^6(\cos^4\theta\,\sin^2\theta + 2\cos^2\theta\,\sin^4\theta/3) \,. \tag{75}$$

The integral on the small circle of radius R is seen to be

$$\int_0^{2\pi} \frac{dV_7}{dt}d\theta \simeq -R^2[\gamma - (1/4)(ap-b)R^2 + (5/24)apqR^4] \,. \tag{76}$$

The quantity in the brackets above yields (67) for $ap \simeq b$, while (65) clearly results from (18) in a similar way.

Yamafuji and others [3] applied a fifth-order perturbation to the self-oscillations of the so-called BVP equation, which can be shown to be expressible by (7) with

$$q = (2/3)\, b p / a \quad . \tag{77}$$

It is hard to make correspondence between their result and ours, because theirs starts from different assumptions and contains complex expressions. However, their result is distinguished by the composition of one equation by adding the solvability conditions for the third, fourth, and fifth balance equations which are respectively multiplied by ϵ^3, ϵ^4, and ϵ^5. Taking the trouble to apply the similar addition to (37)-(39), we have

$$\frac{dK}{dt} = [\gamma - j(\epsilon^2\omega_2 + \epsilon^3\omega_3 + \epsilon^4\omega_4) - (\beta - \epsilon^2\eta)|K|^2 - \zeta|K|^4]K . \tag{78}$$

with higher powers of ϵ neglected, where

$$K = \epsilon W_1 + \epsilon^2 W_2 + \epsilon^3 W_3 \quad . \tag{79}$$

The real part of the quantity in the brackets in (78) is equated with zero to yield

$$\gamma - (1/4)(ap - b)|2K|^2 + (5/24)\, bq|2K|^4 = 0 \quad . \tag{80}$$

This is essentially the same as (67). However, to discuss the applicable range of (80), we have to recall the condition that each balance equation should hold. It is questionable whether the expression (80) in K is significant as it appears to be.

4 Conclusion

The small-amplitude periodic solutions of the quadratic Liénard equation defined by (7) were studied. The existence of solutions is proved and a simple formula for amplitude is given. The above equation seems to be worth notice as a typical equation which can represent a hard or a soft self-oscillation depending on parameter values.

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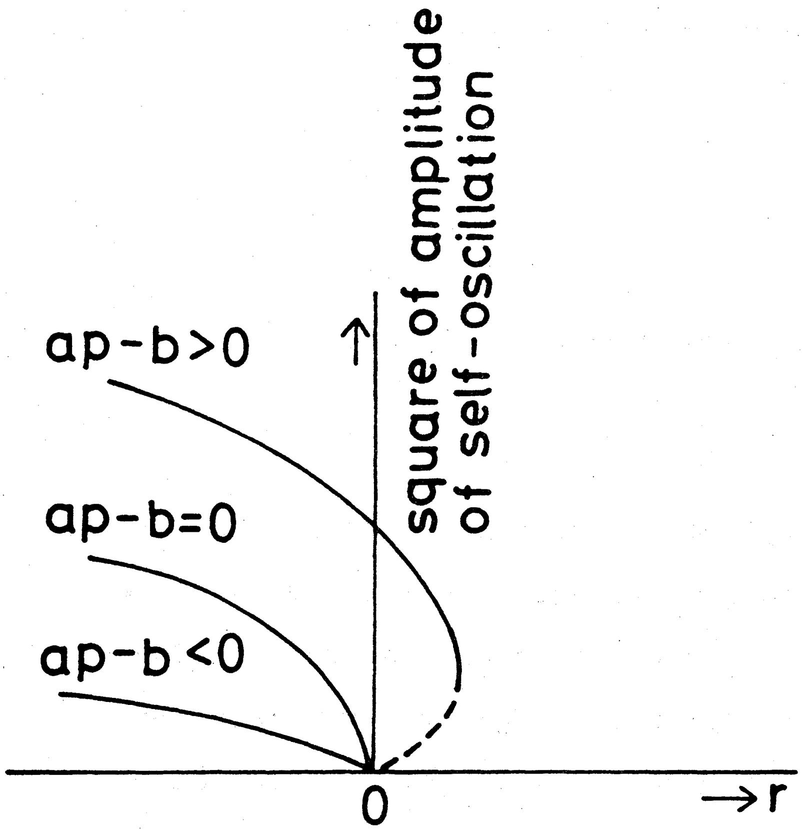

Fig. 1 Schematic illustration of bifurcation with respect to Υ . Solid and broken lines respectively denote the square of the amplitudes of stable and unstable periodic solutions.

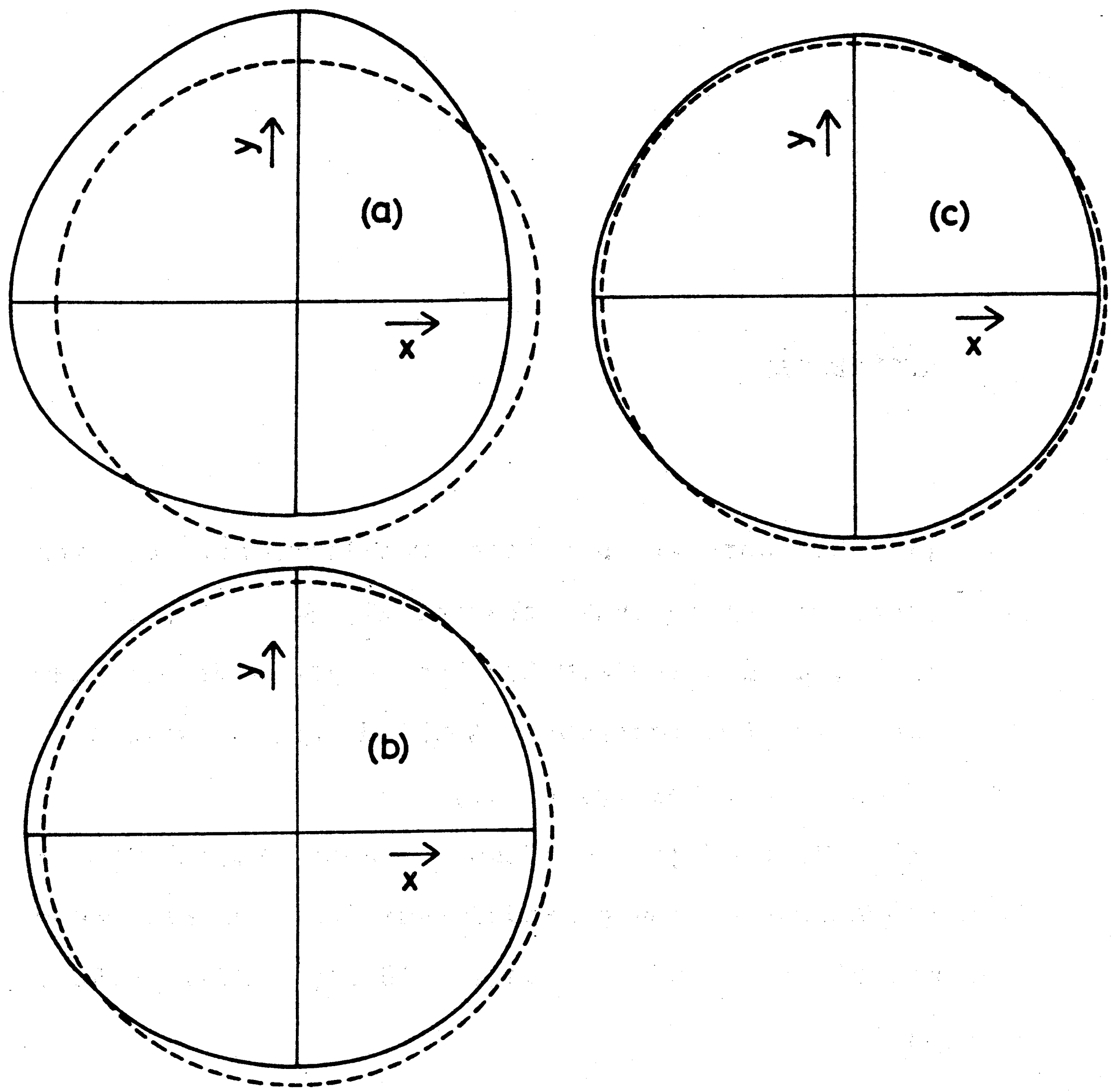

Fig. 2 The fundamentals obtained by (67) (indicated by the dashed circles of radius ρ) and the limit cycles obtained by numerical integration for

$\gamma = -0.1 \times 10^{-1}$ and 0.8×10^{-4} , $a = 1$,

$b = 0.96$, $p = 1$, $q = 1$

(a) Stable limit cycle for
$\gamma = -0.1 \times 10^{-1}$. $\rho = 0.5$.

(b) Stable limit cycle for
$\gamma = 0.8 \times 10^{-4}$. $\rho = 0.2$.

(c) Unstable limit cycle for
$\gamma = 0.8 \times 10^{-4}$. $\rho = 0.1$.

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