数理解析研究所講究録
第545巻 1985年 127-137

Quasilinear Positive Symmetric Systems and Mixed PDEs

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§1. Introduction

The theory on linear positive symmetric system is a well-developed topic[1,2,3] and is very powerful in the study of boundary value problems for varies kinds of linear PDEs, especially, for the linear mixed PDEs[4,5]. It is interesting to extend the theory to the quasilinear case. In this talk we consider quasilinear positive symmetric systems as the perturbation of linear systems. Using the energy estimates for linear systems we establish an existence theorem which is an improvement with some modification of a previous result in [6] .

The application to quasilinear mixed equations is also considered. For the quasilinear case the hyperbolic region and the elliptic region cannot be identified before the solution is obtained. This is the reason why a theory beyond the classical classification of PDEs is desirable.

In §2 we give a brief sketch for the linear theory which is the basis of the quasilinear theory. In §3 we state the main theorem together with the ideas of its proof. §4 is devoted to the application to quasi-linear equations of 2nd order.

The author would like to express his hearty thanks to Prof. H. Fyjita, the organizer of this Symposium, for the hospitalities.

§2. A sketch of linear theory

Let

$$Lu = A^i(x)\frac{\partial u}{\partial x^i} + B(x)u = f(x) \tag{1}$$

be a linear system of PDEs, defined on $\Omega \subset \mathbb{R}^n$. Here u and f are valued in a vector space $\mathbb{R}^m$, $A^i(x)$ and $B(x)$ are $m \times m$ matrices, being sufficiently smooth on $\bar{\Omega}$.

If $A^i(x)$ are symmetric, system (1) is called symmetric. Moreover if the matrix

$$C = B(x) + B^*(x) - \frac{\partial A^i(x)}{\partial x^i} > 0 \tag{2}$$

(1) is called positive symmetric.

Suppose that the boundary of Ω, denoted by $\partial\Omega$, is sufficiently smooth. Let n_i be the outer normal to $\partial\Omega$ and

$$\beta = n_i A^i. \tag{3}$$

Assume that $\partial\Omega$ is noncharacteristic, i.e. $\det \beta \neq 0$. A homogeneous boundary condition can be expressed as

$$u(x)\Big|_{\partial\Omega} \in \mu(x) \subset \mathbb{R}^m \tag{4}$$

where $\mu(x)$ is a sufficiently smooth field of subspace defined on $\partial\Omega$. (4) is called stably admissible if

(i) The restriction of the quadratic form $u \cdot \beta u$ on μ is positive definite;

(ii) dim μ = the number of positive eigenvalues of β.

A sufficiently smooth vector field

$$X = d^i(x)\frac{\partial}{\partial x^i} \tag{5}$$

on $\bar{\Omega}$ is called silted if $n_i(x)\, d^i(x) = 0$ holds on $\partial\Omega$. A set of silted vector field

$$X_\alpha = d_\alpha^i(x)\frac{\partial}{\partial x^i} \qquad (\alpha = 1, 2, \cdots, N) \tag{6}$$

is called complete if any silted vector field X can be expressed as s linear combination of X_α.

Let $I = X_0$ be the unit metrix and consider X_α as the product of X_α in (6) and I. We can write the commutator $[X_\alpha, L]$ in

$$[X_\alpha, L] = p_\alpha^\beta X_\beta + p_\alpha^0 I + t_\alpha L. \tag{7}$$

Let

$$\underset{1}{u} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix} \tag{8}$$

Here $u_0 = u$ and $u_\alpha = X_\alpha u$, satisfying

$$\begin{aligned} &L u_0 = f, \\ &L u_\alpha + p_\alpha^\beta u_\beta + p_\alpha^0 u_0 = (X_\alpha - t_\alpha) f. \end{aligned} \tag{9}$$

(9) is called the first derived system of (1) and denoted by

$$\underset{1}{L}\,\underset{1}{u} = \underset{1}{f}. \tag{10}$$

In a similar way the kth derived systems (k=2,3,...) can be defined successively.

If we write the boundary condition in

$$M(x)\, u = 0, \tag{11}$$

the 1st derived boundary condition is

$$\underset{1}{M}\,\underset{1}{u} = 0 \tag{12}$$

with

$$\underset{1}{M} = \begin{bmatrix} M & 0 & \cdots & 0 \\ \Sigma_1 M & M & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \Sigma_N M & 0 & \cdots & M \end{bmatrix}. \tag{13}$$

We can define the kth derived boundary condition in similar way. We have

<u>Theorem 2.1</u>. If (i) there exists a complete system of silted vector fields such that the kth derived system is positive and symmetric, (ii) the boundary condition is stably admissible, then the boundary value problem (1), (4) admits an H^k strong solution in the Sobolev space $H^k(\Omega)$ uniquely.

The existence and uniqueness of the strong solution in L_2, H^1 was proved firstly by K. O. Friedrichs in his famous paper [1]. For the H^k differentiability see [3].

Theorem 2.1 holds true if the boundary $\partial\Omega$ admits "exceptional corners"[1] such that at least on one side of each corner the matrix β is non-positive or non-negative. Besides, on the part of $\partial\Omega$ where β is non-positive or non-negative Σ_α do not need to be silted.

§3. Existence of solutions to quasilinear systems

Let

$$A^i(x,u)\frac{\partial u}{\partial x^i} + B(x,u)u = \epsilon f(x,u) \tag{14}$$

be a quasilinear symmetric system defined on Ω. Here ϵ is a small parameter. We consider solutions of (14) near some reference function u^0. Without loss of generalities we suppose that $u^0 = 0$. The boundary condition is still linear and homogeneous, being denoted by (4).

Let $\sigma = [\frac{n}{2}]+1$, $s = 2\sigma + \rho$ ($\rho \geq 1$). The linearized approximation of (14) is

$$L(u)U = A^i(x,u)\frac{\partial U}{\partial x^i} + B(x,u)U = \epsilon f(x,u) \tag{15}$$

In particular

$$L(0)U = A^i(x,0)\frac{\partial U}{\partial x^i} + B(x,0)U = \epsilon f(x,0). \tag{16}$$

We have

<u>Theorem 3.1</u>. If (i) the (s+1)th derived system of linear system (16) is positive and symmetric, (ii) the boundary condition (4) is stably admissible with respect to $L(0)$, then there exists a constant $\epsilon_0 > 0$ such that for each ϵ with $|\epsilon| < \epsilon_0$, the boundary value problem (14), (4) admits a solution $u \in H^s(\Omega)$. The ideas of the proof is as follows.

Let

$$K_\eta = \{ u \in H^{s+1}(\Omega) ; \ \|u\|_{s+1} < \eta \} \tag{17}$$

It is seen that if η (>0) is sufficiently small, for each $u \in K_\eta$ the linearized boundary value problem (15), (4) adimits a strong solution $U = T(u) \in H^{s+1}$ uniquely.

There exists an energy estimate

$$\| T(u) \|_{s+1} \leq |\epsilon| C_1(\eta), \tag{18}$$

where $C_1(\eta)$ is a positive function defined in $[0, \eta_0)$ ($\eta_0 > 0$) and is indepent of u, provided that $u \in K_\eta$.

Further we have

$$\| T(u_1) - T(u_2) \|_s \leq |\epsilon| C_2(\eta) \| u_1 - u_2 \|_s \tag{19}$$

for all $u_1, u_2 \in K_\eta$. Here $C_2(\eta)$ is independent of u_1, u_2, provided that $u_1, u_2 \in K_\eta$.

Let Σ_η be the closure of K_η in the space $H^s(\Omega)$. From (19) and (18) it is seen that if $|\epsilon|$ is sufficiently small, $T(u)$ can be extended as a continuous map from Σ_η to itself. Besides if $|\epsilon|$ is small enough the extended map is contractive. Hence there is a fixed point which is nothing else than a solution to the boundary value problem (14), (4). It is also seen that the solution in is unique.

Remark. From the simplest example

$$u = \epsilon f(u) \tag{20}$$

where $f(u)$ is nonlinear, e.g. $f(u) = 1 + (u-1)^2$, it is seen that the condition $|\epsilon| < \epsilon_0$ in Theorem 3.1 cannot be removed. Besides, the global uniqueness does not hold in general.

From this theorem we see that the problem of existence of solutions to the quasilinear case is reduced to considering the linearized problem.

The theorem can be used to prove

(i) The existence of periodic solutions to the system

$$\frac{\partial u}{\partial t} + A^i(t,x,u)\frac{\partial u}{\partial x^i} + B(t,x,u)u = \epsilon f(t,x,u) \tag{21}$$

with stably admissible boundary condition. Here A^i, B and f are periodic functions, having a common period ω.

(ii) The existence and asymptotic stability of the static solution to the equation

$$\frac{\partial u}{\partial t} + A^i(x,u)\frac{\partial u}{\partial x^i} + B(x,u)u = \epsilon f(x,u) \tag{22}$$

with stably admissible boundary condition.

However, we shall consider the application to mixed PDEs only.

It is also noticed that under some conditions the theorem holds for the case that the boundary $\partial\Omega$ is characteristic.

If Ω is unbounded, the similar result holds. In particular, when $\Omega = \mathbb{R}^n$ the problem has been discussed in [7].

§4. Quasilinear mixed PDEs

Let

$$h^{ij}(x,\phi,\partial\phi)\frac{\partial^2\phi}{\partial x^i\partial x^j}+p^i(x,\phi,\partial\phi)\frac{\partial\phi}{\partial x^i}+g(x,\phi,\partial\phi)\phi=\epsilon f(x,\phi,\partial\phi) \quad (23)$$

be a 2nd order quasilinear PDE defined on a bounded and closed region

Suppose that the linearized equation

$$\overset{\circ}{h}{}^{ij}(x)\frac{\partial^2\phi}{\partial x^i\partial x^j}+\overset{\circ}{p}{}^i(x)\frac{\partial\phi}{\partial x^i}+\overset{\circ}{g}(x)\phi=\epsilon\overset{\circ}{f}(x) \quad (24)$$

is a mixed equation in the sense that (n-1) eigenvalues of the matrix $(\overset{\circ}{h}{}^{ij})$ are always positive on $\bar{\Omega}$, whereas the other one changes its sign in $\bar{\Omega}$. Here

$$\overset{\circ}{h}{}^{ij}(x)=h^{ij}(x,0,0) \quad \text{etc.} \quad (25)$$

We can write

$$\overset{\circ}{h}{}^{ij}(x)=\overset{\circ}{a}{}^{ij}(x)-\sigma^i(x)\sigma^j(x) \quad (26)$$

such that $\overset{\circ}{A}=[\overset{\circ}{a}{}^{ij}]$ is positive definite on .

Set

$$\begin{aligned} u_0 &= \lambda\phi-\sigma^i\frac{\partial\phi}{\partial x^i}, \\ u_i &= \frac{\partial\phi}{\partial x^i} \end{aligned} \quad (27)$$

Here $\lambda \neq 0$ is a constant.

We obtain a symmetric system

$$A^i \frac{\partial u}{\partial x^i} + B u = \epsilon F \tag{28}$$

Here

$$u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}, \qquad F = \begin{bmatrix} -f \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$A^i = - \begin{bmatrix} \sigma^i & a^{1i} & \cdots & a^{ni} \\ a^{1i} & & & \\ \vdots & & \sigma^i A & \\ a^{ni} & & & \end{bmatrix}, \tag{29}$$

$$B = - \begin{bmatrix} \frac{g}{\lambda} & \cdots & p^i - (\lambda - \frac{g}{\lambda})\sigma^i + \sigma^j \frac{\partial \sigma^i}{\partial x^j} & \cdots \\ 0 & & & \\ \vdots & & -\lambda a^{ij} + a^{i\ell} \frac{\partial \sigma^j}{\partial x^\ell} & \\ 0 & & & \end{bmatrix},$$

$$A = [a^{ij}] = [h^{ij} + \sigma^i \sigma^j].$$

For the linearized system

$$A^i(x,0) \frac{\partial u}{\partial x^i} + B(x,0)\, u = \epsilon F(x,0) \tag{30}$$

the matrix

$$\beta = n_i \underset{0}{A}^i = - \begin{bmatrix} n_0 & \underset{0}{a}^{1i} n_i & \cdots & \underset{0}{a}^{ni} n_i \\ \underset{0}{a}^{1i} n_i & & & \\ \vdots & & n_0 \underset{0}{A} & \\ \underset{0}{a}^{ni} n_i & & & \end{bmatrix} \tag{31}$$

with $n_0 = \sigma^i n_i$. Let $N^2 = \underset{0}{a}^{ij}\sigma_i\sigma_j$ and suppose that the $\partial\Omega$ is noncharacteristic. The stably admissible boundary condition can be listed as

C	n_0	$1-N^2/n_0^2$	boundary conditions	
+	+	+	$u_i = u_0 = 0$ or $\phi = \partial\phi/\partial n = 0$	(α)
+	+	−	$u_i + (n_i/n_0 - \pi_i)u_0 = 0$ with $a^{ij}\pi_i\pi_j < N^2/n_0^2 - 1$	(β)
+	−	+	/	(γ)
+	−	−	$u_0 = \Sigma\pi_i(u_i + (n_i/n_0)u_0)$ with $[a^{ij}] > [(N^2/n_0^2 - 1)\pi_i\pi_j]$	(δ)
−	+	+	(γ)	
−	+	−	(δ)	
−	−	+	(α)	
−	−	−	(β)	

(32)

Here

$$\underset{0}{C} = \underset{0}{B} + \underset{0}{B}^* - \frac{\partial \underset{0}{A}^i}{\partial x^i} \tag{33}$$

Using Theorem 3.1 we see that if the s+1 th derived system for the linearized system

$$A^i(x,0)\frac{\partial u}{\partial x^i} + B(x,0)\,u = \epsilon F(x,0) \tag{34}$$

is positive (or negative) symmetric and $|\epsilon|$ is sufficiently small then the boundary value problem (28), (32) admits a solution $u \in H^s$.

We need another additional condition: the boundary condition and the part of (28) which is obtained by differentiating the first equation of (27) imply the second set equations of (27).

Then, we obtain the existence of the solution $\phi \in H^{s+1}$ to the boundary value problem (23), (32).

As a special case we suppose that

$$\overset{\circ}{h}{}^{ij}(x,0,0) = \delta^{ij} - x^i x^j, \quad p^i = 2a x^i, \quad q = -a(a+1).$$

The linearized equation is

$$(\delta^{ij} - x^i x^j)\frac{\partial^2 \phi}{\partial x^i \partial x^j} + 2a x^i \frac{\partial \phi}{\partial x^i} - a(a+1)\phi = \epsilon f(x)$$

Let Ω be a region with smooth boundary and tagential planes to $\partial\Omega$ do not meet the unit sphere. For $a > -\frac{n}{2} + s + 1$ the equation (23) and the boundary condition

$$\phi\Big|_{\partial\Omega} = \frac{\partial \phi}{\partial n}\Big|_{\partial\Omega} = 0$$

admit an H^{s+1} solution. For $a < -\frac{n}{2}$ the equation (23) admits a solution in H^{s+1} and the solution is unique if its H^{s+1} norm is sufficiently small.

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