数理解析研究所講究録
第 546 巻 1985 年 51-62

Differential Operators and Congruences for Siegel Modular Forms of Degree Two

東工大·理 佐藤 孝和 (Takakazu Satoh)

1. Introduction

We study congruences between Siegel modular forms of different weight by using differential operators in case of degree two. As an example, we have the following congruence between eigenvalues of Hecke operators T(m) on $\chi_{20}^{(3)}$ (see §3 for definition) and on the Eisenstein series $[\Delta_{18}]$ attached to Δ_{18}:

$$\lambda(m,\chi_{20}^{(3)}) \equiv m^2\lambda(m,[\Delta_{18}]) \mod 7,$$

which was conjectured in Kurokawa [7]. In case of classical elliptic modular forms, such congruences were studied by Serre [12] and Swinnerton-Dyer [14]. We denote by $M_k(\Gamma_n)$ (resp. $S_k(\Gamma_n)$, $M_k^\infty(\Gamma_n)$) the **C**-vector space of holomorphic Siegel modular forms (resp. holomorphic cusp forms, C^∞-modular forms) of degree n and weight k. Let H_n be the Siegel upper half plane of degree n and $Z=\left(z_{jk}\right)$ be a variable on H_n. We put $Y = \frac{1}{2i}(Z-\bar{Z})$. For $f\in M_k^\infty(\Gamma_n)$, let $f(Z)=\sum a(T,Y,f)q^T$ be its Fourier expansion where $q^T=\exp(2\pi i Tr(TZ))$ and T runs over all half-integral matrix. If f is holomorphic, we write $a(T,Y,f)$ as $a(T,f)$ in short since it is independent from Y. Usually $f\in M_k^{\infty}(\Gamma_n)$ is written in the form $f(Z)=\sum a'(T,Y,f)e^{2\pi i\operatorname{Tr}(TX)}$, but it is convenient for our purpose to write as the former. For a subring R of $\mathbf{C}$, we denote by $M_k(\Gamma_n)_R$ the R-submodule of $M_k(\Gamma_n)$, whose element has Fourier expansion with coefficients in R. Further we put $S_k(\Gamma_n)_R = M_k(\Gamma_n)_R \cap S_k(\Gamma_n)$.

We remark that there remains much to be done to obtain systematic results as the elliptic modular case treated by Serre [12] and Swinnerton-Dyer [14], including the study of ℓ-adic representations attached to Siegel modular forms.

2. Statement of results

We put $\frac{d}{dZ} = \left(c_{jk}\frac{\partial}{\partial z_{jk}}\right)$ with $c_{jk} = 1$ for $j=k$ and $c_{jk} = \frac{1}{2}$ for $j\neq k$. For integers $r\geq 0$ and k, we define differential operators on a C^{∞}-function $f(Z)$ on H_n by

$$\delta_{k,n}f=|Y|^{-k+\frac{n-1}{2}}\left|\frac{d}{dZ}\right|\left(|Y|^{k-\frac{n-1}{2}}f\right)$$

and

$$\delta_{k,n}^{r}=\delta_{k+2r-2,n}\cdots\delta_{k+2,n}\delta_{k,n}.$$

We shall omit subscript n when there is no ambiguity. We understand that δ_k^0 is the identity operator. These differential operators were studied by Maass [8]. By Harris [4, 1.5.3], δ_k^r maps $M_k^{\infty}(\Gamma_n)$ to $M_{k+2r}^{\infty}(\Gamma_n)$. If f is a non-zero eigen function of <u>all</u> Hecke operators, we call f an eigen form. For each integer $m \geq 1$, $T(m): M_k(\Gamma_n) \to M_k(\Gamma_n)$ denotes the m-th Hecke operator. We denote the eigenvalue of $T(m)$ by $\lambda(m,f)$. The operator δ has the following properties.

(1) Let $f \in M_k(\Gamma_n)$ be a non-zero holomorphic modular form. Then $\delta_k^r f \neq 0$ for $2k \geq n$ and $\delta_k f = 0$ for $2k<n$. Note that f is necessarily a singular form in the latter case by Resnikoff [10, Theorem 6.4].

(2) If $f \in M_k^\infty(\Gamma_n)$ is an eigenform and $\delta_k^r f \neq 0$, then $\delta_k^r f$ is also an eigen form. As for eigenvalues, we have $\lambda(m,\delta_k^r f) = m^{nr}\lambda(m,f)$. (The same is true for arbitrary double cosets.)

We note a proof of the following congruences:

$$\lambda(m,\chi_{10}) \equiv m^2\lambda(m,\varphi_8) \mod 5 \tag{2.1}$$

and

$$\lambda(m,\chi_{14}) \equiv m^2\lambda(m,\chi_{12}) \mod 23. \tag{2.2}$$

Here we use usual notation for special elements of $M_k(\Gamma_2)$ as follows: $\varphi_k \in M_k(\Gamma_2)$ is (Siegel's) Eisenstein series of weight k. Let $S = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. For $k=10,12,14$, we denote by χ_k the cusp form of weight k normalized as $a\left[S,4\chi_{10}\right] = -1$, $a\left[S,12\chi_{12}\right] = 1$ and $a\left[S,4\chi_{14}\right] = -1$. It is known by Igusa [5, Theorem 1] that $\chi_k \in S_k(\Gamma_2)_{\mathbf{Z}}$ for $k=10$, 12 and 14. Using

$$\delta_{k,2} = -\frac{1}{4}k\left(k-\frac{1}{2}\right)|Y|^{-1}+\frac{1}{2i}\left(k-\frac{1}{2}\right)|Y|^{-1}\mathrm{Tr}\left(Y\frac{d}{dZ}\right)+\left|\frac{d}{dZ}\right|,$$

we obtain

$$-2\varphi_4\tilde{\delta}_4\varphi_4 = 2^9\cdot 3\cdot 7\cdot 4\chi_{10}-7\tilde{\delta}_8\ \frac{1}{15}\varphi_8,$$

$$\frac{23}{33}\ \varphi_6\tilde{\delta}_6\varphi_6-\frac{23}{14}\ \varphi_8\tilde{\delta}_4\varphi_4 = -2^6 3^2\cdot 4\chi_{14}+\frac{2^{10}3^4}{7}\tilde{\delta}_{12}12\chi_{12}-288\tilde{\delta}_{12}[\Delta_{12}]$$

where $\tilde{\delta}_k = (2\pi i)^{-n}\delta_k$ and $[\Delta_{12}]$ is the Eisenstein series attached to the elliptic cusp form Δ_{12}. Noting uniqueness of Fourier coefficients we have

$$-2\sum_{T_1+T_2=T} a(T_1,\varphi_4)|T_2|a(T_2,\varphi_4) = 2^9\cdot 3\cdot 7a(T,4\chi_{10})-7|T|a\left(T,\frac{1}{15}\varphi_8\right)$$

and

$$23\sum_{T_1+T_2=T}|T_2|\left(\frac{1}{33}a(T_1,\varphi_6)a(T_2,\varphi_6)-\frac{1}{14}a(T_1,\varphi_8)a(T_2,\varphi_4)\right)$$

$$= -2^6 3^2 a(T,4\chi_{14})+\frac{2^{10}3^4}{7}|T|a(T,12\chi_{12})-288|T|a(T,[\Delta_{12}]).$$

We note congruences

$$7a(T,[\Delta_{12}]) \equiv 0 \mod 23 \quad \text{for } T>0 \text{ and } 23\nmid|2T|$$

which is proved by Böcherer [2, Satz 5(a)]. Here we note that $7a(T,[\Delta_{12}]) \in \mathbf{Z}$ for all $T>0$. In addition, by Maass [9, Satz 1], we see that

$$a(T,4\chi_{10}) \equiv |T|a\left(T,\frac{1}{15}\varphi_8\right) \mod 5,$$

$$a(T,4\chi_{14}) \equiv 14|T|a(T,12\chi_{12}) \mod 23,$$

for all $T>0$. Noting explicit action of Hecke operators on Fourier coefficients of holomorphic modular forms obtained by Andrianov [1, (2.1.11)], we have (2.1) and (2.2).

Up to now, the lack of knowledge on non-holomorphic modular forms prevents the author from proving further congruences by the above method. To avoid this difficulty, we make use of holomorphic projection defined by Sturm [13]. We put

$$U = \left\{ X = \left(x_{jk}\right) \in M(n,\mathbf{R}) \;\middle|\; X = {}^tX,\ -\frac{1}{2} \le x_{jk} \le \frac{1}{2} \text{ for } 1\le j,k\le n \right\},$$

$$V = \left\{ Y = \left(y_{jk}\right) \in M(n,\mathbf{R}) \;\middle|\; Y = {}^tY,\ Y>0 \right\}$$

and $dX = \prod_{j\le k} dx_{jk}$, $dY = \prod_{j\le k} dy_{jk}$. We put

$$P_w(f) = \sum_{T>0} P(w,T,a(T,Y,f))q^T,$$

where

$$P(w,T,a(T,Y,f)) = \frac{\int_V a(T,Y,f)e^{-4\pi \mathrm{Tr}(TY)}|Y|^{w-1-n}dY}{\int_V e^{-4\pi \mathrm{Tr}(TY)}|Y|^{w-1-n}dY}$$

and T runs over all half-integral positive definite matrices of size two. Then, $P_w(f)$ belongs to the ring of formal power series $\mathbf{C}[q_{jk},q_{jk}^{-1}][[q_{jj}]]_{1\le j,k\le n}$ where $q_{jk}=\exp(2\pi i z_{jk})$. Assume moreover that $w>2n$ and that f is of bounded growth, namely,

$$\int_U\int_V |f(X+iY)||Y|^{w-1-n}e^{-\rho \mathrm{Tr}(Y)}dYdX < \infty$$

for any positive constant ρ. Then, $P_w(f)$ converges for all $Z\in H_n$ and it is a holomorphic cusp form of weight w. (See Sturm[13, Theorem 1].)

In what follows, we treat degree two case. For complex numbers α and β, we put

$$\varepsilon(\alpha,\beta)=\begin{cases}\alpha(\alpha-1)\dots(\beta+1)\beta & \text{if } \alpha-\beta \text{ is a non-negative integer,}\\ 1 & \text{otherwise,}\end{cases}$$

and

$$\eta(\alpha,\beta)=\begin{cases}\alpha\left(\alpha-\frac{1}{2}\right)\dots\left(\beta+\frac{1}{2}\right)\beta & \text{if } 2(\alpha-\beta) \text{ is a non-negative integer,}\\ 1 & \text{otherwise.}\end{cases}$$

Theorem 1. Let R be a subring (not necessarily containing 1) of **C** satisfying $\frac{1}{2}R\subset R$. Let $f\in M_k(\Gamma_2)_R$ and $g\in M_\ell(\Gamma_2)_R$ with $k+\ell>4$. Suppose that I is an ideal of R satisfying

(1) $\frac{1}{2}I\subset I$,

(2) $a(T,g)\in I$ for all $T\neq 0$.

Let s be a non-negative integer and t be a positive integer. We put $r=s+t$ and $w=k+\ell+2r$. Then for any positive integer m,

$$(2\pi i)^{-2r}\xi a\left(mE,P_w\left(\delta_k^s f\cdot\delta_\ell^t g\right)\right)-\nu m^{2r}a(mE,fg)$$

belongs to $(2w-2r-3)I$, where $\xi=\varepsilon(w-3,w-r-2)\varepsilon\left(w-\frac{5}{2},w-r-\frac{3}{2}\right)$ and $\nu=\eta\left(k+s-1,k-\frac{1}{2}\right)\eta\left(\ell+t-1,\ell-\frac{1}{2}\right)$.

Theorem 2. Let $f\in M_k(\Gamma_2)$ and $g\in M_\ell(\Gamma_2)$ with $w>4$ where $k+\ell=w$. Let s and t be non-negative integers. Then we have the following:

(1) $\delta_k^s f \cdot \delta_\ell^t g$ is of bounded growth for $s+t \geqq 3$. Especially, $P_{w+2s}(g\delta_k^s f)$ belongs to $S_{w+2s}(\Gamma_2)$ for $s \geqq 3$.

(2) If at least one of f and g is a cusp form, then $\delta_k^s f \cdot \delta_\ell^t g$ is of bounded growth for all s, $t \geqq 0$.

(3) $P_{w+2}(g\partial_k f + f\partial_\ell g)$ belongs to $S_{w+2}(\Gamma_2)$ where $\partial_k^r = \varepsilon\left(k+r-\frac{3}{2}, k-\frac{1}{2}\right)^{-1}\delta_k^r$. Especially, $P_{2k+2}(f\delta_k f)$ belongs to $S_{2k+2}(\Gamma_2)$.

(4) $P_{w+4}(g\partial_k{}^2 f + 2\partial_k f \cdot \partial_\ell g + f\partial_\ell{}^2 g)$ belongs to $S_{w+4}(\Gamma_2)$.

Theorem 3. Let K be an algebraic number field, O_K be its ring of integers, **p** be its prime ideal not dividing the ideal (2), and R be the localization of O_K at **p**. Let $f \in M_{w-2r}(\Gamma_2)_R$ and $g \in S_w(\Gamma_2)_R$ be eigenforms with $4 < w-2r < w$. Suppose that all the following conditions (1)-(6) are satisfied:

(1) There exist positive integers $m_1, \ldots, m_n$ such that

$$N_{L/K}\left|\left\{\lambda(m_i, f_j)\right\}_{1 \leqq i, j \leqq n}\right| \not\equiv 0 \mod \mathbf{p}$$

where $n = \dim S_w(\Gamma_2)$ and $\{f_1, \ldots, f_n\}$ is an eigen basis of $S_w(\Gamma_2)$ and L is the composite field of K and $\mathbf{Q}(\lambda(m, f_j) | m \geqq 1)$ for $j = 1, \ldots, n$.

(2) There exist a positive integer e and $2s$ $(s\geq 1)$ modular forms $h_{1,t}\in M_{k_{1,t}}(\Gamma_2)_R$, $h_{2,t}\in M_{k_{2,t}}(\Gamma_2)_R$ with $k_{1,t}+k_{2,t}=w-2r$, $r_{1,t}\geq 0$, $r_{2,t}\geq 1$ and $r_{1,t}+r_{2,t}=r$ for $t=1,\ldots,s$ such that

$$a(mE,f)\equiv a\left(mE,\sum_{t=1}^{s}\nu_t h_{1,t}h_{2,t}\right) \mod \mathbf{p}^e$$

for all $m\geq 1$, where

$$\nu_t=\eta\left(k_{1,t}+r_{1,t}-1,k_{1,t}-\tfrac{1}{2}\right)\eta\left(k_{2,t}+r_{2,t}-1,k_{2,t}-\tfrac{1}{2}\right).$$

(3) $\mathbf{p}^e$ divides $(2w-2r-3)I$ where I is the ideal of R generated by $a(T,h_{2,t})$ for $T\geq 0$, $T\neq 0$ and $t=1,\ldots,s$.

(4) $a(E,f)\equiv a(E,g) \mod \mathbf{p}^e$ and $a(E,f)\not\equiv 0 \mod \mathbf{p}$.

(5) $m_i^{2r}\lambda(m_i,f)\equiv\lambda(m_i,g) \mod \mathbf{p}^e$ for $i=1,\ldots,n$.

(6) $\sum_{t=1}^{s}P_w\left(\delta_{k_{1,t}}^{r_{1,t}}h_{1,t}\cdot\delta_{k_{2,t}}^{r_{2,t}}h_{2,t}\right)$ belongs to $S_w(\Gamma_2)$.

Then we have:

$$m^{2r}\lambda(m,f)\equiv\lambda(m,g) \mod \mathbf{p}^e \quad \text{for all } m\geq 1.$$

3. Examples.

We have some congruences between Siegel modular forms of degree two and different weight by using Theorem 3. Let Φ be the Siegel Φ-operator. For an eigen form $f\in M_k(\Gamma_1)$, there is a unique eigen form $[f]\in M_k(\Gamma_2)$ such that $\Phi[f]=f$. Let σ_k be

Saito-Kurokawa lifting $M_{2k-2}(\Gamma_1)\to M_k(\Gamma_2)$. Let $S_k^{II}(\Gamma_2)$ be the orthogonal complement of $\sigma_k(S_{2k-2}(\Gamma_1))$ in $S_k(\Gamma_2)$ with respect to the Petersson inner product. We may call an element of $S_k^{II}(\Gamma_2)$ a generic form since it does not lie in the image of Eisenstein lifting and Saito-Kurokawa lifting. The modular form $\chi_{20}^{(3)}\in S_{20}^{II}(\Gamma_2)$ defined by $4\chi_{10}\varphi_4\varphi_6-12\chi_{12}\varphi_4{}^2+28569600\chi_{10}{}^2$ has the minimal weight 20 among generic forms. (See Kurokawa [6, §5].) By using Theorem 3, we have the following congruences.

Theorem 4. The following congruences hold for all $m\geqq 1$:

$$\lambda(m,\chi_{20}^{(3)}) \equiv m^2\lambda(m,[\Delta_{18}]) \mod 7,$$

$$\lambda(m,\chi_{10}) \equiv m^2\lambda(m,\varphi_8) \mod 5, \qquad (*)$$

$$\lambda(m,\chi_{12}) \equiv m^4\lambda(m,\varphi_8) \mod 17,$$

$$\lambda(m,\chi_{14}) \equiv m^6\lambda(m,\varphi_8) \mod 19.$$

Remark. In the proof of (*), we use Theorem 3 with slight modification.

As to the congruences, we can make the following interpretation. For an eigenform $f\in M_k(\Gamma_n)$, let $\mathbf{Q}(f)$ be the extension of $\mathbf{Q}$ generated by eigenvalues of all Hecke operators on f. Then, $\mathbf{Q}(f)$ is a finite extension of $\mathbf{Q}$. For a prime ideal $\mathbf{l}$ of $\mathbf{Q}(f)$ lying above a rational prime ℓ, we denote by $\mathbf{Q}(f)_{\mathbf{l}}$ the $\mathbf{l}$-adic completion of $\mathbf{Q}(f)$ and by $\mathbf{Z}(f)_{\mathbf{l}}$ the integer ring of $\mathbf{Q}(f)_{\mathbf{l}}$. Let $\overline{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$. For each eigenform $f\in M_k(\Gamma_n)$, assume the existence of an $\mathbf{l}$-adic representation

$$\rho_{\mathbf{l}}(f)\colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL(2^n, \mathbf{Z}(f)_{\mathbf{l}})$$

attached to f in the (naturally generalized) sense of Deligne [3] and Serre [12, 11]. Let $f \in S_k(\Gamma_n)$ and $g \in M_j(\Gamma_n)$ be eigenforms where $k=j+2r$ with a non-negative integer r. For simplicity, we assume $\mathbf{Q}(f) \supset \mathbf{Q}(g)$. Then a congruence

$$\lambda(m,f) \equiv m^{nr}\lambda(m,g) \quad \mod \mathbf{l}^e$$

can be ascribed to

$$\tilde{\rho}_{\mathbf{l}}(f) \cong \tilde{\chi}_{\ell}^{nr} \otimes \tilde{\rho}_{\mathbf{l}}(g)$$

where $\tilde{\rho}$ denotes the reduction modulo $\mathbf{l}^e$ of a representation ρ and χ_ℓ is the cyclotomic ℓ-adic character. In other words, $\mathbf{l}$ would be an exceptional prime for f. Similar to Serre [12], we can expect that $\ell \leq 2k$ for $r>0$. Moreover, a prime ideal $\mathbf{l}$ dividing $(k+j)-(n+1)$ is likely to induce such congruences. It seems natural to work with the operator δ since $\rho_{\mathbf{l}}(\delta_k^r f) = \chi_\ell^{nr} \otimes \rho_{\mathbf{l}}(f)$ (cf. p. 3(2)).

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