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LOCALLY COERCIVE NONLINEAR EQUATIONS

TOSIO KATO

In this lecture we consider abstract nonlinear equations of the form

(1) $Au = f,$

(2a) $(A+\xi)u = f$, (2b) $(1+\kappa A)u = f,$

(3) $\partial u/\partial t + Au = f, \quad t \geqq 0, \quad u(0) = \phi,$

where A is a certain nonlinear opertor. The two equations in (2) are slightly different, in that the same expression f appears on the right-hend side, while the left hand sides differ by a factor of $\kappa = 1/\xi$.

In what follows we shall give some general theorems that are convenient to solve the equations (1) to (3). For this purpose we introduce the usual triplet of real Banach spaces

(4) $V \subset H \subset V^*$

in which H is a Hilbert space with the inner product $(\ |\ )_H$ and the norm $\|\ \|_H$, and V , reflexive and separable, is densely and comtinuously embedded in H . $\langle\ ,\ \rangle$ denotes the pairing between V and V^*. (4) implies that $\langle v,h\rangle = (v|h)_H$ whenever $v \in V$ and $h \in H$. (In problem (1), we may replace H by another pair of Banach spaces $Y \subset Y^*$, but we shall not go into such a generalization here.)

STANDING ASSUMPTION. A is a sequentially weakly continuous map of H into V^*. (In other words, $h_n \underset{w}{\to} h$ in H implies Ah_n

Ah in V^*, where $\underset{w}{\rightarrow}$ denotes weak convergence.)

THEOREM I. Assume that

5) $\langle v, Av \rangle \geqq \beta \geqq 0$ for $v \in V$ with $\|v\|_H = r > 0$.

iven any $f \in H$ with $\|f\|_H \leqq \beta/r$, there is a solution $u \in H$ of

1) with $\|f\|_H \leqq r$.

THEOREM II. Assume that

6) $\langle v, Av \rangle \geqq - \;(\|v\|_H^2)$ for $v \in V$,

/here is a continuous function on $\mathbb{R}_+$ to $\mathbb{R}$. Given any f

I , there is a solution $u \in H$ of (2b) if $|\kappa|$ is sufficiently

small (depending on f).

THEOREM III. Assume (6). Given any $\phi \in H$, there is $T > 0$ and a solution $u \in C_w([0,T];H)$ of (3). (Here C_w indicates weak continuity.) T may be any number such that the ordinary differential equation

(7) $d\rho/dt = 2\mathfrak{g}(\rho)$, $\mathfrak{g}(0) = \|\phi\|_H^2$,

has a solution ρ on $t \in [0,T]$; then the solution u satisfies the inequality

(8) $\|u(t)\| \leqq \rho(t)$, $t \in [0,T]$.

(If the solution to (7) is not unique, let ρ be its maximal solution.)

REMARKS. (a) In these theorems, H is the basic space; V and V^* are auxiliary and may be chosen more or less arbitrarily. By choosing V small (which makes V^* large), the STANDING ASSUMPTION and the conditions (5), (6) are more likely to be satisfied. But V must be dense in H.

(b) It is not assumed that A maps V into H . If this happens, however, the left member of (5) and (6) can be replaced by $(v|Av)_H$ and we can forget V^*.

(c) There is in general no uniqueness in these theorems. But uniqneness may be proved under some additional assumptions. For Theorm I, such a condition is

(9) $$(v-w|Av-Aw)_{H'} \geqq \delta\|v-w\|^2 \quad \text{for} \quad v,\ w \in K,$$

where H' is a Hilbert space such that $V^* \subset H'$ and δ is a positive constant. For Theorems II and III, a corresponding condition is

(10) $$(v-w|Av-Aw)_{H'} \geqq -\psi(\|v\|_H+\|w\|_H)\|v-w\|^2_{H'}, \quad \text{for} \quad v,\ w \in V,$$

where ψ is a function on $\mathbb{R}_+$ to $\mathbb{R}$.

(d) For the proof of these theorems, see [1].

APPLICATIONS. 1. Periodic solutions. Consider a nonlinear equation

(11) $$a(x,u,\partial_1 u,\ldots,\partial_m u) = f(x), \quad x \in T^m,$$

where T^m is the m-dimensional torus, the unknown u is a real-valued function on T^m, and $\partial_j = \partial/\partial x_j$. a is a sufficiently smooth, real-valued function of its arguments, which we denote by $x,\ u,\ p_1,\ldots,p_m$. To describe the assumptions on a, we introduce the notations

(12) $$a_{x_j} = \partial u/\partial x_j, \quad a_u = \partial a/\partial u, \quad a_{x_j p_k} = \partial^2 a/\partial x_j \partial p_k, \text{ etc.,}$$

(13) $$a^0(x) = a(x,0,\ldots,0), \quad a^0_u(x) = a_u(x,0,\ldots,0), \quad \text{etc.}$$

We now assume

(14) $$a^0(x) = 0 \quad \text{identically,}$$

(15) $$a^{00}(x) \equiv a_u^0(x) - \frac{1}{2}\sum_{j=1}^{m} a^0_{x_j p_j}(x) \geqq 2\gamma ,$$

(16) $$a^{00}(x)|\xi|^2 + s \sum_{j,k=1}^{m} a^0_{x_j p_k}(x)\xi_j\xi_k \geqq 2\gamma|\xi|^2 \quad \text{for} \quad \xi = (\xi_j) \in \mathbb{R}^m ,$$

where γ is a positive constant and s is an integer such that

(17) $$s \geqq [m/2] + 3 .$$

If we choose $H = H^s = H^s(T^m)$, $V = H^{s+1}$, $V^* = H^{s-1}$, with

(18) $$(u|v)_H = (\Lambda^s u|\Lambda^s v)_0 + \lambda^2(u|v)_0 ,$$

where $(\ |\ )_0$ is the L^2-inner product. If the constant λ is chosen sufficiently large, it can be shown that condition (5) is satisfied with certain constants β and $r > 0$. Thus it follows from Theorem I that (11) has a solution $u \in H^s$ if $f \in H^s$ with sufficiently small $\|f\|_s$.

Moreover, the solution u is unique, since (9) is seen to hold with $H' = H^{s-1}$ or H^0. Thus we have obtained a partial refinement of a result of Moser [2]. (It is expected that these results can be extended to symmetric systems of the form (11).)

2. Another type of nonlinear equations (see Rabinowitz [3])

(19) $$(1-\Delta)u + \kappa b(x,u,\partial u,\partial^2 u,\partial^3 u) = f , \qquad x \in T^m ,$$

can be handled in the same way, in which κ is a small parameter, b is a smooth function of its arguments, and $\partial^r u$ denotes the aggregate of all the derivatives of u of order r , etc. (19) can be written

(20) $$u + \kappa A u = g = (1-\Delta)^{-1} f , \qquad A u = (1-\Delta)^{-1} b(x,u,\ldots).$$

Theorem II is aplicable to (20) with $H = H^s$, $V = H^{s+1}$, $V^* = H^{s-1}$, provided

(21) $s \geqq [m/2] + 5$.

Indeed, it is not difficult to show that (6) is satisfied. It follows that (19) is solvable for any $f \in H^s$, with $u \in H^s$, if $|\kappa|$ is sufficiently small. There is no restriction on b except smoothness. The solution u is unique; this can be proved by verifying (10) with $H' = H^{s-1}$ (or $H' = H^0$).

3. An equation of evolution

(22) $\partial_t u + a(x,u,\partial u) = 0$, $t \geqq 0$, $x \in \mathbb{R}^m$,

where a is a smooth real-valued function of its arguments. If we set $Au = a(x,u,\partial u)$, Theorem II is applicable with $H = H^s = H^s(\mathbb{R}^m)$, $V = H^{s+1}$, $V^* = H^{s-1}$ with s satisfying (17). Thus (22) has a short-time solution with $u(t) \in H^s$ for any initial value $u(0) = \phi \in H^s$.

4. The Euler equation in $\Omega \subset \mathbb{R}^m$. Theorem II can be applied to the Euler equation. For details see [4].

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