

## Invariant subspace problemについて

北大理 林 実樹広 (Mikihiro Hayashi)

C. J. Read の結果「non trivial to invariant subspace をもたない Banach 空間上の有界線型作用素が存在する」について紹介する。これはすでに [1] に発表されているが、同論文の末尾に「 $\ell^1$  空間上にもこのような作用素  $T$  が作られる」とある。この改良された結果を述べたノートが手に入、たので講演でもこれをもとに紹介を行な、た。

以下は、このノートをもとにしたタイプである。(オリジナルのノートは手書きのため読み難くか、た) なお、正確に欠けていると思われる点は、筆者の理解する範囲で修正し、[Note] と因を補、た。また、内容の順序も一部並べ変、た。このため、最終的に Read に F、て発表されるものは別の形にな、てい、ることと思われる。なおこのノートは伊藤隆司先生(武蔵工大)から越先生(北大)を通じて入手した。記してお礼を述べたい。

C. Read's construction of an operator on the Banach space  $\ell^1$  having no non-trivial invariant subspace.

(Based on a seminar given by Read in Edinburgh on 7/5/84)

Let  $\mathcal{P}$  denote the complex vector space of all polynomials in a single variable  $x$  with complex coefficients, and let  $\mathcal{P}_n$  denote the subspace of polynomials of degree  $\leq n$ , so that  $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$ . We let  $p_n$  denote the natural projection of  $\mathcal{P}$  onto  $\mathcal{P}_n$ , so that  $p_n(\sum a_k x^k) = \sum a_k x^k$ . We also define a norm  $\|\cdot\|_*$  on  $\mathcal{P}$  by  $\|\sum a_k x^k\|_* = \sum |a_k|$ .

Lemma 1. Let  $n$  be a positive integer and let  $\varepsilon, \delta, M > 0$ . Then, there exists  $K = K(\varepsilon, \delta, M, n) > 0$  such that if  $0 \leq m \leq n$  and  $g \in \mathcal{P}_n$  with  $\|g\|_* \leq M$  and  $\|p_m g\|_* \geq \delta$  then we can find  $q \in \mathcal{P}_n$  with  $\|p_n(qg) - x^m\|_* < \varepsilon$  and  $\|q\|_* \leq K$ .

Proof. If  $g = \sum_{j=k}^n \alpha_j x^j$  and  $\alpha_k \neq 0$ , then considering a linear combination of  $g, p_n(xg), \dots, p_n(x^n g)$  we find a polynomial  $q \in \mathcal{P}_n$  with  $p_n(qg) = x^m$  for  $k \leq m \leq n$ . Since the set  $\{g \in \mathcal{P}_n : \|g\|_* \leq M, \|p_m g\|_* \geq \delta\}$  is compact, we need only a finite number of  $q_1, \dots, q_\ell$  to have  $\|p_n(q_j g) - x^m\|_* < \varepsilon$  for some  $q_j$ . So,  $K = \max_j \|q_j\|_*$  has the desired property.

Let  $1 < a_1 < b_1 < a_2 < b_2 \dots$  be a sequence of positive integers which are required to increase rapidly; in the

statements of subsequent lemmas the condition "provided the sequence  $a_1, b_1, \dots$  increases fast enough" will be understood.

We put  $a_0 = b_0 = 1$  and let  $v_1 = v_0 = 0$ ,  $v_n = (n-1)(a_n + b_n)$  for  $n > 1$ .

We define a basis  $f_0, f_1, \dots$  of  $\mathcal{P}$  as follows:

$$f_0 = 1;$$

If  $n = r = 1$ , or  $n \geq 2$  and  $1 \leq r \leq n-1$ , then

(A) for  $ra_n \leq k \leq ra_n + v_{n-r-1}$

$$f_k = a_{n-r} (x^k - x^{k-a_n});$$

(B) for  $a_1 + b_1 < k < a_2$ ,  $(r-1)a_n + v_{n-r} < k < ra_n$  ( $n \neq 2$ )

$$f_k = 2^{[(r - \frac{1}{2})a_n - k]/b_{n-1}} \cdot x^k;$$

(C) for  $k = a_1 + b_1$ ,  $r(a_n + b_n) \leq k \leq (n-1)a_n + rb_n$  ( $n \geq 2$ )

$$f_k = x^k - b_n x^{k-b_n};$$

(D) for  $a_1 < k < a_1 + b_1$ ,  $(n-1)a_n + (r-1)b_n < k < r(a_n + b_n)$  ( $n \geq 2$ )

$$f_k = 2^{[(r - \frac{1}{2})b_n - k]/a_n} \cdot x^k.$$

For given  $n \geq 1$ , formulae (A) - (D) define  $f_k$  for  $v_{n-1} < k < v_n$ . Note that the polynomial  $f_k$  always has a non-zero  $x^k$  term and no terms of higher order; hence  $f_0, \dots, f_n$  span  $\mathcal{P}_n$  for each  $n$ . (See Fig. 1).

We define a norm  $\|\cdot\|_x$  on  $\mathcal{P}$  by  $\|\sum \alpha_k f_k\|_x = \sum |\alpha_k|$  and let  $X$  be the completion of  $\mathcal{P}$  with respect to this norm. So,

$X$  is a Banach space isomorphic to  $\ell^1$ . We also define a linear operator  $T : \mathcal{P} \rightarrow \mathcal{P}$  by  $Tx^k = x^{k+1}$ , i.e.,  $T$  is multiplication by  $x$ . Note that  $p(T)q = pq$  for polynomials  $p, q$ .

We note two consequences of the definition of the norm  $\|\cdot\|_x$ . If  $1 \leq r \leq n-1$  and  $ra_n \leq k \leq ra_n + v_{n-r-1}$ , then from (A) we get

$$x^k - x^{k-ra_n} = a_{n-r}^{-1} f_k + a_{n-r+1}^{-1} f_{k-a_n} + \dots + a_{n-1}^{-1} f_{k-(r-1)a_n}$$

So

$$\|x^k - x^{k-ra_n}\|_x = a_{n-r}^{-1} + \dots + a_{n-1}^{-1} \leq 2/a_{n-r} \quad \dots(1)$$

assuming  $\{a_n\}$  increases fast enough. Similarly, from (C) we get, for  $r(a_n + b_n) \leq k \leq (n-1)a_n + rb_n$ ,

$$\|x^k - b_n^r x^{k-rb_n}\|_x = 1 + b_n + \dots + b_n^{r-1} \leq 2b_n^{r-1} \quad \dots(2)$$

[Note: (1) and (2) show the corresponding finite dimensional subspaces are nearly isometric each other with respect to the coordinate  $x^k$  and the norm  $\|\cdot\|_x$ . See Fig. 2].

Lemma 2.  $\|Tf_k\|_x \leq 2$  for all  $k$ .

From Lemma 2 it follows that  $T$  extends by continuity to a bounded linear operator on  $X$ , with  $\|T\|_x \leq 2$ .

For  $m > 1$ , we define  $\sigma_m = \{k \in \mathbb{N} : \text{for some } n > m \text{ we have } (n-m)a_n \leq k \leq (n-m)a_n + v_{m-1}\}$ . [Note: In other words,  $\sigma_m = \bigcup_{n>m} \{k \in (A) : \text{with } n = n \text{ and } r = n-m \text{ in the definition of } f_k\}$ ].

Lemma 3. Suppose  $m > 2$ ,  $k > (m-1)a_m$  and  $b_m + a_m \leq s \leq b_m + (m-1)a_m$ . Then,

- (a) if  $k \notin \sigma_m$ , we have  $\|T^s f_k\|_X \leq 4$ .  
 (b) if  $k \in \sigma_m$  then, writing  $k = (n-m)a_n + j$ ,  $0 \leq j \leq v_{m-1}$ , we have  $\|T^s f_k + a_m x^{j+s}\|_X \leq 1$ .

We now define a linear mapping  $Q_m : \mathcal{P} \rightarrow \mathcal{P}_{(m-1)a_m}$  for  $m > 2$  [see Fig. 3.3 for  $Q_m$  with  $m = k$ ] by

$$Q_m(f_k) = \begin{cases} f_k & \text{if } 0 \leq k \leq (m-1)a_m \\ 0 & \text{if } k > (m-1)a_m, k \notin \sigma_m \\ -a_m x^j & \text{if } k \in \sigma_m \text{ (i.e., } k = (n-m)a_n + j, j \leq v_{m-1}). \end{cases}$$

The conclusion of Lemma 3 can be restated as follows:

$$m > 2, b_m + a_m \leq s \leq b_m + (m-1)a_m, g \in \mathcal{P} \\ \Rightarrow \|T^s g - T^s Q_m(g)\|_X < 4 \|g\|_X \dots (6)$$

In fact, to prove (6) it suffices to prove it for  $g = f_k$ ; this is trivial if  $k \leq (m-1)a_m$  and follows from Lemma 3 for  $k > (m-1)a_m$ .

We can find  $C_m \geq a_m$ , depending only on  $a_1, b_1, \dots, a_m$ , such that  $\|f_j\|_* \leq C_m$  and  $\|x^j\|_X \leq C_m$  for  $j = 0, \dots, (m-1)a_m$ . Then, by the definition of  $Q_m$ , we have  $\|Q_m(f_k)\|_X \leq a_m C_m$  for all  $k$ , so  $\|Q_m g\|_X \leq a_m C_m \|g\|_X$  for  $g \in \mathcal{P}$ , and so  $Q_m$  extends continuously to  $X$  and (6) holds for  $g \in X$ .

Note also that  $\|Q_m(f_k)\|_* \leq \max(a_m, C_m) \leq C_m$ , so

$$\|Q_m(g)\|_* \leq C_m \|g\|_X, \quad g \in X.$$

Lemma 4. Let  $g \in X$  with  $\|g\|_X = 1$ . Suppose that for some  $m > 2$  and  $1 \leq r < m-2$  we have  $\|p_{ra_m} Q_m(g)\|_* \geq 1/a_m$ . Then, there is a polynomial  $\phi$  with  $\|\phi(T)g - 1\|_X < 3/a_{m-r-1}$ .

THEOREM. Let  $Y$  be a closed subspace of  $X$  with  $TY \subseteq Y$ . Then, either  $Y = \{0\}$  or  $Y = X$ .

Proof of Lemma 2. We consider  $k$  belonging to the four types (A), (B), (C), (D), separately. In the course of the proof we note estimate (3), (4), (5) below for future use.

Case (A):  $ra_n \leq k \leq ra_n + v_{n-r-1}$

In this case  $Tf_k = a_{n-r}(x^{k+1} - x^{k+1-a_n})$ . Now, if  $k < ra_n + v_{n-r-1}$ , we get similarly  $Tf_k = f_{k+1}$ , so  $\|Tf_k\|_X = 1$ . This leaves the case  $k = ra_n + v_{n-r-1}$ . If  $n = r = 1$  or  $n=2$  and  $r=1$ , then  $k+1 = a_n+1 \in (D)$  and  $k+1-a_n = 1 \in (B)$ , so

$$\|x^{k+1}\|_X = 2^{(a_n+1 - \frac{1}{2}b_n)/a_n} < 1/b_n; \text{ and}$$

$$\|x^{k+1-a_n}\|_X = 2^{(1 - \frac{1}{2}a_1)} < 1/a_1^2.$$

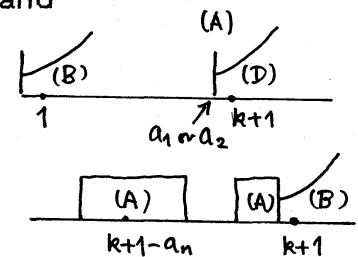
So,  $\|Tf_k\|_X \leq 1/a_{n-r}$ .

If  $n > 2$  and  $r < n-1$ , then  $k+1 \in (B)$ , so

$$\|x^{k+1}\|_X = 2^{(v_{n-r-1}+1 - \frac{1}{2}a_n)/b_{n-1}} < 1/a_n$$

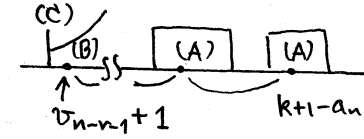
and if  $n > 2$  and  $r = n-1$ , then  $k+1 \in (D)$  and  $k+1-a_n \in (B)$ , so

$$\|x^{k+1}\|_X = 2^{[(n-1)a_n+1 - \frac{1}{2}b_n]/na_n} < 1/b_n.$$



$$\|x^{k+1-a_n}\|_x = 2^{(-\frac{1}{2}a_n+1)/b_{n-1}} < 1/a_n$$

Also, if  $\begin{cases} r < n-1 \\ n > 2 \end{cases}$ , then  $v_{n-r-1}+1 \in (B)$ , and using (1),

$$\begin{aligned} \|x^{k+1-a_n}\|_x &= \|x^{(r-1)a_n+v_{n-r-1}+1}\|_x \\ &\leq 2/a_{n-r+1} + \|x^{v_{n-r-1}+1}\|_x \\ &= 2/a_{n-r+1} + 2^{(v_{n-r-1}+1-\frac{1}{2}a_{n-r})/b_{n-r-1}} \end{aligned}$$


[Note: The above estimate is true even if  $r=1$ ]. So

$$\begin{aligned} \|Tf_{ra_n+v_{n-r-1}}\|_x &\leq a_{n-r} (4/a_{n-r+1} + 2^{(v_{n-r-1}+1-\frac{1}{2}a_{n-r})/b_{n-r-1}}) \\ &\leq 1/a_{n-r} \dots\dots\dots(3) \end{aligned}$$

provided  $\{a_n, b_n\}$  increase fast enough. So, in case (A),

$$\|Tf_k\|_x \leq 1 \text{ always.}$$

Case (B):  $(r-1)a_n + v_{n-r} < k < ra_n$

In this case  $Tf_k = 2^{[(r-\frac{1}{2})a_n - k]/b_{n-1}} x^{k+1}$ .

Now if  $k < ra_n - 1$ , we get  $Tf_k = 2^{1/b_{n-1}} f_{k+1}$ , so

$$\|Tf_k\|_x = 2^{1/b_{n-1}} < 2.$$

This leaves the case  $k = ra_n - 1$ , then  $Tf_k = 2^{(1-\frac{1}{2}a_n)/b_{n-1}} x^{ra_n}$

and  $\|x^{ra_n}\|_x \leq 1 + 2/a_{n-r}$  by (1), so provided  $a_n$  is large enough,

$$\|Tf_{ra_n-1}\|_x \leq 1/a_n \dots\dots\dots(4)$$

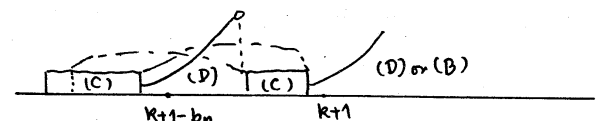
The same proof works for  $a_1 + b_1 < k < a_2$  with  $n = 2$  and  $r = 1$ .

Case (C) :  $r(a_n + b_n) \leq k \leq (n-1)a_n + rb_n$

In this case  $Tf_k = x^{k+1} - b_n x^{k+1-b_n}$ .

If  $k < (n-1)a_n + rb_n$ , then  $Tf_k = f_{k+1}$  so  $\|Tf_k\|_x = 1$ .

This leaves the cases  $k = a_1 + b_1$  ( $n = r = 1$ ) and  $k = (n-1)a_n + rb_n$ . If  $r < n-1$ , then by (D)



$$Tf_k = 2^{\lfloor (n-1)a_n + 1 - \frac{1}{2} b_n \rfloor / na_n} (f_{k+1} - b_n f_{k+1-b_n}).$$

If  $r = n-1$  or  $k = a_1 + b_1$ , then  $k+1 \in (B)$  and  $k+1-b_n \in (D)$ , so

$$Tf_k = 2^{k+1 - \frac{1}{2} a_{n+1}} \cdot f_{k+1} - 2^{\lfloor (n-1)a_n + 1 - \frac{1}{2} b_n \rfloor / na_n} \cdot b_n f_{k+1-b_n}$$

or

$$Tf_k = 2^{k+1 - \frac{1}{2} a_2} f_{k+1} - 2^{\lfloor a_1 + 1 - \frac{1}{2} b_1 \rfloor / a_1} b_1 f_{k+1-b_1},$$

respectively. So,

$$\|Tf_k\|_x = (1+b_n) \cdot 2^{\lfloor (n-1)a_n + 1 - \frac{1}{2} b_n \rfloor / na_n} \leq 1$$

if  $b_n$  and  $a_{n+1}$  large enough.

Case (D) :  $(n-1)a_n + (r-1)b_n < k < r(a_n + b_n)$

In this case  $Tf_k = 2^{\lfloor (r-1)b_n - k \rfloor / na_n} x^{k+1}$ .

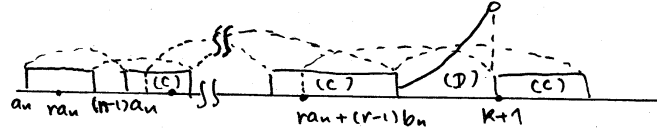
If  $k < r(a_n + b_n) - 1$ , then  $Tf_k = 2^{1/na_n} f_{k+1}$ , so  $\|Tf_k\|_x < 2$ .

This leaves the case  $k = r(a_n + b_n) - 1$ ; then,



$$Tf_k = 2 \frac{(1-ra_n - \frac{1}{2}b_n)/na_n \cdot r(a_n+b_n)}{x}$$

and by (2) and (1) we get



$$\|x^{r(a_n+b_n)}\|_x \leq 2b_n^{r-1} + \|x^{ra_n}\|_x \leq 2b_n^{r-1} + 2/a_{n-r} + 1$$

so if  $b_n$  is large enough, we get

$$\|Tf_k\|_x < 1/b_n \quad \text{when } k = r(a_n+b_n)-1 \quad \dots\dots(5)$$

The same proof works for  $a_1 < k < a_1 + b_1$ . This completes the proof.

### Proof of Lemma 3.

Again we consider separately the cases (A), (B), (C), (D). Case (b) of the statement is covered by case (A)(i) below; the remaining cases cover (a).

Case (A)  $ra_n \leq k \leq ra_n + v_{n-r-1}$

In this case  $Tf_k = a_{n-r} (x^{k+s} - x^{k+s-a_n})$  and

$$a_m + b_m + ra_n \leq k+s \leq (m-1)a_m + b_m + ra_n + v_{n-r-1} \quad \dots(*A)$$

The hypothesis  $k > (m-1)a_m$  implies  $n > m$ . We consider three such cases:

(i)  $r = n - m$  ( $\Leftrightarrow k \in \sigma_m$ ):

Write  $k = ra_n + j$ , where  $0 \leq j \leq v_{m-1}$ ; then

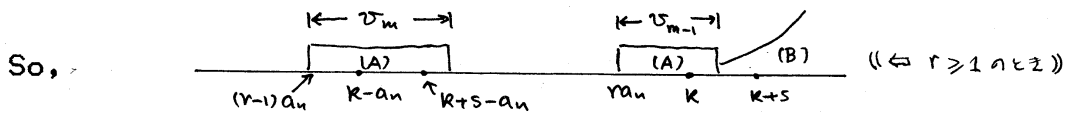
$$b_m < j+s < v_{m-1} + (m-1)a_m + b_m < 2b_m,$$

so  $ra_n + v_{n-r} < k+s < (r+1)a_n$ , i.e.,  $k+s \in (B)$ . We get

$$\|x^{k+s}\|_x = 2^{\lfloor j+s - \frac{1}{2}a_n \rfloor / b_{n-1}} \leq 2^{\lfloor 2b_m - \frac{1}{2}a_n \rfloor / b_{n-1}} \leq 1/a_n$$

if  $a_n$  is large enough. Also, since  $j+s \leq 2b_m < v_m$  (since  $m > 2$ ) [Note:  $(r-1)a_n \leq s+j+(r-1)a_n \leq (r-1)a_n + v_m$ ,  $m = n-(r-1)-1$  ], we get from (1)

$$\|x^{k+s-a_n} - x^{s+j}\|_x = \|x^{s+j+(r-1)a_n} - x^{s+j}\|_x \leq 2/a_{m+1}.$$



$$\|T^s f_k + a_m x^{s+j}\|_x \leq a_m \|x^{k+s}\|_x + a_m \|x^{k+s-a_n} - x^{s+j}\|_x \leq a_m (1/a_n + 2/a_{m+1}) < 1.$$

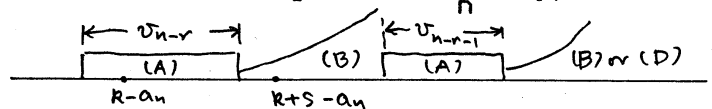
(ii)  $r > n - m$  :

In this case  $s > b_m > v_{n-r}$ , so

$$ra_n + v_{n-r-1} < k+s < (r+1)a_n \text{ and } (r-1)a_n + v_{n-r} < k+s-a_n < ra_n.$$

So,  $k+s \in (B)$  or (D) and  $k+s-a_n \in (B)$ . Writting  $k = ra_n + j$ ,

$0 \leq j \leq v_{n-r-1}$ , we get



$$\|x^{k+s}\|_x = \|x^{k+s-a_n}\|_x = 2^{\lfloor j+s - \frac{1}{2}a_n \rfloor / b_{n-1}} \leq 2^{\lfloor 2b_m - \frac{1}{2}a_n \rfloor / b_{n-1}},$$

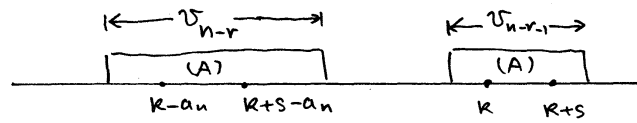
or  $2^{(k+s - \frac{1}{2}b_n) / na_n} \leq 2^{(na_n - \frac{1}{2}b_n) / na_n}$

[Note:  $(r+1 - \frac{1}{2})a_n - (ra_n + j+s) = (r - \frac{1}{2})a_n - (ra_n + j+s - a_n)$  ] since  $j+s \leq v_{n-r-1} + (m-1)a_m + b_m \leq v_{m-1} + (m-1)a_m + b_m < 2b_m$ . So,

$$\|T^s f_k\|_x \leq 2a_{n-r} \cdot 2^{\lfloor 2b_m - \frac{1}{2}a_n \rfloor / b_{n-1}} < 1$$

if  $a_n$  is large enough.

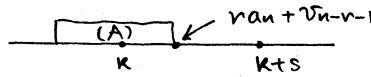
(iii)  $r < n - m$  :



Write  $k = ra_n + j$ ,  $0 \leq j \leq v_{n-r-1}$ .

( $\alpha$ )  $j+s \leq v_{n-r-1}$  : Then [Note:  $k+s \in (A)$ ]  $T^s f_k = f_{k+s}$ , so  $\|T^s f_k\|_x = 1$ .

( $\beta$ )  $j+s > v_{n-r-1}$  : Then [Note:  $T^{v_{n-r-1}-j} f_k = T^{ra_n+v_{n-r-1}-k} f_k = f_{ra_n+v_{n-r-1}}$  as in ( $\alpha$ ), since  $ra_n+v_{n-r-1} \in (A)$ ]  $T^s f_k = T^{j+s-v_{n-r-1}} T f_{ra_n+v_{n-r-1}}$ . Using (3) and  $|T|_x \leq 2$ , we get



$$\|T^s f_k\|_x \leq 2^{j+s-v_{n-r-1}-1} / a_{n-r} < 2^s / a_{n-r} < 2^{2b_m} / a_{n-r} < 1$$

if  $a_{n-r}$  is large enough.

Case (B):  $(r-1)a_n + v_{n-r} < k < ra_n$

$$a_m + b_m + (r-1)a_n + v_{n-r} < k+s < (m-1)a_m + b_m + ra_n \dots (*B)$$

The hypothesis  $k > (m-1)a_m$  implies  $n > m$ . [Note: Hence,  $n \neq 2$ ].

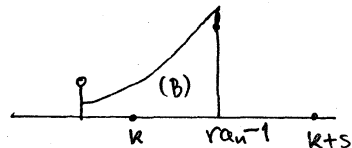
(i)  $k+s < ra_n$  : Then,  $k+s \in (B)$ . Hence,

$$T^s f_k = 2^{[(r-\frac{1}{2})a_n - k]/b_{n-1}} \cdot x^{k+s} = 2^{s/b_{n-1}} f_{k+s}.$$

So,  $\|T^s f_k\|_x = 2^{s/b_{n-1}} \leq 2^{s/b_m} \leq 4$ , since  $s \leq 2b_m$ .

(ii)  $k+s \geq ra_n$  : Since  $ra_n-1 \in (B)$ , as in (i), we have

$$T^s f_k = 2^{(ra_n-1-k)/b_{n-1}} \cdot T^{k+s-ra_n} \cdot T f_{ra_n-1}.$$



So, using (4) and  $\|T\|_x \leq 2$  [Note: and  $ra_n-k \leq s \leq 2b_m$ ], we

get

$$\begin{aligned} \|T^s f_k\|_x &\leq 2^{s/b_{n-1}} \cdot 2^{k+s-ra_n} / a_n \\ &\leq 2^{s/b_m} \cdot 2^s / a_n \leq 4 \cdot 2^{2b_m/a_n} < 1 \end{aligned}$$

if  $a_n$  is large enough, since  $n > m$ .

Case (C):  $r(a_n + b_n) \leq k \leq (n-1)a_n + rb_n$

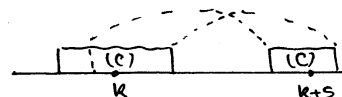
We have  $T^s f_k = x^{k+s} - b_n x^{k+s-b_n}$  and

$$b_m + r(a_n + b_n) \leq k+s \leq (m-1)a_m + b_m + (n-1)a_n + rb_n \dots (*C)$$

The hypothesis implies  $n \geq m$ . We consider separately the cases  $n = m$  and  $n > m$ .

(i)  $n = m$ :

From (\*C),  $k+s \geq (r+1)(a_n + b_n)$ .



(α)  $k+s \leq (n-1)a_n + (r+1)b_n$  (which implies  $r \leq n-2$  and  $k+s \in$

(C) ): We get  $T^s f_k = f_{k+s}$ , so  $\|T^s f_k\|_x = 1$ .

(β)  $k+s > (n-1)a_n + (r-1)b_n$ : Since  $k+s < (r+2)(a_n + b_n)$  from

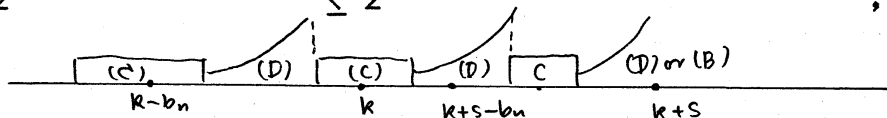
(\*C) [Note: if  $b_n$  is large enough], we have

if  $r < n-2$ , then  $k+s \in (D)$ , so

$$\|x^{k+s}\|_x = 2^{[k+s-(r+\frac{3}{2})b_n]/na_n} \leq 2^{[2(n-1)a_n - \frac{1}{2}b_n]/na_n};$$

if  $r \geq n-2$ , then  $k+s \in (B)$ , so

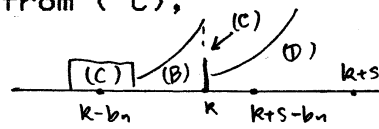
$$\|x^{k+s}\|_x = 2^{[k+s-\frac{1}{2}a_{n+1}]/b_n} \leq 2^{[nb_n+2(n-1)a_n-\frac{1}{2}a_{n+1}]/b_n};$$



similary, since  $k+s \leq 2(n-1)a_n + (n+1)b_n$  from (\*C),

if  $r < n-1$ , then  $k+s-b_n \in (D)$ , so

$$\|x^{k+s-b_n}\|_x = 2^{[k+s-(r+\frac{3}{2})b_n]/na_n} \leq 2^{[2(n-1)a_n - \frac{1}{2}b_n]/na_n};$$



if  $r = n-1$ , then  $k+s-b_n \in (B)$ , so

$$\|x^{k+s-b_n}\|_x = 2^{[k+s-b_n - \frac{1}{2}a_{n+1}]/b_n} \leq 2^{[2(n-1)a_n - \frac{1}{2}a_{n+1}]/b_n}.$$

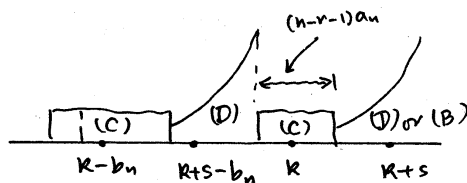
So, if  $b_n, a_{n+1}$  are chosen large enough, we get  $\|T^s f_k\|_x \leq 1$  in each case.

(ii)  $n > m$ :

(α)  $k+s \leq (n-1)a_n + rb_n$ : Then,  $k+s \in (C)$ . So,  $T^s f_k = f_{k+s}$ , and  $\|T^s f_k\|_x = 1$ .

(β)  $k+s > (n-1)a_n + rb_n$ : If  $r < n-1$ , we have  $k+s < (r+1)(a_n + b_n)$ , so  $k+s \in (D)$  and

$$\|x^{k+s}\|_x = 2^{[k+s-(r+\frac{1}{2})b_n]/na_n} \leq 2^{[(n-1)a_n + 2b_m - \frac{1}{2}b_n]/na_n} \leq 1/b_n^2 \quad [\text{Note: by (*C)}]$$



and the same estimate holds for  $\|x^{k+s-b_n}\|_x$ ; while if  $r =$

$n-1$ , then the above estimate holds for  $\|x^{k+s-b_n}\|_x$  and, on the other hand, since  $k+s \in (B)$ ,

$$\|x^{k+s}\|_x = 2^{[k+s - \frac{1}{2}a_{n+1}]/b_n} \leq 2^{[v_n + 2b_m - \frac{1}{2}a_{n+1}]/b_n} < 1/a_{n+1} \quad [\text{Note: by (*C)}].$$

So, in each case we get  $\|T^s f_k\|_x \leq 1$ .

Case (D):  $(n-1)a_n + (r-1)b_n < k < r(a_n + b_n)$

We have  $T^s f_k = 2^{[(r-\frac{1}{2})b_n - k]/na_n} \cdot x^{k+ts}$  and

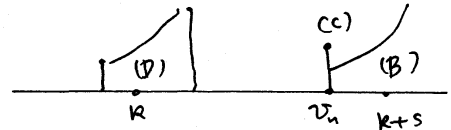
$$a_m + b_m + (n-1)a_n + (r-1)b_n < k+ts < (m-1)a_m + b_m + r(a_n + b_n) \dots (*D)$$

and the hypothesis  $k > (m-1)a_m$  again implies  $n \geq m$  and we consider separately the cases  $n = m$  and  $n > m$ .

(i)  $n = m$  :

From (\*D) we have  $k+ts > (n-1)a_n + rb_n$ . We subdivide this case into four cases (α)-(δ).

(α)  $k+ts > (n-1)(a_n + b_n)$  :



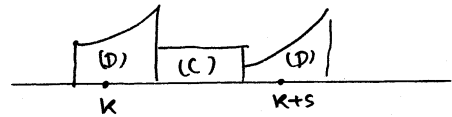
In this case,  $k+ts \in (B)$  and from (\*D)

$$\|x^{k+ts}\|_x = 2^{(k+ts - \frac{1}{2}a_{n+1})/b_n} \leq 2^{[(n+1)b_n - \frac{1}{2}a_{n+1}]/b_n}$$

So,  $\|T^s f_k\|_x \leq 2^{b_n/a_n + [(n+1)b_n - \frac{1}{2}a_{n+1}]/b_n} \leq 1$  if  $a_{n+1}$  is large enough.

In the remaining three cases (β)-(δ), we always assume  $(n-1)a_n + rb_n \leq k+ts \leq (n-1)(a_n + b_n)$ .

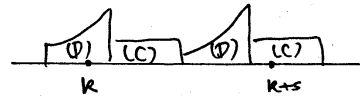
(β)  $k+ts < (r+1)(a_n + b_n)$ ,  $r < n-1$  :



Since  $k+ts \in (D)$ , we have  $\|x^{k+ts}\|_x = 2^{[k+ts - (r+\frac{1}{2})b_n]/na_n}$ ,  
 so  $\|T^s f_k\|_x = 2^{(s-b_n)/na_n} \leq 2$  since  $s-b_n \leq (n-1)a_n$ .

(7)  $(r+1)(a_n + b_n) \leq k+s \leq (n-1)a_n + (r+1)b_n, r < n-1 :$

In this case,  $k+s \in (C)$  and, by (2),



$$\|x^{k+s}\|_x \leq 2b_n^r + b_n^{r+1} \|x^{k+s-(r+1)b_n}\|_x.$$

Now,  $k+s-(r+1)b_n \leq (n-1)a_n$ , so  $\|x^{k+s-(r+1)b_n}\|_x$  is bounded by a function of  $a_n$ , so  $\|x^{s+k}\|_x \leq b_n^{r+2}$  if  $b_n$  is large enough. Also,  $k \geq (r+1)(a_n + b_n) - s \geq rb_n - na_n$ , so

$$\begin{aligned} \|T_k^{s,f}\|_x &\leq 2^{[(r-\frac{1}{2})b_n - k]} \cdot b_n^{r+2} \\ &\leq 2^{(na_n - \frac{1}{2}b_n)/na_n} \cdot b_n^{r+2} \leq 1 \end{aligned}$$

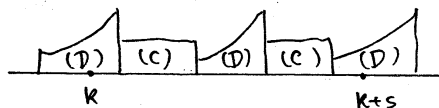
if  $b_n$  is large enough.

(8)  $k+s > (n-1)a_n + (r+1)b_n, r < n-2 :$

Since  $s < 2b_n$ , we have  $k+s < (r+2)(a_n + b_n)$ , i.e.,  $k+s \in$

(D). So,

$$\|x^{k+s}\|_x = 2^{[k+s-(r+\frac{1}{2})b_n]/na_n} \cdot 2^{[k+(n-1)a_n-(r+\frac{1}{2})b_n]/na_n}.$$



So,  $\|T_k^{s,f}\|_x \leq 2^{[(n-1)a_n - b_n]/na_n} \leq 1$  if  $b_n$  is large enough.

(ii)  $n > m$  :



(α)  $k+s < r(a_n + b_n)$  : Then,  $s+k \in (D)$ , so  $x^{k+s} = 2^{[k+s-(r-\frac{1}{2})b_n]/na_n} f_{k+s}$ . So,  $T^s f_k = 2^{s/na_n} f_{k+s}$  and  $\|T^s f_k\|_x = 2^{s/na_n} \leq 2^{2b_m/na_n} \leq 2$  since  $n > m$ .

(β)  $k+s \geq r(a_n + b_n)$  : Using  $\|T\|_x \leq 2$  and the fact that, by (5),  $\|T^s f_{r(a_n+b_n)-1}\|_x \leq 1/b_n$ , we get

$$\|T^s f_k\|_x \leq 2^s/b_n \leq 2^{2b_m}/b_n \leq 1$$

[Note:  $s+k \leq r(a_n + b_n) - 1 + s$ ] if  $b_n$  is large enough, since  $m < n$ . This completes the proof of Lemma 3.

Proof of Lemma 4. Let  $h = Q_m(g)$ , so  $\|h\|_* \leq C_m$ . By Lemma 1, we can find  $q \in P_{(m-2)a_m}$  with

$$\|P_{(m-2)a_m}(qh) - x^{ra_m}\|_* \leq 1/a_m C_m ; \text{ and}$$

$$\|q\|_* \leq K = K(1/a_m C_m, 1/a_m C_m, (m-2)a_m).$$

Let  $\psi = x^{a_m} q \in P_{(m-1)a_m}$ . Then,

$$\|P_{(m-1)a_m}(\psi h) - x^{(r+1)a_m}\|_* \leq 1/a_m C_m ,$$

so

$$\|P_{(m-1)a_m}(\psi h) - x^{(r+1)a_m}\|_x \leq 1/a_m \dots (7)$$

Let  $\phi = b_m^{-1} x^{b_m} \psi$ . Since  $\|x^t - b_m^{-1} x^{t+b_m}\|_x = 1/b_m$  for



$a_m \leq t \leq (m-1)a_m$ , we get

$$\begin{aligned} & \| p_{(m-1)a_m+b_m}(\phi h) - p_{(m-1)a_m}(\phi h) \|_x \\ &= \| b_m^{-1} x^{b_m} p_{(m-1)a_m}(\phi h) - p_{(m-1)a_m}(\phi h) \|_x \\ &\leq \| \phi h \|_* / b_m = \| qh \|_* / b_m \leq KC_m / b_m \leq 1/a_m \dots\dots(8) \end{aligned}$$

provided  $b_m$  is large enough, as  $K$  and  $C_m$  depend only on  $a_1, b_1, \dots, a_m$ . Now the degree of  $\phi h$  does not exceed  $2(m-1)a_m + b_m \leq 2(a_m + b_m)$ , so (D) [Note: the maximum error is at  $k=2(m-1)a_m+b_m$  and we have  $\| \phi h \|_* \leq \| \phi \|_\infty \| h \|_* \leq \| \phi \|_* \| h \|_*$ ] gives

$$\begin{aligned} \| \phi h - p_{(m-1)a_m+b_m}(\phi h) \|_x &\leq 2 \frac{[2(m-1)a_m - \frac{1}{2} b_m] / m a_m}{[2(m-1)a_m - \frac{1}{2} b_m] / a_m} \| \phi h \|_* \\ &\leq 2 \frac{[2(m-1)a_m - \frac{1}{2} b_m] / a_m}{[2(m-1)a_m - \frac{1}{2} b_m] / a_m} \cdot KC_m / b_m \\ &\leq 1/a_m \dots\dots(9) \end{aligned}$$

again if  $b_m$  is large enough. Now apply (6), noting that

$$\phi(x) = \sum_{s=a_m+b_m}^{(m-1)a_m+b_m} \lambda_s x^s$$

where  $\sum |\lambda_s| \leq K/b_m$ , and that  $T^s Q_m(g) = x^s h$  and  $\phi(T)Q_m(g) = \phi h$ , we get

$$\begin{aligned} \| \phi(T)g - \phi h \|_x &= \| \phi(T)g - \phi(T)Q_m(g) \|_x \\ &\leq \sum_{s=a_m+b_m}^{(m-1)a_m+b_m} |\lambda_s| \| T^s g - T^s Q_m(g) \|_x \end{aligned}$$

$$\leq K \cdot 4 \|g\|_x / b_m = 4K/b_m < 1/a_m \dots (10)$$

Now from (1), we get  $\|x^{(r+1)a_m} - 1\|_x \leq 2/a_{m-r-1}$  and combining this with (7), (8), (9), (10) gives

$$\|\phi(T)g - 1\|_x < 2/a_{m-r-1} + 4/a_m < 3/a_{m-r-1}.$$

This proves the lemma.

Proof of THEOREM. Suppose  $Y \neq \{0\}$ . Let  $g \in Y$  with  $\|g\|_x = 1$  and any  $\varepsilon > 0$  be given. To show that  $Y = X$ , it suffices to show that we can find a polynomial  $\phi$  with  $\|\phi(T)g - 1\|_x < \varepsilon$ . We choose  $k > 1$  with  $3/a_k < \varepsilon$ . In view of Lemma 4, it will be sufficient to find  $r \geq 1$  and  $m$  with  $m-r-1 \geq k$  such that  $\|p_{ra_m} Q_m g\|_* \geq 1/a_m$ . So, we suppose on the contrary that

$$r \geq 1, m-r-1 \geq k \implies \|p_{ra_m} Q_m g\|_* < 1/a_m \dots (11)$$

and we shall deduce a contradiction.

For each  $m$  we can find  $D_m > 0$ , depending only on  $a_1, b_1, \dots, b_m$ , such that  $\|x^j\|_x \leq D_m, 0 \leq j \leq v_m$ .

Now we write  $g = \sum_{j=0}^{\infty} \alpha_j f_j$ , where  $\sum |\alpha_j| = \|g\|_x = 1$  and for  $n > 1$  we can write

$$\sum_{j=0}^{(n-1)a_n} \alpha_j f_j = \sum_{j=0}^{(n-1)a_n} \beta_{nj} x^j.$$

We have also, for  $n > 2$ ,

$$Q_n \left( \sum_{j > (n-1)a_n} \alpha_j f_j \right) = \sum_{j=0}^{v_{n-1}} \lambda_{nj} x^j$$

where

$$\lambda_{nj} = -a_n \sum_{m=n+1}^{\infty} \alpha_{j+(m-n)a_m} \dots\dots\dots(12)$$

and so

$$Q_n g = \sum_{j=0}^{(n-1)a_n} \beta_{nj} x^j + \sum_{j=0}^{v_{n-1}} \lambda_{nj} x^j \dots\dots\dots(12)$$

From (11) and (12) we can deduce

$$r \geq 1, m-r-1 \geq k \implies \sum_{j=v_{m-1}+1}^{ra_m} |\beta_{mj}| < 1/a_m \dots\dots(13)$$

From the definition of the polynomial  $f_j$  we can see that if  $ra_m + v_{m-r-2} < j \leq ra_m + v_{m-r-1}$ , then [Note:  $j \in (A)$  and no  $x^j$  term in  $f_\ell$  for  $ra_m + v_{m-r-1} < \ell \leq (m-1)a_m$ ]  $\beta_{mj} = a_{m-r} \alpha_j$ , hence from (13) with  $r$  replaced by  $r+1$  we get

$$r \geq 1, m-r-2 \geq k \implies \sum_{j=ra_m+v_{m-r-2}+1}^{ra_m+v_{m-r-1}} |\alpha_j| < 1/a_{m-r} a_m \dots\dots(14)$$

[Note: (14) covers the shaded area in Fig. 3.1 ] and combining this with (12) we get

$$\begin{aligned} n \geq k+2 \implies \sum_{j=v_{n-2}+1}^{v_{n-1}} |\lambda_{nj}| &< |a_n| \sum_{m=n+1}^{\infty} \sum_j |\alpha_{j+(m-n)a_m}| \\ &< \sum_{m=n+1}^{\infty} 1/a_m < 2/a_{n+1} \dots\dots(15) \end{aligned}$$

From (11) with  $r = 1$  we get, if  $n \geq k+2$ , using (12)

$$\|p_{a_n} Q_n g\|_x < 1/a_n, \text{ so } \sum_{j=0}^{\nu_{n-1}} |\beta_{nj} + \lambda_{nj}| < 1/a_n \dots\dots(16)$$

and from (15) and (16) we get

$$\sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} |\beta_{nj}| < 2/a_n.$$

Now, if we write

$$\sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} \alpha_j f_j = \sum_{j=0}^{\nu_{n-1}} \beta'_{nj} x^j$$

then the correspondence between  $(\alpha_{\nu_{n-2}+1}, \dots, \alpha_{\nu_{n-1}})$  and  $(\beta'_{\nu_{n-2}+1}, \dots, \beta'_{\nu_{n-1}})$  is one-to-one, so there is a constant  $D'_{n-1}$ , depending only on  $a_1, \dots, b_{n-1}$ , such that

$$\sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} |\alpha_j| \leq D'_{n-1} \sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} |\beta'_{nj}|.$$

Now we observe that if  $\nu_{n-1} < j \leq (n-1)a_n$ , then  $f_j$  has no  $x^i$  term for  $\nu_{n-2} < i \leq \nu_{n-1}$ . Hence,  $\beta_j = \beta'_j$  for  $\nu_{n-2} < j \leq \nu_{n-1}$ , whence

$$\sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} |\alpha_j| \leq D'_{n-1} \sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} |\beta_{nj}| < 2D'_{n-1}/a_n < 1/\sqrt{a_n}$$

if  $a_n$  is large enough, as  $D'_{n-1}$  depends only on  $a_1, \dots, b_{n-1}$ . Putting  $m = n-1$ , we conclude that

$$m \geq k+1 \implies \sum_{j=\nu_{m-1}+1}^{\nu_m} |\alpha_j| \leq 1/\sqrt{a_{m+1}} \dots\dots(17)$$

[Note: (17) covers the shaded area in Fig. 3.2 ]. Using (17) in (12) we find

$$\begin{aligned} n \geq k+1 \implies \sum_{j=0}^{\nu_{n-1}} |\lambda_{nj}| &\leq a_n \sum_{m=n+1}^{\infty} \sum_{j=0}^{\nu_{n-1}} |\alpha_{j+(m-n)a_m}| \\ &\leq a_n \sum_{m=n+1}^{\infty} \sum_{j=\nu_{m-1}}^{\nu_m} |\alpha_j| \\ &\leq a_n \sum_{m=n+1}^{(n-1)a_n} 1/\sqrt{a_{m+1}} < 1/a_{n+1} \end{aligned}$$

and combining this with (16) gives

$$\begin{aligned} \sum_{j=0}^{\nu_{n-1}} |\beta_{nj}| &< 2/a_n \text{ for } n \geq k+2 \\ \text{i.e., } \|p_{\nu_{n-1}} \left( \sum_{j=0}^{(n-1)a_n} \alpha_j f_j \right)\|_* &< 2/a_n \dots\dots(18) \end{aligned}$$

Now from (17) we get

$$\sum_{j=\nu_{n-1}+1}^{(n-1)a_n} |\alpha_j| \leq 1/\sqrt{a_{n+1}} < 1/a_n, \quad (n \geq k+1)$$

and noting that if  $\nu_{n-1} < j \leq (n-1)a_n$  then  $\|p_{\nu_{n-1}} f_j\|_* \leq a_{n-1}$  [Note:  $p_{\nu_{n-1}} f_j = 0$  ( $j \in (B)$ ,  $1 \leq r \leq n-1$ ),  $-a_{n-1}$  ( $j \in (A)$ ,  $r = 1$ ),  $0$  ( $j \in (A)$ ,  $2 \leq r \leq n-1$ ) ], we conclude that

$$\|p_{\nu_{n-1}} \left( \sum_{j=\nu_{n-1}+1}^{(n-1)a_n} \alpha_j f_j \right)\|_* \leq \sum_{j=\nu_{n-1}+1}^{(n-1)a_n} |\alpha_j| \|p_{\nu_{n-1}} f_j\|_*$$

$$< a_{n-1}/a_n$$

and combining this with (18) gives

$$\| \sum_{j=0}^{n-1} \alpha_j f_j \|_* < (a_{n-1} + 2)/a_n, \text{ if } n \geq k+2$$

whence

$$\sum_{j=0}^{n-1} |\alpha_j| = \| \sum_{j=0}^{n-1} \alpha_j f_j \|_x < D_{n-1}(a_{n-1} + 2)/a_n,$$

which tends to zero as  $n \rightarrow \infty$  if  $\{a_n\}$  increasing fast enough.

Since  $\sum |\alpha_j| = 1$ , this is the desired contradiction.

#### References

1. C.J. Read, A solution of the invariant subspace problem on a Banach space, Bull. London Math. Soc. 16 (1984), 337-401.

Case	k	Case	k	(左下の続き)
$f_0$	$[ 0$	(C)	$[ v_{n-1}$	(D) $\left[ \begin{matrix} \vdots \\ \vdots \end{matrix} \right.$
(B)	$\left. \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$	(B)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$	(C) $\left[ \begin{matrix} a_n + b_n \\ (n-1)a_n + b_n \end{matrix} \right.$
(A)	$[ a_1$	(A)	$[ a_n$	(D) $\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$
(B)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$	(B)	$[ a_n + v_{n-2}$	(C) $\left[ \begin{matrix} 2(a_n + b_n) \\ (n-1)a_n + 2b_n \end{matrix} \right.$
(C)	$[ a_1 + b_1$	(B)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$	(D) $\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$
(B)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$	(A)	$[ 2a_n$	(C) $\left[ \begin{matrix} 3(a_n + b_n) \\ (n-1)a_n + 3b_n \end{matrix} \right.$
(A)	$[ a_2$	(A)	$[ 2a_n + v_{n-3}$	(D) $\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$
(D)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$		$\bullet$	(C) $\left[ \begin{matrix} (n-4)a_n \\ (n-4)a_n + v_3 \end{matrix} \right.$
(C)	$[ v_2$		$\bullet$	(B) $\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$
(B)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$	(A)	$[ (n-4)a_n$	(C) $\left[ \begin{matrix} (n-3)(a_n + b_n) \\ \vdots \\ (n-1)a_n + (n-3)b_n \end{matrix} \right.$
(A)	$[ a_3$	(B)	$[ (n-4)a_n + v_3$	(D) $\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$
(B)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$	(A)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$	(C) $\left[ \begin{matrix} (n-2)(a_n + b_n) \\ (n-1)a_n + (n-2)b_n \end{matrix} \right.$
(A)	$[ 2a_3$	(A)	$[ (n-3)a_n$	(D) $\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$
(D)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$	(B)	$[ (n-3)a_n + v_2$	(C) $\left[ \begin{matrix} (n-2)(a_n + b_n) \\ (n-1)a_n + (n-2)b_n \end{matrix} \right.$
(C)	$\left[ \begin{matrix} a_3 + b_3 \\ \vdots \\ 2a_3 + b_3 \end{matrix} \right.$	(C)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$	(D) $\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$
(D)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$	(A)	$[ (n-2)a_n$	(C) $[ v_n$
(C)	$[ v_3$	(C)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$	
(B)	$\left( \begin{matrix} \vdots \\ \vdots \end{matrix} \right)$			

Fig. 1

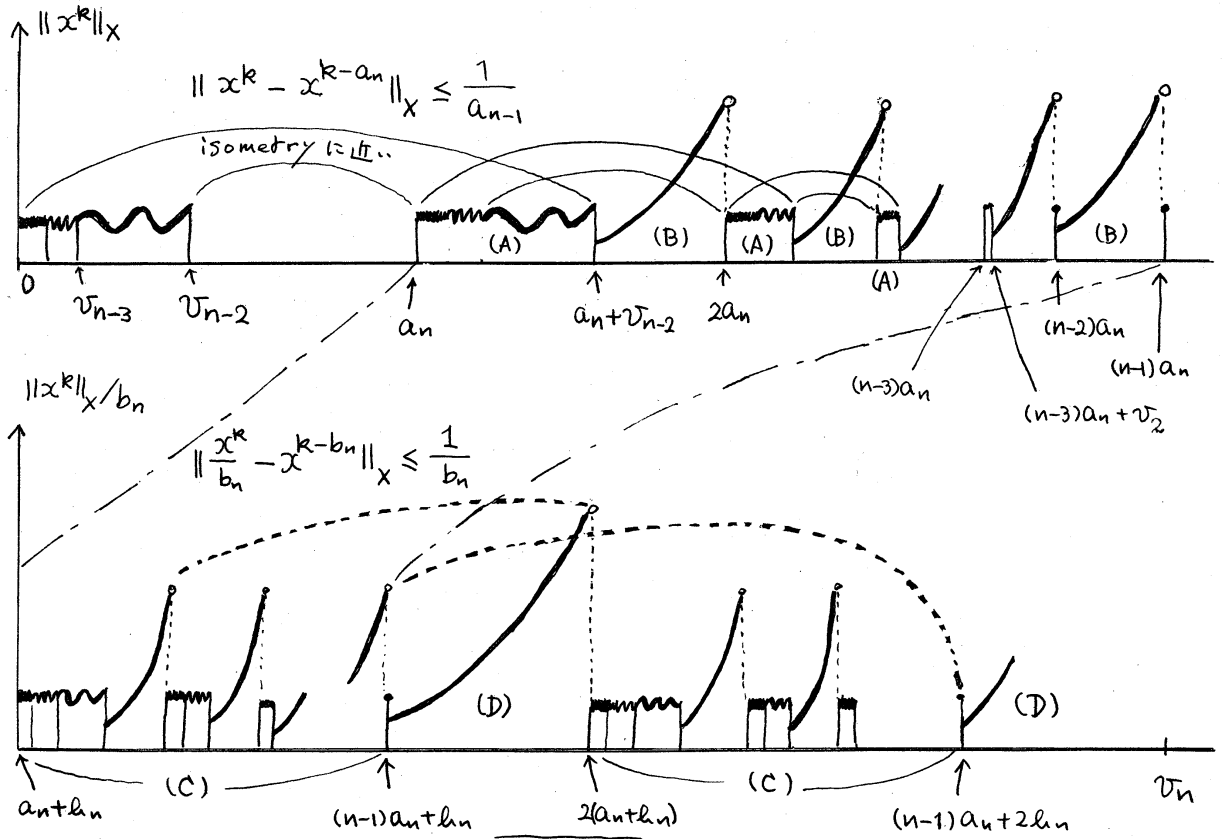


Fig. 2

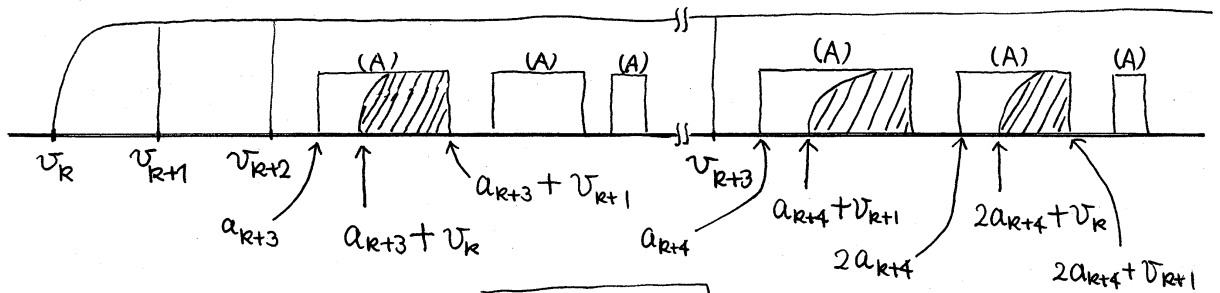


Fig. 3. 1

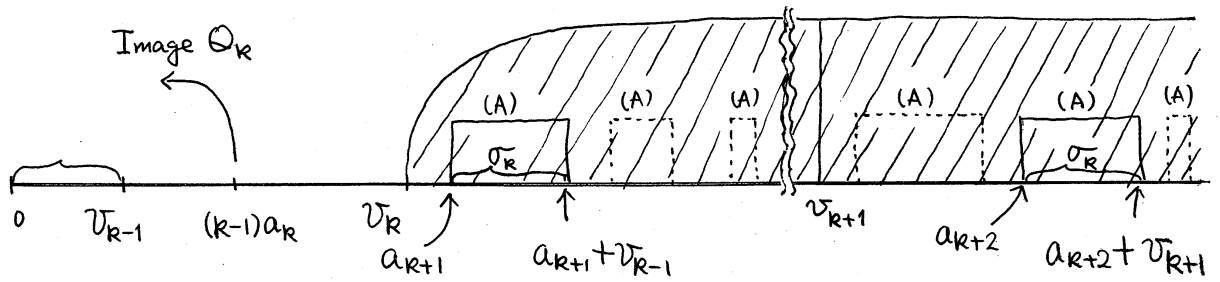


Fig. 3. 2