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Harmonic forms, Eisenstein series and arithmetic quotients attached to Sp_4

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§ 0 Introduction

In this paper we describe some recent results concerning arithmetic quotients $\Gamma\backslash X$ of the Siegel upper half space of degree two, the de Rham cohomology groups $H^*(\Gamma\backslash X,E)$ of these spaces and their relation with the theory of automorphic forms. This study is motivated by arithmetic-geometric applications we have in mind. For background we refer to [10], [24], [26].

Let $G = Sp_4(\mathbb{R})$ be the symplectic group of degree two, X the associated symmetric space and $\Gamma \subset Sp_4(\mathbb{Z})$ a torsion free arithmetic subgroup of G. The group Γ acts properly and freely on X, and the arithmetic quotient $\Gamma\backslash X$ is a non compact complete Riemannian manifold of finite volume. The space $\Gamma\backslash X$ may be viewed as the interior of a compact manifold with corners $\Gamma\backslash\bar{X}$ where $\bar{X}$ is a suitable completion of X on which Γ acts freely [4]. The inclusion $\Gamma\backslash X \to \Gamma\backslash\bar{X}$ is a homotopy equivalence. Given a finite dimensional complex representation (τ,E) of G (defined over $\mathbb{Q}$) the de Rham cohomology groups $H^*(\Gamma\backslash X,E)$ of $\Gamma\backslash X$ with coefficients in the associated local system $\tilde{E}$ are defined.

Using the theory of Eisenstein series in the sense of Langlands [14] we construct a subspace $H^*_{Eis}(\Gamma\backslash X,E)$ in $H^*(\Gamma\backslash X,E) = H^*(\Gamma\backslash\overline{X},E)$ which describes the cohomology of $\Gamma\backslash X$ 'at infinity', i.e. which restricts isomorphically onto the image of the natural restriction $r^* : H^*(\Gamma\backslash\overline{X},E) \to H^*(\partial(\Gamma\backslash\overline{X}),E)$. More precisely (cf. 3.2.), there is a direct sum decomposition

$$H^*(\Gamma\backslash X,E) = H^*_!(\Gamma\backslash X,E) \oplus H^*_{Eis}(\Gamma\backslash X,E)$$

where $H^*_!(\Gamma\backslash X,E)$ is the image in $H^*(\Gamma\backslash X,E)$ of the cohomology of $\Gamma\backslash X$ with compact supports. By taking the analytic continuation of suitable Eisenstein series attached to elements in $H^*(\partial(\Gamma\backslash\overline{X}),E)$ the Eisenstein cohomology $H^*_{Eis}(\Gamma\backslash X,E)$ is generated by classes which have as a representative a closed harmonic form on $\Gamma\backslash X$ given by a regular value of such an Eisenstein series or a residue of such at a special point. The size of the Eisenstein cohomology can be completely determined (cf. 2.7.). A precise description how the Eisenstein cohomology is built up is given in 3.4.; the proof of these results combines analytic, arithmetic and geometric methods. It requires an explicit knowledge of the constant terms of the Eisenstein series and certain intertwining operators involved.

Let $H^*_{(2)}(\Gamma\backslash X,E)$ be the subgroup of classes in $H^*(\Gamma\backslash X,E)$ represented by closed square integrable forms on $\Gamma\backslash X$; by a representation theoretical interpretation it is related with the discrete spectrum in the space $L^2(\Gamma\backslash G)$ of square integrable functions on $\Gamma\backslash G$. It is also the space of classes represented by a square integrable harmonic form and contains $H^*_!(\Gamma\backslash X,E)$. Thus, our result implies that each class in $H^*(\Gamma\backslash X,E)$

has a closed harmonic form as a representative; moreover it can be chosen among the harmonic automorphic forms.

This work has to be seen as part of the attempt to understand various cohomology groups attached to an arithmetic group and their relation with the arithmetic theory of automorphic forms. The results of Langlands [12] and Harder [7], [10] in the case of an arithmetic subgroup Γ of the special linear group $SL_2(k)$ over an algebraic number field k are basic for this approach; it is pursued in [22], [24]. In particular, the algebraic and arithmetic properties of the cohomology classes are of interest (cf. [9], [10], [26]). However, if the underlying algebraic $\mathbb{Q}$ - group has $\mathbb{Q}$ - rank greater than one the situation is not investigated thoroughly. In the case $\Gamma \subset Sp_4(\mathbb{Z})$ considered here we will discuss these questions and some relations with other aspects as developed in [19], [20], [32] elsewhere.

§ 1 Generalities

1.1. Let $\underline{G}$ be a connected reductive algebraic $\mathbb{Q}$ - group with $\mathrm{rank}_{\mathbb{Q}}\underline{G} \gneq 0$ and without non-trivial rational character defined over $\mathbb{Q}$. Let K be a maximal compact subgroup of the group $G = \underline{G}(\mathbb{R})$ of real points of $\underline{G}$ and $X = G/K$. Then X is a complete Riemannian manifold with negative curvature. Let (τ,E) be a finite dimensional complex rational representation of G , and let $\Gamma \subset G$ be a torsionfree arithmetic subgroup of $\underline{G}$. The group Γ acts properly and freely on X , and Γ operates also on the space $\Omega^*(X,E)$ of smooth E - valued differential forms on X . The quotient space $\Gamma\backslash X$ is a non-compact $K(\Gamma,1)$ - manifold of finite volume. Our object of concern is the de Rham cohomology group $H^*(\Gamma\backslash X,E)$ of Γ , which is, by definition, the cohomology of the subcomplex $\Omega^*(X,E)^\Gamma$ of Γ - invariant elements in $\Omega^*(X,E)$. It can be naturally identified with the singular cohomology of $\Gamma\backslash X$ with coefficients in the local system $\tilde{E}$ defined by (τ,E) , i.e. we have (cf. [5], VII,2)

$$H^*(\Gamma\backslash X,E) = H^*(\Gamma\backslash X,\tilde{E})$$

1.2. There is also an interpretation of the cohomology $H^*(\Gamma\backslash X,E)$ of the arithmetic quotient $\Gamma\backslash X$ in representation-theoretical terms. Denote by $\underline{g}$ resp. $\underline{k}$ the Liealgebra of G resp. K , and let (π,V) be a $(\underline{g},K)$ - module. The relative Liealgebra cohomology of $\underline{g}$ mod K with coefficients in V is then defined as the cohomology of the complex $D^*(\underline{g},K;V) = \mathrm{Hom}_K(\Lambda^*(\underline{g}/\underline{k}),V)$ with the usual differential as in [5] ,I,§ 1.

Since the space $F_{(K)}$ of K - finite vectors in a differentiable G - module F is a $(\underline{g},K)$ - module in a natural way the above notion makes also sense for F if one puts $D^*(\underline{g},K;F) = D^*(g,K;F_{(K)})$.

The space of smooth functions on $\Gamma\backslash G$ with values in $\mathbb{C}$ will be denoted by $C^\infty(\Gamma\backslash G)$. The lifting of forms via the projection $G \to G/K = X$ induces then an isomorphism of differential complexes $\Omega^*(X,E)^\Gamma = D^*(\underline{g},K;C^\infty(\Gamma\backslash G)\otimes E)$, whence an isomorphism in cohomology (cf. [5] ,VII,2.7)

$$H^*(\Gamma\backslash X,E) = H^*(\underline{g},K;C^\infty(\Gamma\backslash G) \otimes E) .$$

1.3. Given a smooth G - module (π,V) and an intertwining operator $\alpha : V \to C^\infty(\Gamma\backslash G)$ one can study the induced map in cohomology

$$H(\alpha) : H^*(\underline{g},K;V \otimes E) \to H^*(\underline{g},K;C^\infty(\Gamma\backslash G) \otimes E) = H^*(\Gamma\backslash X,E)$$

Two cases are of interest for us. Let $L^2(\Gamma\backslash G)$ be the space of complex valued square integrable functions on $\Gamma\backslash G$, viewed as usual as a unitary G - module via right translations. The space $L^2(\Gamma\backslash G)$ is the direct sum of the discrete spectrum $L^2_{dis}(\Gamma\backslash G)$ and the continuous spectrum $L^2_{ct}(\Gamma\backslash G)$; and the space $L^2_{dis}(\Gamma\backslash G)$ decomposes into a direct Hilbert sum of closed irreducible G - modules H_π with finite multiplicities. The space $L^2_o(\Gamma\backslash G)$ of square integrable cuspidal functions on $\Gamma\backslash G$ (cf. [11] for this notion) is a G - invariant subspace of $L^2_{dis}(\Gamma\backslash G)$ and we may then write

$$(1) \qquad L^2_o(\Gamma\backslash G) = \widetilde{\bigoplus_{\pi\in\hat{G}}} m(\pi,\Gamma)H_\pi$$

where $m(\pi,\Gamma)$ denotes the multiplicity with which H_π occurs in $L^2_o(\Gamma\backslash G)$. The inclusion $L^2_o(\Gamma\backslash G) \to C^\infty(\Gamma\backslash G)$ induces then an homomorphism

(2) $$j^* : H^*(\underline{g},K;L^2_o(\Gamma\backslash G) \otimes E) \to H\ (\underline{g},K;C^\infty(\Gamma\backslash G) \otimes E)$$

which is injective by [1] , 5.5. By definition, the cusp cohomology $H^*_{cusp}(\Gamma\backslash X;E)$ of $\Gamma\backslash X$ with coefficients in E is the image of $H^*(\underline{g},K;L^2_o(\Gamma\backslash G) \otimes E)$ under j^* in (2).

Let $H^*_{(2)}(\Gamma\backslash X;E)$ be the subspace of $H^*(\Gamma\backslash X,E)$ given by classes represented by closed square integrable forms. It can be viewed as the image of the map

(3) $$H^*(\underline{g},K;L^2_{dis}(\Gamma\backslash G) \otimes E) \to H^*(\underline{g},K;C^\infty(\Gamma\backslash G) \otimes E) = H^*(\Gamma\backslash X,E)$$

We denote by $H^*_c(\Gamma\backslash X,\tilde{E})$ the cohomology with compact supports of $\Gamma\backslash X$ and by $H^*_!(\Gamma\backslash X;E)$ the image of $H^*_c(\Gamma\backslash X,\tilde{E})$ into $H^*(\Gamma\backslash X;E)$ under the natural map. Then the space $H^*_{(2)}(\Gamma\backslash X,E)$ contains the socalled interior cohomology $H^*_!(\Gamma\backslash X,E)$ (cf.[24], 1.8). On the other hand, the cusp cohomology of $\Gamma\backslash X$ is contained in $H^*_!(\Gamma\backslash X,E)$ hence we get

(4) $$H^*_{cusp}(\Gamma\backslash X,E) \subset H^*_!(\Gamma\backslash X,E) \subset H^*_{(2)}(\Gamma\backslash X,E)$$

We remark that, by a theorem of Kodaira, $H^*_{(2)}(\Gamma\backslash X,E)$ is also the space of classes represented by a square integrable harmonic form.

1.4. As in 1.3.(1) the discrete spectrum $L^2_{dis}(\Gamma\backslash G)$ decomposes as a direct Hilbert sum of closed irreducible G - subspaces with finite multiplicities. Let V_π denote the isotypic component in $L^2_{dis}(\Gamma\backslash G)$ corresponding to a given irreducible unitary representation π of G. Then it is shown in [3] that

$H^*(\underline{g},K;L^2_{dis}(\Gamma\backslash G)\otimes E)$ is a finite algebraic direct sum

(1) $$H^*(\underline{g},K;L^2_{dis}(\Gamma\backslash G)\otimes E) = \bigoplus_{\pi\in\hat{G}} H^*(\underline{g},K;V_\pi^\infty\otimes E)$$

where $\pi\in\hat{G}$ runs over the finite set of equivalence classes of all irreducible unitary representations of G whose infinitesimal character χ_π is equal to the infinitesimal character χ_{τ^*} of the representation (τ^*,E^*) contragredient to E (cf. [5] ,I,Thm. 5.3). This finite set (up to equivalence) of irreducible unitary representations of G with non-trivial $(\underline{g},K)$ - cohomology has been conveniently parametrized by Vogan and Zuckerman [29] , [30] .

Of course, by 1.3.(1) we have also a decomposition for the cuspidal cohomology of Γ as a finite direct sum

(2) $$H^*_{cusp}(\Gamma\backslash X;E) = \bigoplus_{\pi\in\hat{G}} m(\pi,\Gamma)H^*(\underline{g},K;H_\pi^\infty\otimes E)$$

These two relations (1) and (2) imply that results in relative Lie algebra cohomology with coefficients in irreducible unitary representations of G apply to the cuspidal cohomology of Γ and to the groups $H^*_{(2)}(\Gamma\backslash X,E)$ as well (cf. [2] , [5] , [25]). However, one still has to determine the multiplicities $m(\pi,\Gamma)$. On the other hand, the study of $H^*(\Gamma\backslash X,E)$ can also sometimes used to get information on these multiplicities (see [6] , [16] for examples).

1.5. Examples

(1) Let $\underline{G}$ be the special linear group $SL_n/\mathbb{Q}$ and Γ an arithmetic subgroup of $SL_n(\mathbb{Q})$. Then there is (up to equivalence) at most one (resp. two for n even) irreducible unitary representation (π_o, H_{π_o}) of G such that

$H^*(\underline{g},K;H^\infty_{\pi_o} \otimes E) \neq \{0\}$ and (π_o, H_{π_o}) occurs with non-zero multiplicity $m(\pi_o,\Gamma)$ in the cuspidal spectrum $L^2_o(\Gamma\backslash G)$. This result (due to Casselman, cf. [25] , § 3) implies a strong vanishing result for the cusp cohomology $H^*_{cusp}(\Gamma\backslash X,E)$ of Γ outside a range $[C_n(n), C_o(n)]$ of length $\mathrm{rk}\, SL_n(\mathbb{R}) - \mathrm{rk}\, SO(n)$ centered around the middle dimension $(1/2)\cdot \dim X$.

(2) For G equal to $SL_2(\mathbb{R})$, $SL_3(\mathbb{R})$ resp. $SL_2(\mathbb{C})$ one can show that one has equality in 1.3.(4), i.e. for $q \geqslant 1$

$$H^q_{cusp}(\Gamma\backslash X,E) = H^q_!(\Gamma\backslash X,E) = H^q_{(2)}(\Gamma\backslash X,E)$$

(cf. [6] , [15]). As examples show this is not true in general.

For example, it follows from (1) that in the case $G = SL_3(\mathbb{R})$ (with $E = \mathbb{C}$) one has

$$H^q_{cusp}(\Gamma\backslash X;\mathbb{C}) = H^q(\underline{g},K;m(\pi_o,\Gamma)H^\infty_\pi)$$

where the righthand side vanishes for $q \neq 2,3$. For $\Gamma = \Gamma(m)$, $m \geqslant 3$, a congruence subgroup of level m it is shown in [16] that $\dim H^q_{cusp}(\Gamma(m)\backslash X;,\mathbb{C})$, $q = 2,3$, and therefore $m(\pi_o,\Gamma(m))$ is greater than $m(m+1)$.

1.6. Eisenstein cohomology

The non-compact quotient $\Gamma\backslash X$ may be identified to the interior of a compact manifold $\Gamma\backslash\overline{X}$ with corners, where $\overline{X}$ is a suitable completion of X on which Γ acts freely. This compactification is due to Borel and Serre [4] . The inclusion $\Gamma\backslash X \to \Gamma\backslash\overline{X}$ is a homotopy equivalence. The boundary $\partial(\Gamma\backslash\overline{X})$ is a disjoint union of a finite number of faces $e'(Q)$ which correspond bijectively to the Γ - conjugacy classes of proper parabolic $\mathbb{Q}$ - subgroups of G . Denote by $\underline{\underline{P}}$ the set of parabolic $\mathbb{Q}$ - subgroups of G . For a given P in $\underline{\underline{P}}$, $P \neq G$, we denote the natural restriction of the cohomology of $\Gamma\backslash\bar{X}$ onto the cohomology of the corresponding face $e'(P)$ in $\partial(\Gamma\backslash\bar{X})$ by

(1) $$r_P^* : H^*(\Gamma\backslash\bar{X}, E) \to H^*(e'(P), E)$$

Via Eisenstein series in the sense of Langlands [14] one tries to construct classes in $H^*(\Gamma\backslash X, E)$ with a non-zero restriction to $\partial(\Gamma\backslash\bar{X})$ and to get hold of cross-sections to (suitable families of) the restrictions in (1) or, ultimately, the restriction

(2) $$r^* : H^*(\Gamma\backslash\bar{X}, E) \to H^*(\partial(\Gamma\backslash\bar{X}), E)$$

in this way. The use of Eisenstein series to construct cohomology classes which describe the cohomology of $\Gamma\backslash X$ "at infinity" was initiated by Harder [7], [8] and pursued in [22], [24], [10]. If the $\mathbb{Q}$ - rank of $\underline{G}$ is one he has shown the existence of a subspace $H^*_{Eis}(\Gamma\backslash X, E)$ in $H^*(\Gamma\backslash\bar{X}, E)$ which restricts isomorphically onto Im r and whose elements are obtained either by evaluating suitable Eisenstein series at special points or by taking residues of such at simple poles. Since there is almost no information concerning the behavior of Eisenstein series at certain values which are of interest here the result of Harder has to be seen as an answer up to the existence of poles. It can be made more precise in the case SL_2/k defined over an algebraic number field k where one gets out of this a complete description of Im r^*. Moreover in this case Eisenstein cohomology classes have interesting algebraic or arithmetic properties which led, for example, to some rationality results on special values of L - functions attached to algebraic Hecke characters [9], [10].

For groups of higher rank the situation is not investigated thoroughly. However, as a first step, there is a general result (cf. [24], § 4) describing in which way an Eisenstein series $E(\phi,\Lambda)$ which is associated to a cuspidal differential form on a face $e'(P)$ and depends on a complex parameter provides us with a closed harmonic form on $\Gamma\backslash X$ and with a non-trivial class in $H^{*}(\Gamma\backslash X;E)$ if $E(\phi,\Lambda)$ is holomorphic at a special point Λ_o uniquely determined by ϕ . As examples in [10] , [24] show $E(\phi,\Lambda)$ may very well have poles at such points. In this case we have to take residues of Eisenstein series. However, in order to understand to cohomology of Γ "at infinity" a detailed study of the behavior of Eisenstein series at special points, of the corresponding cohomology classes and its images under the various restrictions r_P^{*} resp. r^{*} is necessary.

§ 2 The case $Sp_4/\mathbb{Q}$: The boundary of the Borel-Serre compactification and its cohomology

We begin now with the study of arithmetic quotients of the Siegel upper half space of degree two and the corresponding cohomology groups of arithmetic subgroups Γ of the symplectic group $Sp_4(\mathbb{Q})$. In this section we give a convenient description of the boundary $\partial(\Gamma\backslash\overline{X})$ of the Borel-Serre compactification $\Gamma\backslash\overline{X}$ of the quotient $\Gamma\backslash X$ in this case and determine its cohomology. Out of this we derive a formula for the size of the cohomology of Γ at infinity. For details we refer the reader to § 2 in [26].

2.1. Now let $\underline{G}$ be the $\mathbb{Q}$ - split algebraic $\mathbb{Q}$ - group $Sp_4/\mathbb{Q}$, i.e. the symplectic group of degree two. The group $G = \underline{G}(\mathbb{R})$ of real points of $\underline{G}$ is then

$$G = Sp_4(\mathbb{R}) = \{\alpha \in GL_4(\mathbb{R}) \mid \alpha^t J \alpha = J\} \text{ with } J = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix} .$$

We fix as maximal compact subgroup of G the group $K = G \cap O(4)$, i.e. the group of orthogonal symplectic matrices. If Γ is a torsionfree arithmetic subgroup of $\underline{G}(\mathbb{Q}) = Sp_4(\mathbb{Q})$ the quotient $\Gamma\backslash X$ is a 6 - dimensional non-compact $K(\Gamma,1)$ - manifold of finite volume. In particular we consider for a given $m \geqslant 3$ the full congruence subgroup $\Gamma(m) = \{A \in Sp_4(\mathbb{Z}) \mid A \equiv Id \bmod m\}$ of $Sp_4(\mathbb{Z})$. We fix m once and for all. This justifies the notation ${}_fG = \Gamma(m)\backslash Sp_4(\mathbb{Z})$ for the finite group which depends on m . The group $Sp_4(\mathbb{Z})$ operates in a natural way on the Borel-Serre compactification $\Gamma\backslash\overline{X}$ and so also ${}_fG$ acts on $H^*(\Gamma\backslash\overline{X},E)$ resp. $H^*(\partial(\Gamma\backslash\overline{X}),E)$. In a similiar way the faces $e'(P)$ in $\partial(\Gamma\backslash\overline{X})$ $(P \in \underline{\underline{P}})$ are acted upon by $P \cap Sp_4(\mathbb{Z})$, and we put ${}_fP = (\Gamma(m) \cap P)\backslash(Sp_4(\mathbb{Z})\cap P) \subset {}_fG$. As a first step towards the cohomology of Γ at infinity we give now a description of $H^*(\partial(\Gamma(m)\backslash \overline{X}),E)$ as a representation space for the finite group ${}_fG$. Out of this we derive a formula for the dimension of the image of the restriction $r^* : H^*(\Gamma(m)\backslash\overline{X},E) \to H^*(\partial(\Gamma(m)\backslash\overline{X}),E)$.

For simplicity we will mainly assume that $E = \mathbb{C}$ is the trivial representation, and in this case $E = \mathbb{C}$ will be omitted in the notation of the cohomology groups dealt with. However, the methods work as well for arbitrary coefficients.

2.2. The $\underline{G}(\mathbb{Q})$ - conjugacy classes of proper parabolic $\mathbb{Q}$ - subgroups of G fall into three classes, two conjugacy classes $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ of maximal parabolic subgroups and one class $\underline{\underline{P}}_o$ of minimal ones. We say a proper parabolic subgroup P of G is of type i , $i=0,1,2$, if $P \in \underline{\underline{P}}_i$. A given P in $\underline{\underline{P}}$ may be written as a semidirect product $P = M \cdot N$ where N denotes the unipotent radical of P and M the unique Levi subgroup of P which is stable under the Cartan involution θ associated to K . For each $\underline{\underline{P}}_i$, $i=0,1,2$, we fix a representative P_i with $P_i = M_i \cdot N_i$ and such that $P_o = P_1 \cap P_2$. The maximal parabolic $\mathbb{Q}$ - subgroups are labelled in such a way that M_1 is isomorphic to the direct product $SL_2(\mathbb{R}) \times GL_1(\mathbb{R})$ and N_1 is non-abelian resp. M_2 is isomorphic to $GL_2(\mathbb{R})$ and N_2 is abelian.

For a given P in $\underline{\underline{P}}$ with Langlands decomposition $P = {}^oMAN$ [where $M = {}^oMA$ (resp. A) denotes the unique Levi subgroup of P (resp. split component) which is stable under θ)

the face $e'(P)$ in $\partial(\overline{\Gamma\backslash X})$ corresponding to the Γ - conjugacy class of P is defined as $e'(P) = \Gamma_P\backslash {}^oP/K \cap P$ with ${}^oP = {}^oMN$ and $\Gamma_P := \Gamma \cap P$. The projection $\kappa : P \to P/N$ induces a fibration

$$\Gamma_N\backslash N \to e'(P) \to \Gamma_M\backslash Z_M \tag{1}$$

of $e'(P)$ over the locally symmetric space $\Gamma_M\backslash Z_M =$ $\kappa(\Gamma \cap P) \backslash ({}^oP/N)/\kappa(K \cap P)$ with fiber the compact manifold $\Gamma_N\backslash N$ where $\Gamma_N = \Gamma \cap N$. If P is a maximal parabolic $\mathbb{Q}$ - subgroup of G of type $i=1,2$, then $e'(P)$ is a 5 - dimensional manifold fibered over a non-compact 2 - dimensional manifold $\Gamma_M\backslash Z_M$ (homeomorphic to some arithmetic quotient $\Gamma_i^*\backslash SL_2(\mathbb{R})/SO(2)$) with fiber a 3 - dimensional nil-manifold for $i=1$ (resp. torus for $i=2$).

According to [4] there is a natural compactification $\overline{e'(P_i)}$ of $e'(P_i)$, $i=1,2$, which adds over each cusp of the base $\Gamma_{M_i}\backslash Z_{M_i}$ a 4 - dimensional nilmanifold. In particular, the action of $P_i \cap Sp_4(\mathbb{Z})$ extends to one on $\overline{e'(P_i)}$, and the group ${}_fP_i$ acts transitively on the boundary components of $\overline{e'(P_i)}$. Indeed, one has an ${}_fP_i$ - equivariant diffeomorphism

$$\beta_i : {}_fP_i \times_{{}_fP_o} e'(P_o) \xrightarrow{\sim} \partial(\overline{e'(P_i)}) . \tag{2}$$

As here, we put now for $i=0,1,2$ resp. $i=1,2$

$$Y_i = {}_fG \times_{{}_fP_i} e'(P_i) \quad \text{resp.} \quad \overline{Y}_i = {}_fG \times_{{}_fP_i} \overline{e'(P_i)} \tag{3}$$

which is a disjoint union of copies of $e'(P_i)$ resp. $\overline{e'(P_i)}$ and has a natural action of ${}_fG$ which extends the one of ${}_fP_i$

on $e'(P_i)$ resp. $\overline{e'(P_i)}$. The manifolds $\overline{Y}_i$ are compact and one has ${}_fG$ - equivariant diffeomorphism

(4) $$\alpha_i : Y_o \xrightarrow{\sim} \partial(\overline{Y}_i) = {}_fG \times_{{}_fP_i} \partial(\overline{e'(P_i)}) \quad , \ i=1,2 \quad ,$$

defined by $Id \times \beta_i$ onto the boundary $\partial(\overline{Y}_i)$. Using the α_i we get a closed 5 - dimensional manifold (with an action by ${}_fG$)

(5) $$Y = \overline{Y}_1 \cup_{Y_o} \overline{Y}_2$$

by gluing together $\overline{Y}_1$ and $\overline{Y}_2$ along their common boundaries. One obtains then

2.3. Proposition. - The boundary $\partial(\Gamma(m)\backslash\overline{X})$ of the Borel-Serre compactification $\Gamma(m)\backslash\overline{X}$ of $\Gamma(m)\backslash X$ is equivariant diffeomorph with respect to the action of ${}_fG$ to the manifold Y given in 2.2.(5) .

2.4. In order to describe $H^*(\partial(\Gamma(m)\backslash\overline{X}))$ and to determine the ${}_fG$ - action on this one has to analyse the Mayer-Vietoris sequence in cohomology attached to the decomposition $\partial(\Gamma(m)\backslash\overline{X}) = \overline{Y}_1 \cup \overline{Y}_2$. Since we have as ${}_fG$ - modules

(1) $$H^*(\overline{Y}_i) = H^*({}_fG \times_{{}_fP_i} \overline{e'(P_i)}) = \mathrm{Ind}_{{}_fP_i}^{{}_fG}[H^*(\overline{e'(P_i)})]$$

(where $\mathrm{Ind}_{{}_fP_i}^{{}_fG}[\ ]$ denotes the representation of ${}_fG$ induced from the representation of ${}_fP_i$ on $H^*(\overline{e'(P_i)})$) this involves to understand the cohomology $H^*(e'(P_i)) = H^*(\overline{e'(P_i)})$ as ${}_fP_i$ - module.

Associated to the fibration 2.2.(1) of the faces $e'(P_i)$, $i=1,2$, there is a spectral sequence which converges to the cohomology of $e'(P_i)$, and whose E_2 - term is given by

$$E_2^{p,q} = H^p(\Gamma_{M_i} \setminus Z_{M_i}, H^q(\Gamma_{N_i} \setminus N_i)) .$$

Since the base space is of type $\Gamma' \setminus H$ (H = upper half plane, $\Gamma' \subset SL_2(\mathbb{Z})$ of finite index) and $H^p(\Gamma' \setminus H, \tilde{E}) = 0$, $p > 1$, for an arbitrary coefficient system $\tilde{E}$ we have $E_2^{p,q} = 0$ for $p > 1$, $q > 3$ and the spectral sequence degenerates at E_2. We obtain

$$H^r(e'(P_i)) = \bigoplus_{p+q=r} H^p(\Gamma_{M_i} \setminus Z_{M_i}, H^q(\Gamma_{N_i} \setminus N_i)) .$$

The natural M_i - module structure of $H^*(\Gamma_{N_i} \setminus N_i)$ can be determined by a general result of Kostant using the identification $H^*(\Gamma_{N_i} \setminus N_i) = H^*(\underline{n}_i)$ with the Liealgebra cohomology of the Liealgebra $\underline{n}_i$ of N_i (cf. [24] , 2.2. - 2.4.).

If we put ${}_fM_i = (\Gamma(m) \cap M_i) \setminus (Sp_4(\mathbb{Z}) \cap M_i)$ resp. ${}_fN_i = (\Gamma(m) \cap N_i) \setminus (Sp_4(\mathbb{Z}) \cap N_i)$ then ${}_fP_i$ is a split group extension of ${}_fM_i$ by ${}_fN_i$. Then it turns out that the action of ${}_fP_i$ on $H^p(\Gamma_{M_i} \setminus Z_{M_i}, H^q(\underline{n}_i))$ is the pullback of the action of ${}_fM_i$ on these cohomology spaces obtained by the natural action of ${}^oM_i \cap Sp_4(\mathbb{Z})$ on $\Gamma_{M_i} \setminus Z_{M_i}$ resp. the action on $H^*(\underline{n}_i) = H^*(\Gamma_{N_i} \setminus N_i)$ just mentioned. We refer to 2.5. in [26] for details.

A detailed study of the kernel resp. cokernel of the ${}_fG$ - morphism $\alpha_1^* \oplus \alpha_2^*$ in the Mayer-Vietoris sequence

$$\to H^q(\partial(\Gamma \setminus \overline{X})) \to H^q(\overline{Y}_1) \oplus H^q(\overline{Y}_2) \xrightarrow{\alpha_1 \oplus \alpha_2} H^q(Y_o) \to H^{q+1}(\partial(\Gamma \setminus \overline{X}))$$

attached to $\partial(\Gamma \setminus \overline{X}) = \overline{Y}_1 \cup \overline{Y}_2$ leads then to the following result describing $H^*(\partial(\Gamma \setminus \overline{X}))$ as ${}_fG$ - module. For later purposes we give a description of the cohomology which reflects the geometric source of the various summands.

2.5. Theorem. - For a given congruence subgroup $\Gamma = \Gamma(m)$ of $Sp_4(\mathbb{Z})$, $m \geq 3$, the cohomology $H^*(\partial(\Gamma(m)\backslash\overline{X}),\mathbb{C})$ of the boundary $\partial(\Gamma(m)\backslash X)$ of the Borel-Serre compactification of $\Gamma(m)\backslash X$ is described as representation space of ${}_fG = Sp_4(\mathbb{Z}/m\mathbb{Z})$ by

$$(1) \qquad H^q(\partial(\Gamma\backslash\overline{X})) \cong \begin{cases} \mathbb{C} & q = 0,5 \\ 0 & q > 5 \end{cases}$$

$$(2) \qquad H^1(\partial(\Gamma\backslash\overline{X})) \cong \bigoplus_{i=1}^{2} \operatorname{Ind}_{{}_fP_i}^{{}_fG}[H^1_{cusp}(\Gamma_{M_i}\backslash Z_{M_i})] \oplus St(m)$$

$$(3) \qquad H^2(\partial(\Gamma\backslash\overline{X})) \cong \bigoplus_{i=1}^{2} \operatorname{Ind}_{{}_fP_i}^{{}_fG}[H^1_{cusp}(\Gamma_{M_i}\backslash Z_{M_i}, H^1(\underline{n}_i))] \oplus \bigoplus_{i=1}^{2} \operatorname{Ind}_{{}_fP_i}^{{}_fG}[\tau_i]$$

$$(4) \qquad H^3(\partial(\Gamma\backslash\overline{X})) \cong \bigoplus_{i=1}^{2} \operatorname{Ind}_{{}_fP_i}^{{}_fG}[H^1_{cusp}(\Gamma_{M_i}\backslash Z_{M_i}, H^2(\underline{n}_i))] \oplus \bigoplus_{i=1}^{2} \operatorname{Ind}_{{}_fP_i}^{{}_fG}[H^0(\Gamma_{M_i}\backslash Z_{M_i}, H^3(\underline{n}_i))]$$

$$(5) \qquad H^4(\partial(\Gamma\backslash\overline{X})) \cong \bigoplus_{i=1}^{2} \operatorname{Ind}_{{}_fP_i}^{{}_fG}[H^1_{cusp}(\Gamma_{M_i}\backslash Z_{M_i}, H^3(\underline{n}_i))] \oplus St(m) .$$

(where the action of ${}_fP_i$ on the various terms $H^p_{cusp}(\Gamma_{M_i}\backslash Z_{M_i}, H^q(\underline{n}_i))$ is described above)

The ${}_fG$ - module $St(m)$ is defined by the short exact sequence

$$(6) \qquad 0 \to \mathbb{C} \to \bigoplus_{i=1}^{2} \operatorname{Ind}_{{}_fP_i}^{{}_fG}[\mathbb{C}] \to \operatorname{Ind}_{{}_fP_o}^{{}_fG}[\mathbb{C}] \to St(m) \to 0 .$$

and τ_i is a certain one-dimensional representation of ${}_fP_i$ which corresponds to the one-dimensional representation $H^o(\Gamma_{M_i} \setminus Z_{M_i}, H^3(\underline{n}_i))$ via Poincare duality for $\partial(\Gamma(m) \setminus \overline{X})$

Remarks: (1) If we deal with a non-trivial coefficient system $\tilde{E}$ given by a rational representation (τ, E) a similiar procedure allows also to determine $H^*(\partial(\Gamma \setminus X, \tilde{E})$ as ${}_fG$ - module. The final result depends on the highest weight μ of the given representation (τ, E), and is simpler than the one above (for $E = \mathbb{C}$) if μ is sufficiently regular.

(2) By taking $\Gamma/\Gamma(m)$ - invariants (with $\Gamma(m)$ appropriatly chosen) on both sides of 2.5. one obtains also $H^*(\partial(\Gamma \setminus \overline{X}), \mathbb{C})$ for an arbitrary torsionfree subgroup Γ of finite index in $Sp_4(\mathbb{Z})$.

2.6. We consider now the natural restriction $r^* : H^*(\Gamma \setminus \overline{X}) \to H^*(\partial(\Gamma \setminus \overline{X}))$. Recall that the cohomological dimension $cd(\Gamma)$ of Γ is equal to $\dim \Gamma \setminus X - rk_{\mathbb{Q}} Sp_4 = 4$, i.e. we have $H^i(\Gamma \setminus \overline{X}) = 0$ for $i > 4$ by [4] , § 9. Obviously r^o is an isomorphism. The positive solution of the congruence subgroup problem for Sp_4 (cf. [17]) shows that the commutator factor group $\Gamma / [\Gamma : \Gamma]$ is finite. On one hand, this implies that $H_1(\Gamma \setminus X) = \Gamma/[\Gamma:\Gamma] \otimes \mathbb{C}$ vanishes and via Poincaré duality one gets that r^4 is surjective. On the other hand, one obtains $H^1(\Gamma \setminus \overline{X}) = 0$, and therefore r^1 is trivial. Moreover, there is a dual pairing on $H^*(\partial(\Gamma \setminus \overline{X}))$ induced by duality such that the image of $H^*(\Gamma \setminus \overline{X})$ under r^* is its own annihilator; in particular one gets for $i=0,1,\ldots$

(1) $$\dim \operatorname{Im} r^{5-i} = \dim H^i(\partial(\Gamma\backslash\overline{X})) - \dim \operatorname{Im} r^i$$

which shows that $\operatorname{Im} r^3$ and $\operatorname{Im} r^2$ are related to eachother.

For $\Gamma = \Gamma(m)$, $m \geq 3$, the group Γ_{M_i} can be viewed as the full congruence subgroup $\Gamma(2,m)$ of level m in $SL_2(\mathbb{Z})$. The Eichler-Shimura isomorphism (cf.[28], 8.2)

(2) $$H^1_{cusp}(\Gamma(2,m)\backslash H, E_k) \cong S_{k+1}(\Gamma(2,m)) \oplus \overline{S_{k+1}(\Gamma(2,m))}$$

relates the cusp cohomology of $\Gamma(2,m)$ with coefficients in the representation of $SL_2(\mathbb{R})$ of dimension k to the space $S_{k+1}(\Gamma(2,m))$ of holomorphic resp. antiholomorphic cuspidal forms of weight $k+1$ on the upper half plane H with respect to $\Gamma(2,m)$. The dimension of the spaces on the right hand side are known ([28] , § 2). Using this and (1) we obtain the following dimension formulas for the image of the restriction map r^* .

2.7. Proposition. - *Let* $\Gamma = \Gamma(m)$, $m \geq 3$, *be the congruence subgroup of level* m *of* $Sp_4(\mathbb{Z})$. *The dimensions of the images of the restrictions* $r^* : H^*(\Gamma\backslash\overline{X}) \to H(\partial(\Gamma\backslash\overline{X}))$ *are given by*

(1) $$\dim \operatorname{Im} r^q = \begin{cases} 1 & q = 0 \\ 0 & q = 1 \text{ or } q \geq 5 \end{cases}$$

(2) $$\dim \operatorname{Im} r^2 + \dim \operatorname{Im} r^3 = \sum_{i=1}^{2} p_i(m)(1+2\dim S_{k_i}(\Gamma(2,m)))$$

$$\text{with } k_i = \begin{cases} 3 & i=1 \\ 4 & i=2 \end{cases}$$

(3) $$\dim \operatorname{Im} r^4 = \sum_{i=1}^{2} (p_i(m)\cdot 2\cdot \dim S_2(\Gamma(2,m)))$$

$$+ (p_o(m) - (p_1(m)+p_2(m))+1)$$

where $p_j(m) = |{}_fP_j \backslash {}_fG|$ *denotes the number of* $\Gamma(m)$ - *conjugacy classes of parabolic* $\mathbb{Q}$ - *subgroups of* $Sp_4(\mathbb{R})$ *of type* j $(j=0,1,2)$.

§ 3 Eisenstein cohomology for $Sp_4/\mathbb{Q}$

The theory of Eisenstein series ([11], [14]) can now be used to construct a section to the natural restriction $r^* : H^*(\Gamma\backslash\bar{X};E) \to H^*(\partial(\Gamma\backslash\bar{X}),E)$ where $\Gamma \subset Sp_4(\mathbb{Z})$ is a torsion-free congruence subgroup and (τ,E) a rational representation of $G = Sp_4(\mathbb{R})$ as in § 1 . In explaining the main results we have to assume some familiarity with this approach as explained in [8], [10], [22], [24], [25].

3.1. Eisenstein cohomology classes. Let us briefly recall the construction of Eisenstein cohomology classes in a simple setting. Given a parabolic $\mathbb{Q}$ - subgroup P of G with $P = {}^{O}MAN$ we consider the cusp cohomology

(1) $$H^*_{cusp}(e'(P),E) := H^*_{cusp}(\Gamma_M\backslash Z_M, H^*(\underline{n},E))$$

of the corresponding face $e'(P)$ in the boundary of the Borel-Serre compactification $\Gamma\backslash\bar{X}$. For a definition in representation theoretical terms we refer to § 1 . However, this space may also be interpreted in terms of $H^*(\underline{n})$ - valued differential forms on $\Gamma_M\backslash Z_M$ whose coefficients are $H^*(\underline{n})$ - valued cuspidal functions. In particular, the cusp cohomology $H^*_{cusp}(e'(P))$ can be identified with the space of harmonic cuspidal $\mathbb{C}$ - valued differential forms on $e'(P)$, i.e. those whose coefficients are cuspidal (see [2], § 5).

Let $0 \neq [\phi] \in H^*_{cusp}(e'(P),E)$ be a non-trivial cuspidal cohomology class represented by a harmonic cuspidal form $\phi \in \Omega^*(e'(P),E)$. We have topologically $\Gamma_P\backslash X = e'(P) \times A_P$. For a given Λ in the dual $\underline{a}^*_{\mathbb{C}}$ of the complexified Liealgebra

$\underline{a}_{\mathbb{C}}$ of $A = A_P$ we can associate to ϕ via the differential form $\phi_\Lambda = \phi a^{\Lambda+\rho}$ in $\Omega^*(\Gamma_P\backslash X)$ the Eisenstein series

(2) $$E(\phi,\Lambda) = \sum_{\Gamma_P\backslash\Gamma} \gamma \circ \phi_\Lambda \, .$$

This Eisenstein series is first defined for all Λ in

(3) $$(\underline{a}^*_{\mathbb{C}})^+ = \{\Lambda \in \underline{a}_{\mathbb{C}} \mid (\mathrm{Re}\,\Lambda,\alpha) > (\rho_P,\alpha),\ \alpha \in \Delta(P,A)\}$$

and is holomorphic in that tube where $\Delta(P,A)$ denotes the set of simple roots of P with respect to A and the element $\rho_P \in \underline{a}^*$ is defined by $\rho_P(a) = (\det \mathrm{Ad}\, a|_{\underline{n}})^{1/2}$, $a \in A$. Via analytic continuation it admits a meromorphic extension to all of $\underline{a}^*_{\mathbb{C}}$. We refer to [14], [11] for the general theory of Eisenstein series. If $\Lambda_o \in \underline{a}^*_{\mathbb{C}}$ is fixed and $E(\phi,\Lambda)$ is holomorphic at this point, then evaluating the Eisenstein series in Λ_o gives an E - valued, Γ - invariant differential form on X , i.e. we obtain $E(\phi,\Lambda_o) \in \Omega^*(\Gamma\backslash X,E)$. In fact, by 4.11. [24] there is a special point Λ_ϕ uniquely determined by ϕ such that this construction provides us with a <u>closed harmonic</u> form $E(\phi,\Lambda_\phi)$ if $E(\phi,\Lambda)$ is holomorphic at this point Λ_ϕ . In particular, this form represents a non-trivial cohomology class $[E(\phi,\Lambda_\phi)]$ in $H^*(\Gamma\backslash X,E)$.

However, since Λ_ϕ does not necessarily lie in the region $(\underline{a}^*_{\mathbb{C}})^+$ of absolute convergence of the defining series $E(\phi,\Lambda)$ may very well have poles at such points. In this case, we have to take residues of these Eisenstein series, and the situation is much harder to describe. It requires an explicit knowledge of the constant Fourier coefficient of $E(\phi,\Lambda)$ along the various parabolic $\mathbb{Q}$ - subgroups Q and certain intertwining operators involved. In order to prove that these values of Eisenstein series represent non-trivial cohomology classes in $H^*(\Gamma\backslash\bar{X},E)$ one considers its images under the various restrictions $r_Q^* : H^*(\Gamma\backslash\bar{X},E) \to H^*(e'(Q),E)$. By [24], 1.10. this reduces to study the image of the constant Fourier coefficient $E(\phi,\Lambda)_Q$ along Q under r_Q^* .

There is the following result, where we retain the notations of § 1 and § 2 .

3.2. Theorem.- Let $\Gamma \subset Sp_4(\mathbb{Z})$ be a torsionfree subgroup of finite index of the Siegel modular group of degree two, $X = Sp_4(\mathbb{R})/K$ the associated symmetric space (K a maximal compact subgroup of $G = Sp_4(\mathbb{R})$) and (τ,E) a rational representation of G (as in § 1).

(1) There is a direct sum decomposition of the cohomology of Γ as

$$H^*(\Gamma\backslash X,E) = H_!^*(\Gamma\backslash X,E) \oplus H_{Eis}^*(\Gamma\backslash X,E)$$

where $H_!^*(\Gamma\backslash X,E)$ is the image in $H^*(\Gamma\backslash X,E)$ of the cohomology with compact supports (cf. § 1), and $H_{Eis}^*(\Gamma\backslash X;E)$ is the space generated by Eisenstein cohomology classes (i.e. classes with a representative given by a regular value of an Eisenstein series attached to the classes in $H^*(\partial(\Gamma\backslash\bar{X}),E)$ or a residue of such at a point Λ_o). The space $H_{Eis}^*(\Gamma\backslash X,E)$ maps under the restriction $r^* : H^*(\Gamma\backslash\bar{X},E) \to H^*(\partial(\Gamma\backslash\bar{X}),E)$ isomorphically onto the image of r^* . (For a description of $\mathrm{Im}\, r^*$ and the construction of $H_{Eis}^*(\Gamma\ X,E)$ we refer to 3.4.)

(2) Each class in $H^*(\Gamma\backslash X,E)$ has a harmonic differential

form as a representative.

(3) The following spaces coincide (for notation we refer to § 1):

$$H^3_{cusp}(\Gamma\backslash X,E) = H^3_!(\Gamma\backslash X,E) = H^3_{(2)}(\Gamma\backslash X,E)$$

resp.

$$H^4_!(\Gamma\backslash X,E) = H^4_{(2)}(\Gamma\backslash X,E) \quad .$$

3.3. Corollary. - Each class in $H^*(\Gamma\backslash X,E)$ can be represented by a differential form whose coefficient functions are automorphic forms with respect to Γ, i.e. if $\underline{A}(\Gamma,G)$ denotes the space of automorphic forms with respect to Γ (cf. 1.4. [24]) then the morphism

$$H^*(\underline{g},K;\underline{A}(\Gamma,G) \otimes E) \to H^*(\underline{g},K;C^\infty(\Gamma\backslash G) \otimes E = H^*(\Gamma\backslash X,E)$$

induced by the inclusion $\underline{A}(\Gamma,G) \to C^\infty(\Gamma\backslash G)$ is surjective.

3.4. We describe mor precisely the image of the restriction map $r^* : H^*(\Gamma\backslash\overline{X},E) \to H^*(\partial(\Gamma\backslash\overline{X}),E)$ and in which way the space $H^*_{Eis}(\Gamma\backslash X,E)$ is built up. We indicate briefly also some of the methods of proof. For simplicity we assume that $\Gamma = \Gamma(m)$, $m \geqslant 3$, is a full congruence subgroup and that $E = \mathbb{C}$ is the trivial representation;we will omit the trivial coefficients in the notation. Recall that $H^q(\Gamma) = 0$ for $q \geqslant 5$ or $q = 1$ (cf. 2.6.).

The set $\Gamma\backslash\underline{\underline{P}}$ of Γ - conjugacy classes of parabolic

$\mathbb{Q}$ - subgroups of G falls into two subsets, the set $\Gamma\backslash(\underline{\underline{P}}_1 \cup \underline{\underline{P}}_2)$ of maximal ones and the set $\Gamma\backslash\underline{\underline{P}}_o$ of minimal ones. Denote a set of representatives for these by $\underline{P}_{max}$ resp. $\underline{P}_{min}$.

At first, we consider the cusp cohomology spaces

$$(1) \qquad \oplus \; H^q_{cusp}(e'(Q)) \;, \quad Q \in \underline{P}_{max}$$

of the faces $e'(Q)$ corresponding to the elements in $\underline{P}_{max}$. For $q = 1,\ldots,4$ the space in (1) corresponds to the ${}_fG$ - submodule

$$\bigoplus_{i=1}^{2} \mathrm{Ind}_{{}_fP_i}^{{}_fG} \, [H^1_{cusp}(\Gamma_{M_i}\backslash Z_{M_i}, H^{q-1}(\underline{n}_i))]$$

in the cohomology $H^q(\partial(\Gamma\backslash X))$ of the boundary (cf. 2.5.). The Eisenstein series $E(\phi,\Lambda)$ attached as in 3.1. to a given non-trivial cuspidal class of degree 4 $[\phi] \in H^4_{cusp}(e'(Q))$, $Q \in \underline{P}_{max}$, is holomorphic at the point $\Lambda_\phi = \rho_Q \in \underline{a}^*_{\mathbb{C}}$, and the form $E(\phi,\Lambda_\phi)$ is closed and harmonic and represents a non-trivial class in $H^4(\Gamma\backslash X)$ whose image under the restriction $r^4_R : H^4(\Gamma\backslash\overline{X}) \to H^4(e'(R))$ to a face $e'(R)$ in $\partial(\Gamma\backslash\overline{X})$ is given by

$$(2) \qquad r^4_R([E(\phi,\Lambda_\phi)]) = \begin{cases} [\phi] & \text{for } R \ \Gamma \text{ - conjugate to } Q \\ 0 & \text{otherwise} \end{cases}$$

This is shown in [26], § 3. As a consequence, these regular Eisenstein cohomology classes generate a subspace $H^4_{max}(\Gamma\backslash X) \subset H^4_{Eis}(\Gamma\backslash X)$ of the Eisenstein cohomology of Γ which restricts isomorphically onto the first term (1) in the cohomology $H^4(\partial(\Gamma\backslash\overline{X}))$. Its dimension can be read off from formula 2.7.(3).

For a minimal parabolic P in $\underline{P}_{min}$ the face $e'(P)$ is compact, whence $H^*_{cusp}(e'(P)) = H^*(e'(P))$. If we consider then in degree 4 the cohomology

$$(3) \qquad H^4(Y_o) = \oplus\, H^4(e'(P)) \;,\; P \in \underline{P}_{min}$$

of the faces corresponding to $P \in \underline{P}_{min}$ it turns out that not all of this space is in the image of r^4 . In order to deal with this question we have to consider all faces $e'(P)$, $P \in \underline{P}_{min}$, simultaneously, i.e. we have to work in an adelic frame-work (cf. [10], [22], [24], [27]). Again, there is an Eisenstein series $E(\phi_{\mathbb{A}},\Lambda)$ associated to an element $\phi_{\mathbb{A}}$ in $H^*(Y_o)$ which depends on the 2-dimensional complex parameter Λ ; the possible singularities of $E(\phi_{\mathbb{A}},\Lambda)$ lie along certain hyperplanes. However, if a class $\phi_{\mathbb{A}}$ of degree 4 satisfies certain conditions (which can be formulated in terms of Größencharakters, cf. [22], [23]) the corresponding Eisenstein series gives rise to a regular Eisenstein cohomology class (represented by an harmonic form) in $H^4(\Gamma\backslash X)$ which restricts under r^4 to the class $\phi_{\mathbb{A}}$ we started with. These classes span a subspace $H^4_{min}(\Gamma\backslash X)$ in $H^4_{Eis}(\Gamma\backslash X)$ of dimension $|\underline{P}_{min}| - |\underline{P}_{max}| + 1$ such that we have finally

$$(4) \qquad H^4_{Eis}(\Gamma\backslash X) = H^4_{max}(\Gamma\backslash X) \oplus H^4_{min}(\Gamma\backslash X)$$

and this space is generated by regular, non-square integrable Eisenstein cohomology classes. (By 2.5.(5) this proves also 3.2.(3) in degree 4). The proof of this result uses an explicit computation of the adelic intertwining operators occuring in the constant Fourier coefficient of $E(\phi_{\mathbb{A}},\Lambda)$ along P and the arithmetic information contained in there.

This adelic procedure plays also a very important role if we are dealing with the Eisenstein cohomology of Γ in degree three. As above, there is a decomposition

$$(5) \qquad H^3_{Eis}(\Gamma\backslash X) = H^3_{max}(\Gamma\backslash X) \oplus H^3_{min}(\Gamma\backslash X)$$

where $H^3_{min}(\Gamma\backslash X)$ is built up in the following way: Starting with a class $\phi_{\mathbb{A}}$ in $H^3(Y_o)$ which satisfies certain arithmetic conditions there is a uniquely determined singular hyperplane $\underline{r}$ for the associated Eisenstein series $E(\phi_{\mathbb{A}},\Lambda)$ such that the residue $\mathrm{Res}_{\underline{r}}\, E(\phi_{\mathbb{A}},\Lambda)$ along $\underline{r}$ gives rise to a closed and harmonic form on $\Gamma\backslash X$ which represents a non-trivial class in $H^3(\Gamma\backslash X)$. The space $H^3_{min}(\Gamma\backslash X)$ generated by these classes is of dimension $|\underline{P}_{max}|$ and restricts isomorphically under $r^3 : H^3(\Gamma\backslash\overline{X}) \to H^3(\partial(\Gamma\backslash\overline{X}))$ onto the second term in the right hand side of 2.5.(4). These residue classes can also be interpreted as classes obtained from Eisenstein series associated to non-cuspidal forms on the faces $e'(Q)$ with Q a maximal parabolic $\mathbb{Q}$ - subgroup of G . The first term in the description 2.5.(4) of $H^3(\partial(\Gamma\backslash\overline{X}))$ is partly lifted up by regular Eisenstein cohomology classes attached to $\psi_{\mathbb{A}}$ in $\oplus\, H^3_{cusp}(e'(Q))$, $Q \in \underline{P}_{max}$, in an adelic setting; these classes generate $H^3_{max}(\Gamma\backslash X)$. We recall the formula $\dim \mathrm{Im}\, r^2 + \dim \mathrm{Im}\, r^3 = \dim H^2(\partial(\Gamma\backslash X)) = \dim H^3(\partial(\Gamma\backslash X))$. In fact, it turns out that the part missed in $H^3(\partial(\Gamma\backslash X))$ by $\mathrm{Im}\, r^3$ is detected by classes in $H^2_{Eis}(\Gamma\backslash X)$ generated by residues of Eisenstein series attached to $\psi_{\mathbb{A}}$ in $\oplus\, H^2_{cusp}(e'(q))$, $Q \in \underline{P}_{max}$. This game is controlled by certain L - functions in the sense of Langlands

([13]) which occur in the constant Fourier coefficient of $E(\psi_A, \Lambda)$ and which have to be explicitly determined (cf. [27]). One needs some arithmetic information on the vanishing or non-vanishing of these Euler products attached to ψ_A at special points. As a consequence, the space $H^2_{Eis}(\Gamma\backslash X)$ consists out of square-integrable cohomology classes. Moreover, by a dimension argument one gets that the second term in the right hand side of 2.5.(3) is not in the image of r^2. In particular, we have $H^2_{Eis}(\Gamma\backslash X) = H^2_{max}(\Gamma\backslash X)$ with the usual notation.

By the relation 2.6.(1) and the result in degree 4 we see that $r^1 : H^1(\Gamma\backslash\overline{X}) \to H^1(\partial(\Gamma\backslash\overline{X}))$ is the trivial map. In degree zero we have that $H^o(\Gamma\backslash\overline{X}) = \mathbb{C} \to H^o(\partial(\Gamma\backslash\overline{X})) = \mathbb{C}$ is injective and a non-trivial class in $H^o(\Gamma\backslash\overline{X})$ is given by the successive residue of the Eisenstein series attached to a generating class in $H^o(Y_o)$.

Assertion 3.2.(2) and corollary 3.3. follow easily from this discussion. For the first assertion of 3.2.(3) we refer to 3.5.

Remark: The proof of 3.3. in the case of a non-trivial coefficient system $\tilde{E}$ given by a rational representation (τ, E) of G whose highest weight is sufficiently regular is simpler than the one described above for trivial coefficients. In particular, the Eisenstein cohomology occurs only in degrees three and four.

3.5. Following the general discussion in 1.3., 1.4. it is of interest to determine the list of irreducible unitary representations of $G = Sp_4(\mathbb{R})$ with non-zero relative Lie algebra cohomology. We consider the case of trivial coefficients $E = \mathbb{C}$. One derives from the general result of Vogan and Zuckerman [29], [30] that there are up to equivalence exactly eight distinct irreducible unitary representations (ω, H_ω) of G such that the relative Lie algebra cohomology with trivial coefficients

(1) $$H^*(\underline{g}, K; H_\omega^\infty) \neq 0$$

does not vanish. First of all, there are (up to equivalence) four discrete series representations which satisfy (1), two holomorphic ones and two antiholomorphic ones; they are characterized by the fact that they have the same infinitesimal character as the trivial representation. For a realisation we refer to [18]. For the relative Lie algebra cohomology with respect to such a discrete series representation ω_{dis} of G one has

(2) $$H^q(\underline{g}, K; H^\infty_{\omega_{dis}}) = 0 \quad \text{for} \quad q \neq 3$$

Then there are (up to equivalence) three other irreducible unitary representations; they have only non-trivial relative Lie algebra cohomology in degree two and four. Their Langlands parameters are given in [26], 3.5. The list is completed by the trivial representation $\mathbb{C}$.

Due to a result of Harish-Chandra and Wallach ([31], 4.3) the discrete series representations ω_{dis} occur already, if they occur as an irreducible constituent in the discrete spectrum $L^2_{dis}(\Gamma\backslash G)$, with its full multiplicity in the cuspidal spectrum $L^2_o(\Gamma\backslash G)$. Since there are as unitary representations only the discrete series ones which contribute non-trivially to cohomology in degree three we get (by 1.6. in [15])

(3) $$H^3_{cusp}(\Gamma\backslash X) = H^3_!(\Gamma\ X) = H^3_{(2)}(\Gamma\backslash X)$$

A similiar argument works for non-trivial coefficients E . The multiplicities $m(\omega_{dis},\Gamma)$ with which the ω_{dis} occur in $L^2_o(\Gamma\backslash G)$ can be interpreted as dimensions of certain spaces of automorphic forms with respect to Γ .

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