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AVERAGE VOLTAGE OF CHAOTIC RESPONSE IN A JOSEPHSON JUNCTION CIRCUIT

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ABSTRACT

Qualitative behavior of a current driven Josephson junction circuit is investigated. The circuit dynamics is expressed by a nonautonomous periodic equation on a cylindrical phase space. By using the Poincaré mapping on the phase space the bifurcation of periodic and chaotic response is considered. It is found that the average voltage of the responses is invariant under some bifurcational processes.

1. INTRODUCTION

A current driven Josephson junction circuit[1] described by the nonautonomous periodic differential equation

$$\ddot{x} + k\dot{x} + \sin x = B_0 + B\cos \nu t \qquad (1)$$

exhabits many interesting properties, such as the jump and hysteresis behavior of various states, the frequency entrainments, the appearance of chaotic states, etc. In particular, the time average of $\dot{x}$ is kept constant under some bifurcational processes, for example, the period doubling bifurcations. This corresponds to the well-known constant voltage steps appeared in the current-voltage characteristics of the circuit[2].

In this paper we make use of the qualitative theory of ordinary differential equations and numerical methods for calculating bifurcations of periodic solutions and consider the global feature of bifurcation diagrams. Invariance property of the time average is clarified theoretically and numerically.

2. PERIODIC SOLUTION AND ITS BIFURCATION

Equation (1) is equivalent to

$$\begin{aligned} \dot{x} &= y &&= f(t, x, y) \\ \dot{y} &= -ky - \sin x + B_0 + B\cos \nu t &&= g(t, x, y) \end{aligned} \tag{2}$$

where the state belongs to a cylindrical phase space $(x, y) \in S^1 \times R$, which is homeomorphic to the punctured plane, $R^2 - \{0\}$. By using the solution of Eqs. (2) the Poincaré mapping T is defined as:

$$T: S^1 \times R \longrightarrow S^1 \times R; \quad P \longmapsto T(P) = \varphi(2\pi/\nu, P) \tag{3}$$

where $\varphi(t, P)$ is the solution of Eqs. (2) with $\varphi(0, P) = P$.

A periodic solution $\varphi(t, P)$ with period $2N\pi/\nu$ corresponds to a fixed point of T^N or an N-periodic point of T:

$$T^N(P) - P = 0 \tag{4}$$

A quasi periodic solution of Eqs. (2) corresponds to an invariant closed curve (abbr. ICC) of the mapping T.

An N-periodic point is called hyperbolic, if the absolute values of the roots μ of the characteristic equation:

$$\det(DT^N(P) - \mu E) = 0 \tag{5}$$

are both different from unity. Topological type of a periodic solution is then classified by the characteristic roots of Eq. (5) and the winding number L of the solution around the cylindrical phase space. Hence we call a hyperbolic L times winding N-periodic point of T, or simply (N, L)-periodic point, as:

(i) completely stable, if $|\mu_1|, |\mu_2| < 1$,

(ii) completely unstable, if $|\mu_1|, |\mu_2| > 1$,

(iii) directly unstable, if $0 < \mu_1 < 1 < \mu_2$,

(iv) inversely unstable, if $\mu_1 < -1 < \mu_2 < 0$,

which we denote $S^{N,L}$, $U^{N,L}$, $D^{N,L}$ and $I^{N,L}$, respectively. Note that an (N, L)-periodic point corresponds to a periodic solution of the form

$$\begin{aligned} x(t + 2N\pi) &= x(t) + 2L\pi \\ y(t + 2N\pi) &= y(t) \end{aligned} \tag{6}$$

which is wrapped around the phase space L times. We call the solution as an (N, L)-periodic solution.

PROPERTY 1. Let $\varphi(t, \phi, \psi)$ be an (N, L)-periodic solution of Eqs.(2). Then

$$\int_0^{2N\pi} f(t, \phi, \psi)dt = 2L\pi, \tag{7}$$

and

$$\int_0^{2L\pi} dx/f(t, \phi, \psi) = 2N\pi, \tag{8}$$

if $f(t, \phi, \psi) \neq 0$ for $t \in [0, 2N\pi]$.

Proof.

$$\int_0^{2N\pi} f(t, \phi, \psi)dt = \int_0^{2N\pi} \frac{dx}{dt}\, dt = \oint dx = 2L\pi,$$

$$\int_0^{2L\pi} dx/f(t, \phi, \psi) = \int_0^{2L\pi} \frac{dt}{dx}\, dx = \int_0^{2N\pi} dt = 2N\pi,$$

where $\oint dx$ denotes the line integral along the solution curve. QED.

COROLLARY 1. If $f(t, \phi, \psi) = y$ as in Eqs. (2), then

$$\bar{y} = \frac{1}{2N\pi} \int_0^{2N} y(t)\, dt = \frac{1}{2N\pi} \oint dx = \frac{L}{N} \tag{9}$$

This means the time average of $y(t)$ is a rational number for a periodic solution of Eqs. (2).

When a system parameter varies, the bifurcation of a fixed or periodic point may occur at some bifurcation parameter[3]. An appearance or disappearance of a couple of fixed point is observed if one of the roots of Eq. (5) satisfies the condition: $\mu_1 = 1$, or equivalently,

$$\det(DT^N(P) - E) = 0 \tag{10}$$

This bifurcation is symbolically expressed as:

$$S^{N,L} + D^{N,L} \rightleftharpoons \emptyset \tag{11}$$

where $\emptyset$ denotes the disappearance of the periodic points. This type of bifurcation is also called the tangent bifurcation of (N, L)-periodic points.

The branching of periodic points is obtained if $\mu_1 = -1$, or equivalently,

$$\det(DT^N(P) + E) = 0 \tag{12}$$

The branching of periodic points is expressed as:

$$\begin{aligned} S^{N,L} &\rightleftharpoons I^{N,L} + 2\times S^{2N,2L} \\ D^{N,L} &\rightleftharpoons S^{N,L} + 2\times D^{2N,2L} \end{aligned} \tag{13}$$

This type of bifurcation is also called the period doubling bifurcation. Under certain variation of parameters the branching may proceed successively as:

$$S^{2^m N, 2^m L} \rightleftharpoons I^{2^m N, 2^m L} + 2 \times S^{2^{m+1} N, 2^{m+1} L} \qquad (14)$$

for m = 0, 1, 2, ... The time average of y(t) in Eqs. (2) is kept constant under the branching process, because the period and winding number alter simultaneously by factor 2^m, m = 1, 2, ...

PROPERTY 2. Let $(x_m(t), y_m(t))$ be the m-th branching solution, i.e., a $(2^m N, 2^m L)$-periodic solution of Eqs. (2), which bifurcates from an (N, L)-periodic solution $(x_0(t), y_0(t))$. Then the time average of $y_m(t)$ is invariant under the branching process (14):

$$\bar{y}_m = \frac{2^m L\pi}{2^m N\pi} = \frac{L}{N}, \quad \text{for } m = 0, 1, \ldots \qquad (15)$$

We omit the Hopf bifurcation and the bifurcation related with the completely unstable periodic solution[3]. These bifurcations cannot occur in Eqs. (2), because the damping parameter k is positive.

3. BIFURCATION DIAGRAMS AND AVERAGE VOLTAGES

In this section we illustrate some numerical results of bifurcation diagrams and the time average y(t). In the following the system parameters k and ν are fixed as

$$k = 0.5, \quad \nu = 1.0 \qquad (16)$$

and two-parameter problem for B and B_0 is considered. For notational convenience, we use the symbols:

$G_m^{N,L}$: the tangent bifurcation set of (N, L)-periodic solutions,

$I_m^{N,L}$: the branching set of (N, L)-periodic solutions,

in bifurcation diagrams, where m denotes the number of separate component of the bifurcation set.

The bifurcation value of parameter and the location of (N, L)-periodic point can be calculated by solving Eq. (4) and the condition (10) or (12), simultaneously[3].

The bifurcation diagram illustrated in Fig. 1 shows the region in which different types of fixed points are obtained. The boundary curves constitute the bifurcation sets of parameters. The bifurcation set $G_m^{1,L}$ forms in the shape of leaf and aligns parallel to the B axis for m = 1, 2, ..., although we omit the sets for m ≧ 2 in the figure. In the region surrounded by $G_m^{1,L}$ there exist a directly unstable (1, L)-periodic solution and a completely stable or an inversely unstable (1, L)-periodic solution. Therefore in the overlapped regions, we can always find even number of periodic solutions. At the point where $G_1^{1,L}$ intersects the B_0 axis, we observe the phenomenon of frequency entrainment, i.e., the higher harmonic entrainment. A limit cycle of the second kind with period $2\pi/L$ synchronizes with the external periodic force B cos t.

In the shaded region encircled by $I_m^{1,L}$, we have an inversely unstable (1, L)-periodic point and a pair of (2, 2L)-periodic points.

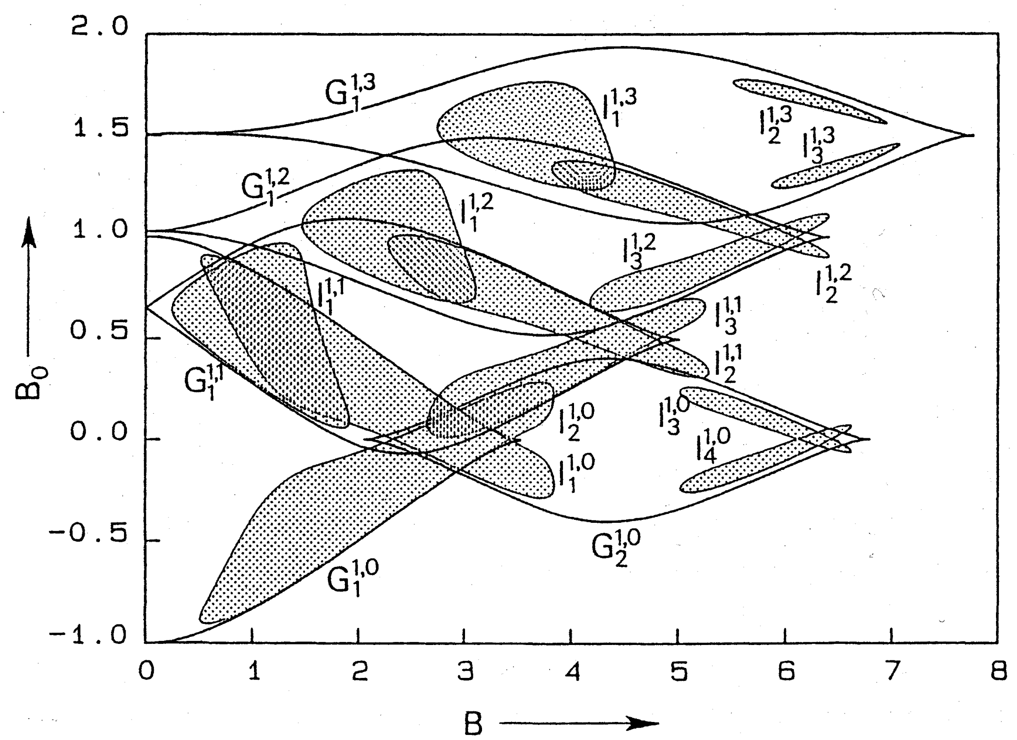

Fig. 1. Bifurcation diagram for (1, L)-fixed point of T.

The bifurcation diagram for (2, 2L)-periodic points is schematically illustrated in Fig. 2. In the shaded portion encircled by $I_1^{1,0}$, tangent bifurcations $G_1^{2,0}$, $G_2^{2,0}$ and branching $I_1^{2,0}$ are obtained. In the region encircled by $I_1^{2,0}$, the period doubling process (14) proceeds successively and finally causes chaotic states.

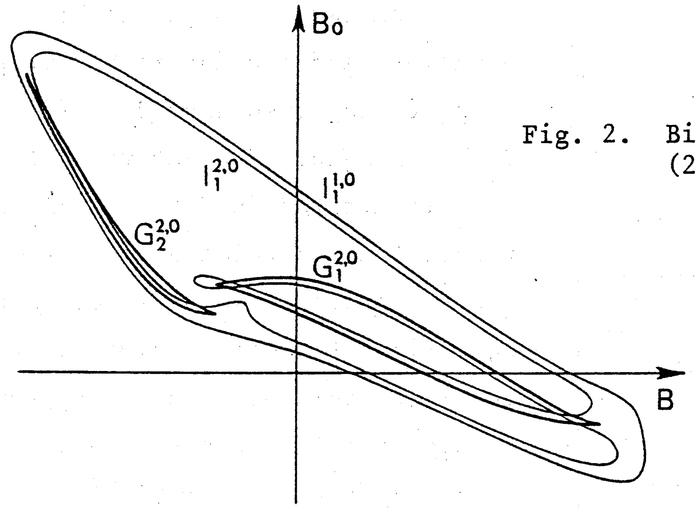

Fig. 2. Bifurcation diagram for (2, 0)-periodic points.

Now we consider the time average of y(t):

$$\bar{y} = \lim_{T\to\infty} \int_0^T y(t)dt \tag{17}$$

which corresponds to the average voltage across a Josephson junction element. Figure 3 shows numerical results obtained by the Poincaré mapping and the computation of Eq. (17). By increasing B_0 from 0.06, the branching processes of $S_1^{1,0}$: $S_1^{1,0} \longrightarrow I_1^{1,0} + 2\times S^{2,0}$, $S_1^{2,0} \longrightarrow I^{2,0} + 2\times S^{4,0}$, ..., proceed and finally chaotic states are obtained. At $B_0 \fallingdotseq 0.106$ the chaotic state abruptly changes into (3, 0)-periodic points. This transition is caused by the appearance of heteroclinic points. The (3, 0)-periodic points disappear at $B_0 \fallingdotseq 0.111$ by the tangent bifurcation and the state jumps to (1, 1)-fixed point $S_1^{1,1}$ whose $\bar{y} = 1$. By decreasing B_0, $S_1^{1,1}$ remains to exist up to the boundary curve $G_1^{1,1}$ in Fig. 1. From this results, we see that the average $\bar{y}$ is equal to zero under the period doubling bifurcation as expected from PROPERTY 2.

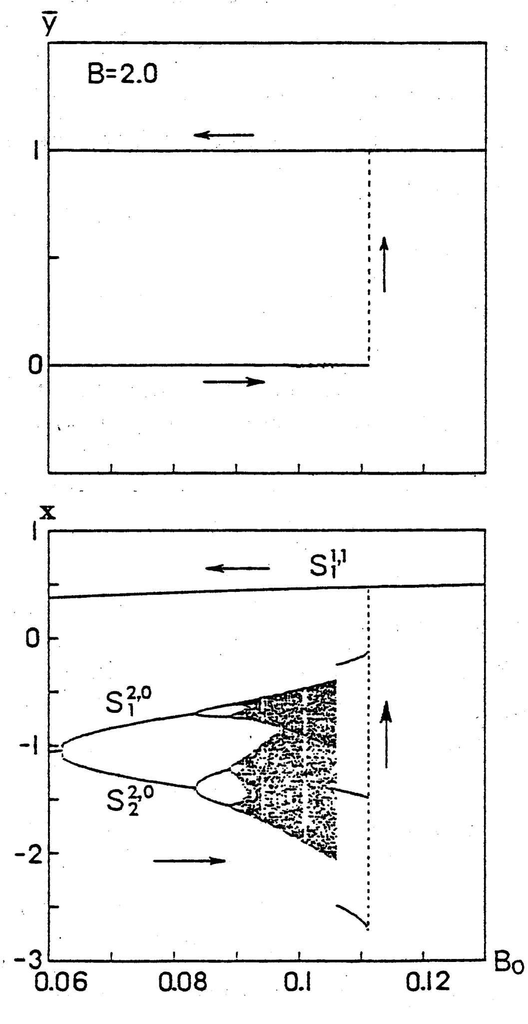

Fig. 3. Phase diagram and $\bar{y}$ obtained by numerical analysis.

Even if a state transition takes place from chaotic to periodic, $\bar{y}$ is still invariant if the final state has the same winding number as the original one.

Similarly, jumps and hysteresis behaviors and the invariance property of $\bar{y}$ are observed when B is fixed and B_0 is changed. Two numerical examples are illustrated in Fig. 4. In the figure ICC denotes the average (17) for the quasi-periodic solution of Eqs. (2). For $\min(1, B) \ll B_0$, the quasi-periodic solution is approximately calculated as $y(t) = B_0/k$, $x(t) = x_0 + B_0 t/k$. Hence $\bar{y} = B_0/k$. Figure 4 shows a satisfactory agreement with the approximation of ICC. Note that in Fig. 4(b), we see fractional steps of $\bar{y}$ in $\bar{y} \in (4, 5)$. Under certain variation of parameter, irregular change of $\bar{y}$ is also observed by transitions between chaotic states. They are interesting problems left for further investigation.

Finally, we append the bifurcation diagrams of (1, L)-fixed points for k = 0.2 and ν = 1.0 in Fig. 5.

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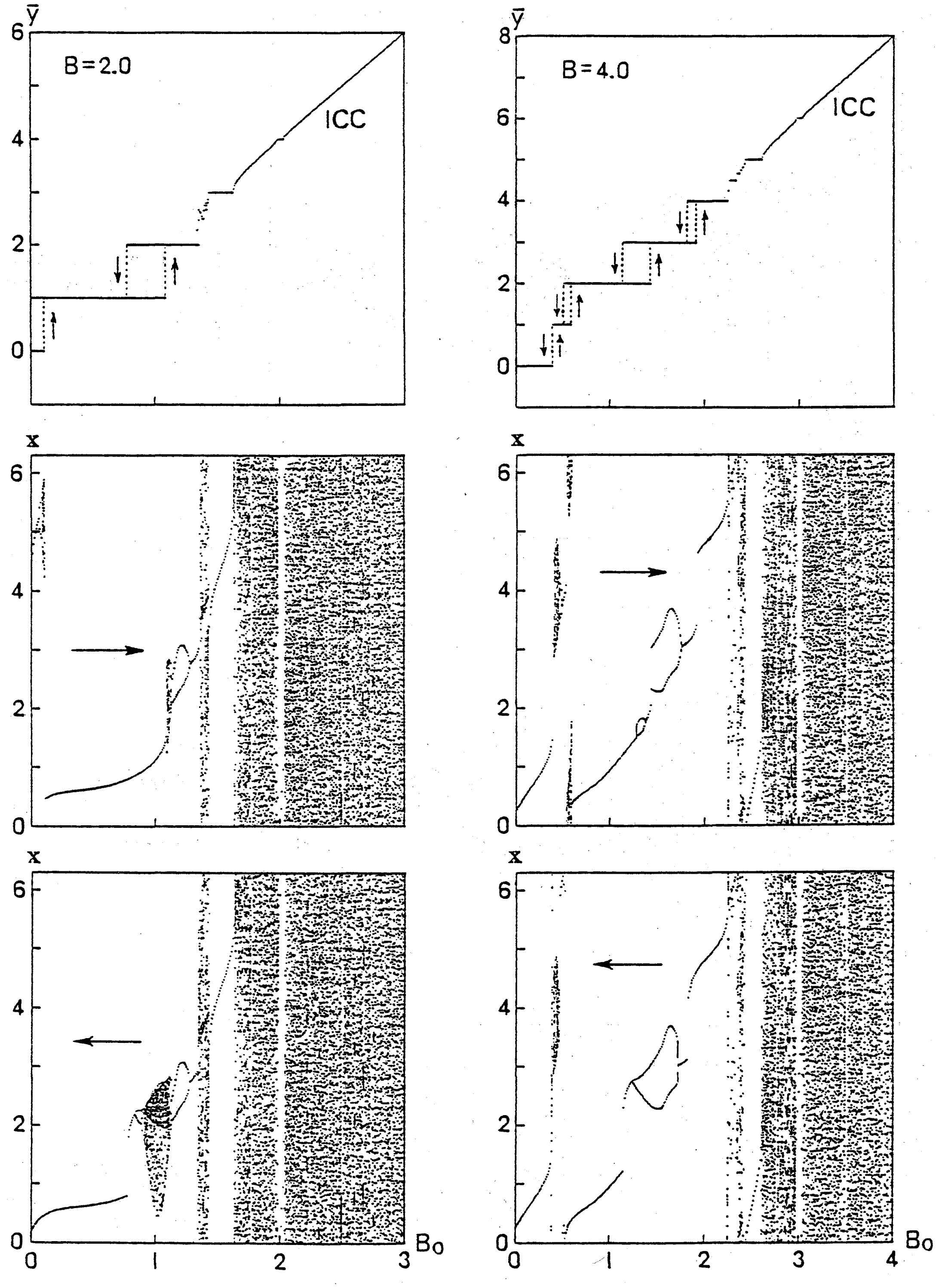
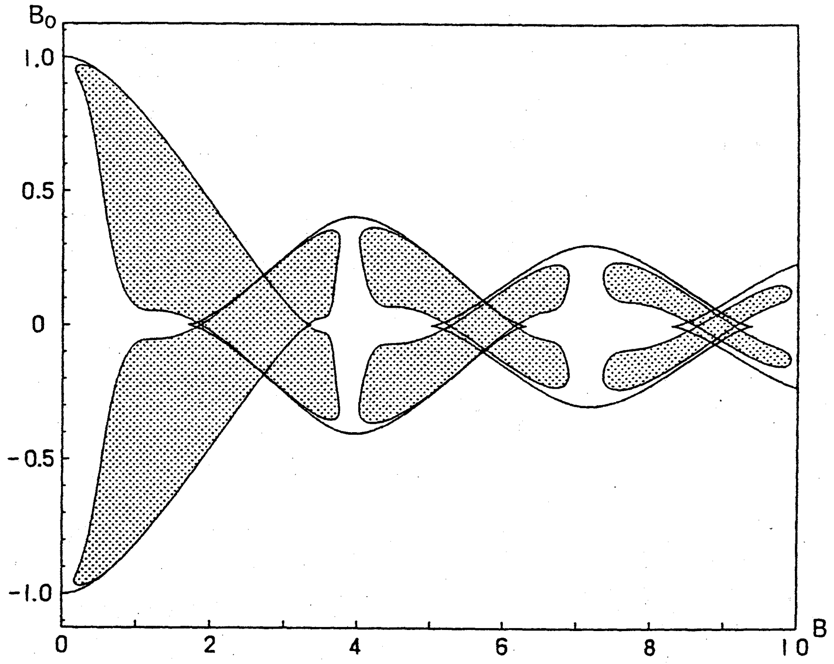
2) V. Belykh, N. Pedersen and O. Soerensen, Phys. Rev. B16, 4860, 1977.
3) H. Kawakami, IEEE Trans. Circuits Syst., CAS-31, 248, 1984.



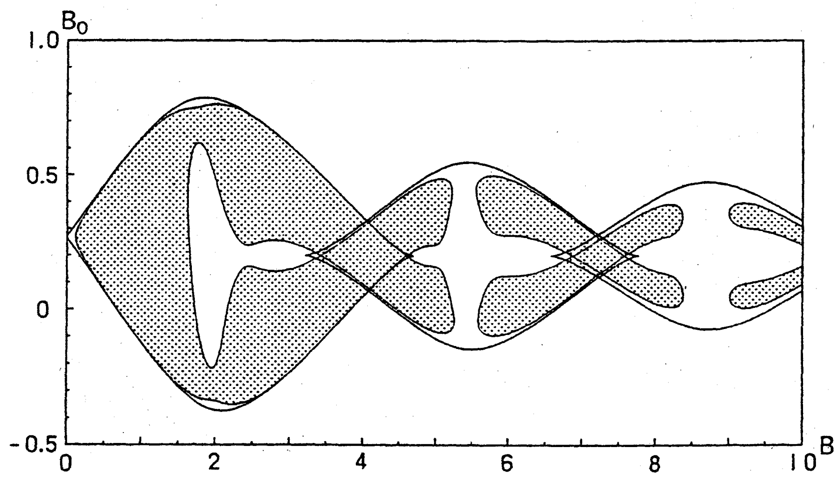
(a) k = 0.5, B = 2.0 (b) k = 0.5, B = 4.0

Fig. 4. Phase diagram of x and $\overline{y}$. The arrow indicates the direction for variation of B_0.



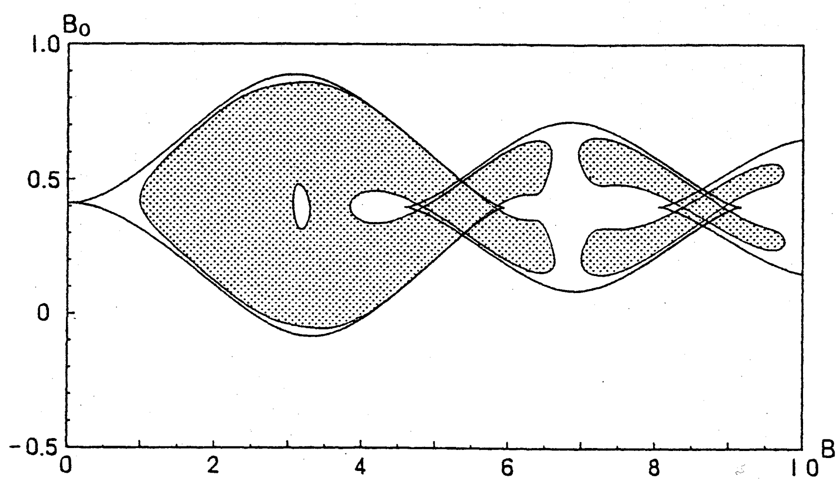
(a)

Bifurcation sets for (1, 0)-fixed points.



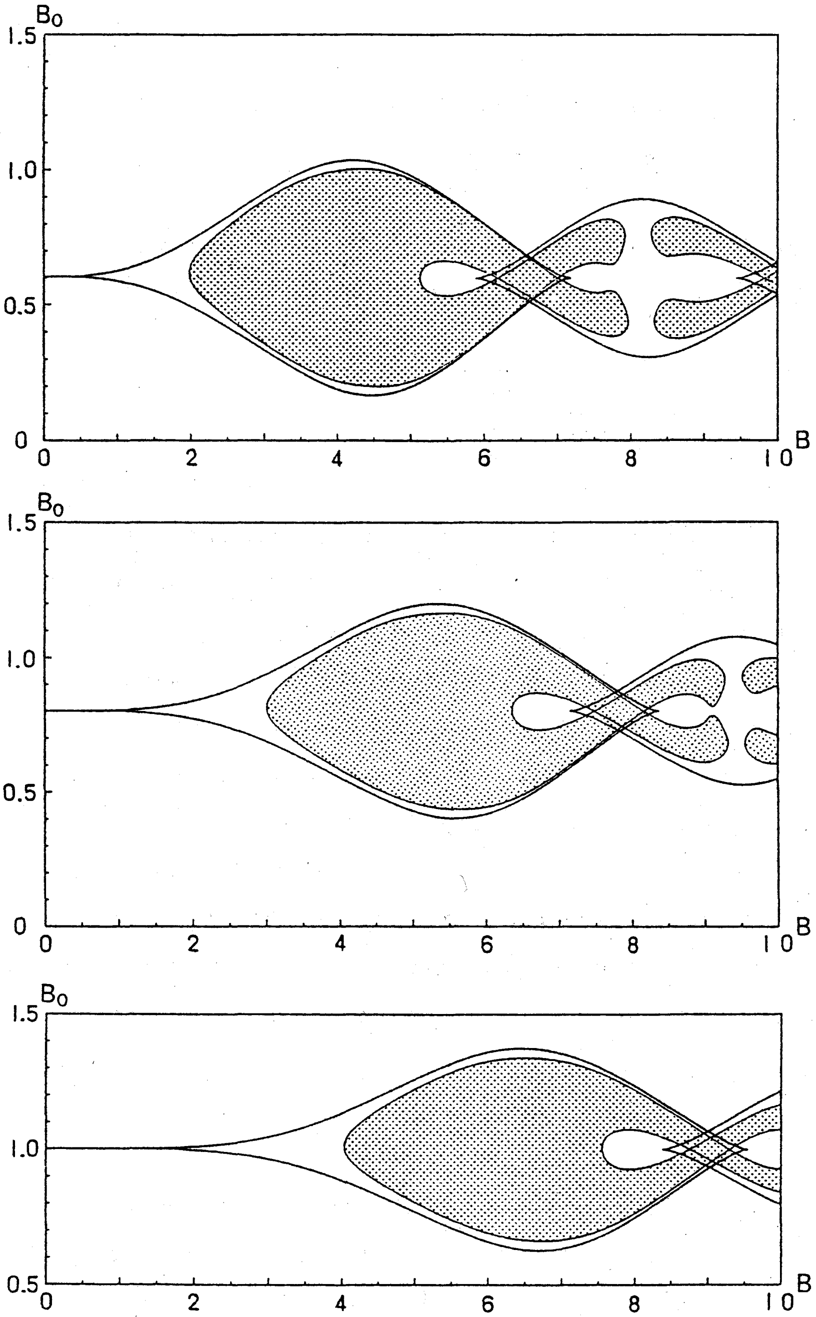
(b)

Bifurcation sets for (1, 1)-fixed points.



(c)

Bifurcation sets for (1, 2)-fixed points.



(d)

Bifurcation sets for (1, 3)-fixed points.

(e)

Bifurcation sets for (1, 4)-fixed points.

(f)

Bifurcation sets for (1, 5)-fixed points.

Fig. 5. Bifurcation diagrams for (1, L)-fixed point of T. $k = 0.2$, $\nu = 1.0$. Outer boundary curve denotes the set for tangent bifurcation. Boundary curve of shaded portion indicates the set for period doubling bifurcation; see Fig. 1.