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Topological types of topologically finitely determined map-germs

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§0. Introduction

In this article we investigate the following two problems.

Problem(I). Is finite-C^0-$\mathcal{K}$-determinacy a topological invariant among analytic map-germs?

Problem(II). Do the topological types of all finitely-C^0-$\mathcal{K}$-determined map-germs have topological moduli, i.e. do they have infinitely many topological types with the cardinal number of continuum?

Let K = R or C. Two map-germs f and g : $(K^n, 0) \to (K^p, 0)$ are <u>topologically equivalent</u> or <u>C^0-$\mathcal{A}$-equivalent</u> if there exist germs of homeomorphisms $h_1 : (K^n, 0) \to (K^n, 0)$ and $h_2 : (K^p, 0) \to (K^p, 0)$ such that $g = h_2 \circ f \circ h_1$. A map-germ f is <u>finitely-C^0-$\mathcal{A}$-determined</u> (or <u>C^0-$\mathcal{A}$-finite</u> for short) if there is an integer k such that any germ g with $j^k(g) = j^k(f)$ is C^0-$\mathcal{A}$-equivalent to f. This is the topological version of J.Mather's $\mathcal{A}$-equivalence and $\mathcal{A}$-determinacy. We can also define C^0-$\mathcal{K}$, C^0-$\mathcal{R}$, C^0-$\mathcal{L}$, C^0-$\mathcal{C}$ equivalence and their determinacies in a similar way replacing diffeomorphisms in the C^∞ version by homeomorphisms.

We will give a precise definition of C^0-$\mathcal{K}$-equivalence at the end of Introduction.

Let $J^k_{\mathbb{K}}(n,p)$ denote the set of all polynomial map-germs : $(\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ with degree $\leq k$ and let $J^k_{\mathbb{K}}(n,p)_{C^0-\mathcal{K}}$ denote the set of all finitely-C^0-$\mathcal{K}$-determined elements of $J^k_{\mathbb{K}}(n,p)$. Let $J^k_{\mathbb{K}}(n,p)_{C^0-\mathcal{K}} / C^0$-$\mathcal{A}$ denote the set of topological equivalence classes of elements of $J^k_{\mathbb{K}}(n,p)_{C^0-\mathcal{K}}$. Then our main results are

<u>Theorem</u> 1. <u>Let</u> $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ <u>be holomorphic map-germs satisfying the followings;</u>

(1) f <u>is</u> C^0-$\mathcal{K}$-<u>finite</u>.

(2) f <u>and</u> g <u>are</u> C^0-$\mathcal{A}$-<u>equivalent</u>.

<u>Then</u> g <u>is also</u> C^0-$\mathcal{K}$-<u>finite</u>.

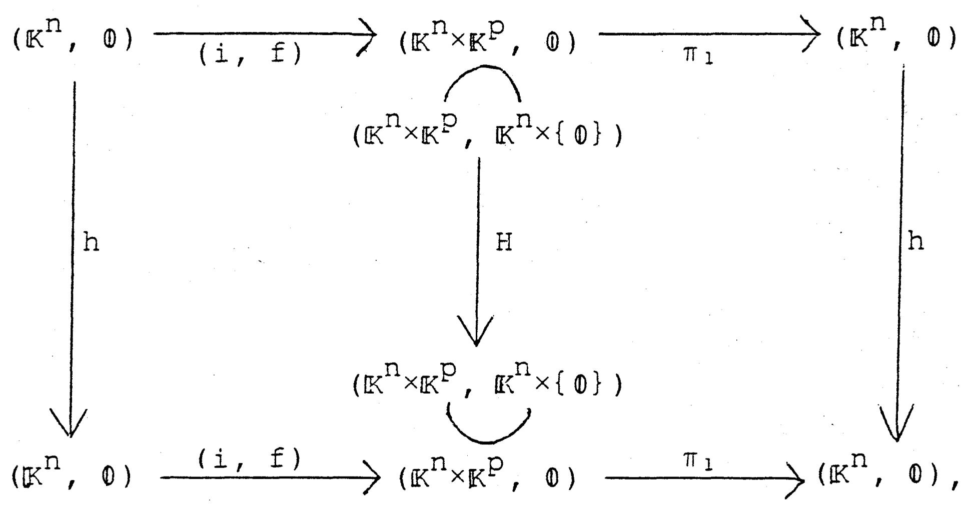
<u>Theorem</u> 2. $J^k_{\mathbb{C}}(n,p)_{C^0-\mathcal{K}} / C^0$-$\mathcal{A}$ <u>is a finite set for any positive integer</u> n, p, k.

<u>Theorem</u> 3. $J^k_{\mathbb{R}}(n,p)_{C^0-\mathcal{K}} / C^0$-$\mathcal{A}$ <u>is a finite set for</u> p = 1, 2, <u>any positive integer</u> n, k.

(2) $J^k_{\mathbb{R}}(n,p)_{C^0-\mathcal{K}} / C^0$-$\mathcal{A}$ <u>is an infinite set if</u> $n \geq 4$, $p \geq 4$, $k \geq 12$. <u>In fact they have topological moduli.</u>

When we compare our theorem 2 and theorem 3(2) with the results in [3], [2] and [10], it is interesting that there is a difference of cardinal numbers between the real case and the complex case. Theorem 1 and 2, combined with the fact that C^0-$\mathcal{K}$-finiteness is a generic property, tell us that finitely-C^0-$\mathcal{K}$-determined holomorphic map-germs are fascinating objects to study from the topological view point.

<u>Definition</u>. Two map-germs f and g : $(\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ are <u>C^0-$\mathcal{K}$-equivalent</u> if there exist germs of homeomorphisms $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ and $H : (\mathbb{K}^n \times \mathbb{K}^p, 0) \to (\mathbb{K}^n \times \mathbb{K}^p, 0)$ such that the following diagram commutes:



where $(i, f)(x) = (x, f(x))$ and $\pi_1(x, y) = x$.

§1. Remarks/Related topics

(1) The following simple example shows that the real version of theorem 1 does not hold.

<u>Example</u>. $f(x, y) = xy$, $g(x, y) = x^3 y$.

Function f is finitely-C^0-$\mathcal{K}$-determined but g is not, although f and g are topologically equivalent as real functions.

(2) Let $\mathcal{G}$ be any of $\mathcal{A}$, $\mathcal{K}$, $\mathcal{R}$, $\mathcal{L}$, $\mathcal{C}$. Then the following questions are more natural to be asked than our problems (I) and (II).

Problem(III). Is finite C^0-$\mathcal{G}$-determinacy a C^0-$\mathcal{G}$-invariant among analytic map-germs?

Problem(IV). Is $J^k_{\mathbb{K}}(n,p)_{C^0\text{-}\mathcal{G}} / C^0\text{-}\mathcal{G}$ a finite set for any positive integer n, p, k?

We have easily the following answers to these problems.

Problem (III)		$\mathcal{G}=\mathcal{A}$	$\mathcal{K}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{C}$
Answer	$\mathbb{K} = \mathbb{R}$	No	No	No	No	Yes
	$\mathbb{K} = \mathbb{C}$	?	?	Yes*)	?	Yes

*) This is a corollary of Theorem 1 (see the end of §3).

<table>
<tr><th colspan="2">Problem (IV)</th><th>$\mathcal{G} = \mathcal{A}$</th><th>$\mathcal{K}$</th><th>$\mathcal{R}$</th><th>$\mathcal{L}$</th><th>$\mathcal{C}$</th></tr>
<tr><td rowspan="2">Answer</td><td>$\mathbb{K} = \mathbb{R}$</td><td>finite [12]</td><td rowspan="2">finite</td><td rowspan="2">finite [6,7]</td><td>finite</td><td rowspan="2">finite</td></tr>
<tr><td>$\mathbb{K} = \mathbb{C}$</td><td>finite**)</td><td>?</td></tr>
</table>

**) This is a corollary of Theorem 2.

(3) In his paper [11] in which his second isotopy lemma and his condition a_f were announced for the first time, Thom gave an example of a family of polynomial mappings of $\mathbb{R}^3$ into $\mathbb{R}^3$ which contains continuously many topological types. Fukuda [3] and Aoki [2] showed that every family of polynomial functions of several variables or of polynomial map-germs of $\mathbb{R}^2$ into $\mathbb{R}^2$ (or $\mathbb{C}^2$ into $\mathbb{C}^2$) has only finitely many topological types. Recently Nakai [10] gave examples of families of polynomial map-germs of $\mathbb{R}^n$ into $\mathbb{R}^p$ (or $\mathbb{C}^n$ into $\mathbb{C}^p$) of degree k with n, p, k $\geq$ 3 or n $\geq$ 3, p $\geq$ 2, k $\geq$ 4 which contain continuously many topological types.

The examples of Thom and Nakai motivate to consider what will happen if we restrict objects of study within better map-germs, for example finitely-C^0-$\mathcal{K}$-determined ones? Thus arise our problems (I) and (II).

§2. Proof of theorem 1

Theorem 1. Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be holomorphic map-germs satisfying the followings;

(1) f is C^0-$\mathcal{K}$-finite.

(2) f and g are C^0-$\mathcal{A}$-equivalent.

Then g is also C^0-$\mathcal{K}$-finite.

Proof. If $n < p$, all points $x \in \mathbb{C}^n$ are singular points of f. If f is C^0-$\mathcal{K}$-finite, $f^{-1}(0) = \{0\}$ by geometric characterization of C^0-$\mathcal{K}$-finiteness ([14]). Hence by hypothesis (2), $g^{-1}(0)$ is also $\{0\}$ as germ. Therefore g is C^0-$\mathcal{K}$-finite by geometric characterization of C^0-$\mathcal{K}$-finiteness.

Hence our interest is essentially in the case $n \geq p$. By the hypothesis (2), we can put

$$g = (h')^{-1} \circ f \circ h$$

on a certain open neighborhood U of 0 in $\mathbb{C}^n$, where $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and $h' : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ are germs of homeomorphisms.

Suppose that g is not C^0-$\mathcal{K}$-finite. Then by geometric characterization of C^0-$\mathcal{K}$-finiteness,

$$\overline{\mathrm{Sing}(g) \cap g^{-1}(0) - \{0\}} \ni \{0\},$$

where $\mathrm{Sing}(g) = \{x \in \mathbb{C}^n : x \text{ is a singular point of } g\}$. Since f is C^0-$\mathcal{K}$-finite, for any $z^0 \in \mathrm{Sing}(g) \cap g^{-1}(0) - \{0\}$ which is sufficiently close to 0, there exists a sufficiently small positive number ε such that for any positive number $r (r < \varepsilon)$

$$h(r \cdot D^{2n}) \subset \{\text{regular point of } f\}$$

where $r \cdot D^{2n}$ is an open r-disk centered at z^0 in U.

We have

<u>Lemma</u> 1. $h(r\cdot D^{2n})$ <u>is</u> <u>homeomorphic</u> <u>to</u> $(f^{-1}(h'(z'))\cap h(r\cdot D^{2n})) \times f(h(r\cdot D^{2n}))$ <u>where</u> $h'(z')$ <u>is</u> <u>an</u> <u>arbitrary</u> <u>point</u> <u>in</u> $f(h\cdot D^{2n})) - \{0\}$.

We put $f = (f_1, \cdots, f_p)$, $g = (g_1, \cdots, g_p)$, $h' = (h'_1, \cdots, h'_p)$ and $z' = (z'_1, \cdots, z'_p)$.
Since $n \geqq p$, we may take z' in lemma 1 such that $z'_j \neq 0$ and $h'_j(z') \neq 0$ for any j $(1 \leqq j \leqq p)$.

$$\begin{aligned}
& h((r\cdot D^{2n})\cap g_j^{-1}(z'_j)) \\
&= h(r\cdot D^{2n})\cap(h)^{\circ}(g^{-1})(\{(z_1, \cdots, z_{j-1}, z'_j, z_{j+1}, \cdots, z_p) : \\
&\quad z_i \in \mathbb{C}\ (i \neq j)\}) \\
&= h(r\cdot D^{2n})\cap(f^{-1})\circ(h')(\{z_1, \cdots, z_{j-1}, z'_j, z_{j=1}, \cdots, z_p) : \\
&\quad z_i \in \mathbb{C}\ (i \neq j)\}).
\end{aligned}$$

By lemma 1, this space is homeomorphic to

$$\begin{aligned}
& h(r\cdot D^{2n})\cap f^{-1}(\{(z_1, \cdots, z_{j-1}, h'_j(z'), z_{j+1}, \cdots, z_p) : \\
&\quad z_i \in \mathbb{C}\ (i \neq j),\ h'_j(z') \neq 0\}) \\
&= h(r\cdot D^{2n})\cap f_j^{-1}(h'_j(z')).
\end{aligned}$$

In particular we have

Lemma 2. *The homology of the fiber of Milnor fibration of f_j at $h(z^0)$ and the homology of the fiber of Milnor fibration of g_j at z^0 are isomorphic for any j $(1 \leqq j \leqq p)$.*

On the other hand, after a suitable coordinate transformation we have

Lemma 3. (1) *z^0 is a singular point of g_j for a certain* j $(1 \leqq j \leqq p)$ *for* $z^0 \in \mathrm{Sing}(g) \cap g^{-1}(0) - \{0\}$.
(2) $h(z^0)$ *is a regular point of f_j for any* j $(1 \leqq j \leqq p)$.

Lemma 2 and lemma 3 contradict to A'campo's result ([1]). The map-germ g must be C^0-$\mathcal{K}$-finite. □

Corollary. *Let* $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ *be holomorphic map-germs satisfying the followings;*

(1) f *is* C^0-$\mathcal{R}$-*finite*.

(2) f *and* g *are* C^0-$\mathcal{R}$-*equivalent*.

Then g *is also* C^0-$\mathcal{R}$-*finite*.

Proof of corollary. In fact the above proof of theorem 1 shows that for all holomorphic map-germs $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ such that $f = (h')^{-1} \circ g \circ h$ as germs at 0, where $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ $h' : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ are germs of homeomorphisms, $h(\mathrm{Sing}(f)) = \mathrm{Sing}(g)$ as germs at 0. Hence our corollary follows from theorem 1 by geometric characterization of C^0-$\mathcal{R}$-finiteness ([14]). □

§ 3. Thom-Mather's stratification theory

Let X^r and Y^s be differentiable submanifolds of $\mathbb{R}^n$ having dimension r and s respectively. We say the pair (X, Y) satisfies Whitney's condition (a) at a point of Y if for any sequence of points x_i of X such that $x_i \to y$ and the tangent space $T_{x_i}(X)$ to X at x_i converge to some r-plane $\tau(\subset \mathbb{R}^n)$, we have $T_y(Y) \subset \tau$. We say that (X, Y) satisfies Whitney's condition (b) at a point y of Y if for any sequences $\{x_i \in X\}$ and $\{y_i \in Y\}$ such that $x_i \neq y_i$, $x_i \to y$ and $y_i \to y$ and such that $T_{x_i}(X)$ converge to some r-plane $\tau(\subset \mathbb{R}^n)$ and the secants $\widehat{x_i y_i}$ joining x_i with y_i converge to some line $\ell(\subset \mathbb{R}^n)$, we have $\ell \subset \tau$. Note that condition (b) is stronger than condition (a).

We say (X, Y) satisfies condition (a) (resp. (b)) if it satisfies condition (a) (resp. (b)) at every point y of Y.

A Whitney stratification of a subset E of $\mathbb{R}^n$ is a family $S = \{X_i\}$ of connected smooth submanifolds of $\mathbb{R}^n$, called strata of S, such that the strata are pairwise disjoint, any pair (X, Y) of strata of S satisfies Whitney's condition (a) and (b), the family S is locally finite and for any pair X and Y of strata of S if $\overline{X} \cap Y \neq \phi$, then we have $\partial X \supset Y$.

A set with one of its stratification is called a stratified set. Let S(E) and S(F) be Whitney stratifications of sets $E(\subset \mathbb{R}^n)$ and $F(\subset \mathbb{R}^p)$. A continuous mapping $f : E \to F$ is a stratified mapping if it is extendable to a smooth mapping of a neighborhood of E in $\mathbb{R}^n$ into $\mathbb{R}^p$ and if for any stratam X of S(E), f(X) is contained in a stratam Y of S(F) and $f|X : X \to Y$ is a submersion.

Let X and Y be smooth submanifolds of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^p$ be a smooth mapping defined in a neighborhood U of $X \cup Y$ in $\mathbb{R}^n$. Suppose that the restricted mapping $f|X : X \to \mathbb{R}^p$ and $f|Y : Y \to \mathbb{R}^p$ are of constant ranks. We say that the pair (X, Y) satisfies <u>condition</u> a_f if for any point y of Y and for any sequence $\{x_i \in X\}$ converging to y such that the sequence of the planes $\ker(d(f|X)_{x_i})$ converges to a plane κ, we have $\ker(d(f|Y)_y) \subset \kappa$. Where $\ker(d(f|X)_x)$ denotes the kernel of the differential

$$d(f|X)_x : T_x(X) \to T_{f(x)}(\mathbb{R}^p)$$

of $f|X$ at x.

A <u>Thom mapping</u> $f : E \to F$ is a stratified mapping such that any pair of strata of S(E) satisfies condition a_f.

<u>Proposition</u> 1. (<u>Thom's local isotopy lemma</u>) <u>Let</u> $f : E \to F$ <u>be a Thom mapping and let</u> $g : F \to V$ <u>be a stratified mapping with respect to stratifications</u> S(E), S(F) <u>and</u> {V}, <u>where</u> V <u>is a connected smooth manifold and</u> E <u>and</u> F <u>are locally compact. If points</u> p <u>and</u> q <u>of</u> E <u>belong to the same stratum of</u> E, <u>then the germ at</u> p <u>of the restriction</u> $f|A_{g(f(p))} : A_{g(f(p))} \to B_{g(f(p))}$ <u>and the germ at</u> q <u>of the restriction</u> $f|A_{g(f(q))} : A_{g(f(q))} \to B_{g(f(q))}$ <u>are</u> C^0-$\mathcal{A}$<u>-equivalent</u>, <u>where</u> $A_{g(f(p))} = (g \circ f)^{-1}(g(f(p)))$ <u>and</u> $B_{g(f(p))} = g^{-1}(g(f(p)))$.

Let A be a semi-algebraic set. Then a *semi-algebraic stratification* of A is a Whiyney stratification of A such that each stratum of A is a semi-algebraic set and the number of these strata is finite.

Proposition 2. *Let* $A, C \subset \mathbb{R}^n$ *and* $B, D \subset \mathbb{R}^p$ *be semi-algebraic sets such that* $A \subset C$ *and* $B \subset D$. *Let* $f : \mathbb{R}^n \to \mathbb{R}^p$ *be a polynomial map with* $f(C) \subset D$. *Then there exist semi-algebraic stratifications* $S(C)$ *and* $S(D)$ *such that the map* $f|C : C \to D$ *is a stratified map and* A *and* B *are stratified subsets of* C *and* D *respectively. Moreover given any semi-algebraic stratifications* $S(C)$ *and* $S(D)$, *there exist semi-algebraic refinements* $S'(C)$ *of* $S(C)$ *and* $S'(D)$ *of* $S(D)$ *such that the map* $f|C : C \to D$ *is a stratified map and* A *and* B *are stratified subsets of* C *and* D *respectively*.

For the proof of proposition 1, see [8] or [4] and for the proof of proposiiton 2, see [3].

§ 4. Proofs of theorem 2 and theorem 3(1)

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. We also identify $J^k_{\mathbb{K}}(n,p)$ not only with the set of polynomial mappings of $(\mathbb{K}^n, 0)$ into $(\mathbb{K}^p, 0)$ with degree $\leq k$, but also with an Euclidean space $\mathbb{R}^{\varepsilon pN}$ of a suitable dimension εpN ($\varepsilon = 1$ if $\mathbb{K} = \mathbb{R}$ and $\varepsilon = 2$ if $\mathbb{K} = \mathbb{C}$) as usual.

Under these identification the mapping

$$F : J^k_{\mathbb{K}}(n,p) \times \mathbb{R}^{\varepsilon n} \longrightarrow J^k_{\mathbb{K}}(n,p) \times \mathbb{R}^{\varepsilon p}$$

defined by $F(f, x) = (f, f(x))$ can be considered as a real polynomial mapping, where $f \in J^k_{\mathbb{K}}(n,p)$, $x \in \mathbb{R}^{\varepsilon n}$ and $\varepsilon = 1$ if $\mathbb{K} = \mathbb{R}$ or $\varepsilon = 2$ if $\mathbb{K} = \mathbb{C}$.

<u>Lemma</u> 4. $J^k_{\mathbb{K}}(n,p)_{C^0\text{-}\mathcal{K}}$ <u>is a semi-algebraic subset in</u> $J^k_{\mathbb{K}}(n,p) \cong \mathbb{R}^{\varepsilon pN}$.

<u>Proof of lemma</u> 4. By geometric characterization, for each f in $J^k_{\mathbb{K}}(n,p)$, f is contained in $J^k_{\mathbb{K}}(n,p)_{C^0\text{-}\mathcal{K}}$ if and only if there exists a neighborhood V of 0 in $\mathbb{K}^n$ such that $V \cap \mathrm{Sing}(f) \cap f^{-1}(0) - \{0\} = \phi$, which is equivalent that there exists a neighborhood V of 0 in $\mathbb{R}^{\varepsilon n}$ such that

$$(\{f\} \times V) \cap \mathrm{Sing}(F) \cap F^{-1}(J^k_{\mathbb{K}}(n,p)) \times \{0\}) - \{f \times 0\} = \phi.$$

Clearly $A \subset \mathbb{R}\times\mathbb{R}^{\varepsilon n}\times J^k_{\mathbb{K}}(n,p)\times\mathbb{R}^{\varepsilon n}$ comprising all quadruplets (t, y, f, x) with $(f, x) \in F^{-1}(0) \cap \mathrm{Sing}(F) - J^k_{\mathbb{K}}(n,p)\times\{0\}$ and $|x-y| < t$ is semi-aljebraic. Now consider the following polynomial projections;

$$(\mathbb{R}\times\mathbb{R}^{\varepsilon n}\times J^k_{\mathbb{K}}(n,p))\times\mathbb{R}^{\varepsilon n} \xrightarrow{p_1} \mathbb{R}\times\mathbb{R}^{\varepsilon n}\times J^k_{\mathbb{K}}(n,p) \xrightarrow{p_2} \mathbb{R}^{\varepsilon n}\times J^k_{\mathbb{K}}(n,p) \xrightarrow{p_3} J^k_{\mathbb{K}}(n,p) \ .$$

Tarski-Seidenberg theorem implies

$$(\mathbb{R}^{\varepsilon n}\times J^k_{\mathbb{K}}(n,p) - p_2(\mathbb{R}\times\mathbb{R}^{\varepsilon n}\times J^k_{\mathbb{K}}(n,p) - p_1(A))) \cap (\{0\}\times J^k_{\mathbb{K}}(n,p))$$

is semi-algebraic. This set is denoted by B.
A minor computation verifies that

$$J^k_{\mathbb{K}}(n,p)_{C^0-\mathcal{X}} = J^k_{\mathbb{K}}(n,p) - p_3(B),$$

which is also semi-algebraic. □

Now we consider the following sequence;

$$J^k_{\mathbb{K}}(n,p)\times\mathbb{R}^{\varepsilon n} \xrightarrow{F} J^k_{\mathbb{K}}(n,p)\times\mathbb{R}^{\varepsilon p} \xrightarrow{\pi} J^k_{\mathbb{K}}(n,p)$$

where π is the canonical projection. Since F and π are polynomial mappings, by proposition 2 and lemma 4 there exist semi-algebraic stratifications $S(J^k_{\mathbb{K}}(n,p)\times\mathbb{R}^{\varepsilon n})$, $S(J^k_{\mathbb{K}}(n,p)\times\mathbb{R}^{\varepsilon p})$ and $S(J^k_{\mathbb{K}}(n,p))$ with which F and π are stratified mappings and $J^k_{\mathbb{K}}(n,p)\times\{0\}$, $J^k_{\mathbb{K}}(n,p)\times\{0\}$ and $J^k_{\mathbb{K}}(n,p)_{C^0-\mathcal{X}}$ are startified subsets of $J^k_{\mathbb{K}}(n,p)\times\mathbb{R}^{\varepsilon n}$, $J^k_{\mathbb{K}}(n,p)\times\mathbb{R}^{\varepsilon p}$ and $J^k_{\mathbb{K}}(n,p)$ respectively.

Remark that Sing(F) is a stratified subset of the set $J^k_{\mathbb{K}}(n,p)\times\mathbb{R}^{\varepsilon n}$. Then for each stratum Z of $S(J^k_{\mathbb{K}}(n,p)_{C^0-\mathcal{K}})$, the sequence of restricted mappings,

$$(*) \qquad Z\times\mathbb{R}^{\varepsilon n} \xrightarrow{F} Z\times\mathbb{R}^{\varepsilon p} \xrightarrow{\pi} Z$$

is also a sequence of stratified maps with the canonically induced semi-algebraic stratifications $S(Z\times\mathbb{R}^{\varepsilon n})$, $S(Z\times\mathbb{R}^{\varepsilon p})$ and $\{Z\}$ from $S(J^k_{\mathbb{K}}(n,p)\times\mathbb{R}^{\varepsilon n})$, $S(J^k_{\mathbb{K}}(n,p)\times\mathbb{R}^{\varepsilon p})$ and $S(J^k_{\mathbb{K}}(n,p))$ respectively, where F and π in (*) stand for $F|_{Z\times\mathbb{R}^{\varepsilon n}}$ and $\pi|_{Z\times\mathbb{R}^{\varepsilon p}}$ respectively. We use this sequence (*) to prove theorem 2 and theorem 3(1).

Proof of theorem 2.

Theorem 2. $J^k_{\mathbb{C}}(n,p)_{C^0-\mathcal{K}}/C^0\text{-}\mathcal{A}$ is a finite set for any positive integer n, p, k.

Proof. We consider the stratified sequence (*). We want to state that for each stratum Z of $S(J^k_{\mathbb{C}}(n,p)_{C^0-\mathcal{K}})$ there exists a semi-algebraic stratification S′(Z) of Z such that for each stratum W of S′(Z) there exists a semi-algebraic neighborhood U_W of $W\times\{0\}$ in $W\times\mathbb{R}^{2n}$ and the restricted mapping

$$U_W \xrightarrow{F} W\times\mathbb{R}^{2p}$$

s a Thom mapping with respect to the canonically induced semi-algebraic tratifications $S((W\times\mathbb{R}^{2n})\cap U_W)$, $S(W\times\mathbb{R}^{2p})$.

By geometric characterization, any mapping $f\in Z$ has the ondition that $\mathrm{Sing}(f)\cap f^{-1}(0) - \{0\} = \phi$ as germs. It is well-nown that if $\mathrm{Sing}(f)\cap f^{-1}(0) - \{0\} = \phi$ as germs then there xists a neighborhood U of 0 in $\mathbb{C}^n$ such that the restriction

$$f|_{U\cap \mathrm{Sing}(f)} : U\cap\mathrm{Sing}(f) \longrightarrow \mathbb{C}^p$$

s proper and finite to one. As $\mathrm{Sing}(F) = \{(f, \mathrm{Sing}(f))\,|\,f\in J^k_{\mathbb{K}}(n,p)\}$, e can deduce that there exists a semi-algebraic stratification ′(Z) of Z such that for any stratum W of S′(Z) there exists semi-algebraic neighborhood U_W of $W\times\{0\}$ in $W\times\mathbb{R}^{2n}$ and the estricted mapping

$$U_W\cap\mathrm{Sing}(F) \xrightarrow{F} W\times\mathbb{R}^{2p}$$

s proper and finite to one.

Also the restricted mapping

$$U_W \xrightarrow{F} W\times\mathbb{R}^{2p}$$

is a stratified mapping with respect to the canonically induced semi-algebraic stratifications $S((W\times\mathbb{R}^{2n})\cap U_W)$, $S(W\times\mathbb{R}^{2p})$ and $U_W\cap \mathrm{Sing}(F)$ is a stratified subset of $(W\times\mathbb{R}^{2n})\cap U_W$.

For any point $(f, x)\in U_W\cap\mathrm{Sing}(F)$, as the restricted mapping $U_W\cap\mathrm{Sing}(F)\xrightarrow{F} W\times\mathbb{R}^{2p}$ is proper and finite to one, $\ker(d(F|X)_{(f,x)}$ $= 0$, where X is a stratum of the startification $S((W\times\mathbb{R}^{2n})\cap U_W)$ which contains (f, x). For any pair of non-singular strata (X,Y) such that $X, Y\in S(W\times\mathbb{R}^{2n})\cap U_W)$ and $\bar{X}\supset Y$, where non-singular means that for any point $(f, x)\in Y$ $(f,x)\notin U_W\cap\mathrm{Sing}(F)$, the pair (X, Y) always satisfies condiiton a_f.

These obser vations show that the restricted mapping

$$U_W \xrightarrow{F} W\times\mathbb{R}^{2p}$$

is a Thom mapping with respect to the canonically induced semi-algebraic stratifications $S((W\times\mathbb{R}^{2n})\cap U_W)$, $S(W\times\mathbb{R}^{2p})$.

Now the proof of theorem 2 follows from proposition 1. □

Proof of theorem 3(1).

Theorem 3(1). $J^k_{\mathbb{R}}(n,p)_{C^0-\mathcal{K}}/C^0-\mathcal{A}$ is a finite set for $p = 1, 2$, and for any positive integers n, k.

Proof. In the function case, that is $p = 1$, for any positive integers n, k, our theorem is contained in the local case of Fukuda's theorem [3]. So we prove our theorem only in the case $p = 2$.

Consider the stratified sequence (*). Let X, Y be strata of $S(Z\times\mathbb{R}^n)$ such that $\overline{X} - X \supset Z\times\{0\}$, $\overline{Y} - Y \supset Z\times\{0\}$ and $\overline{X} \supset Y$, where $\overline{X}$ denotes the closure of X in $Z\times\mathbb{R}^n$. Let $\widetilde{X}$, $\widetilde{Y}$ be strata of $S(Z\times\mathbb{R}^2)$ such that $F(X) \subset \widetilde{X}$ and $F(Y) \subset \widetilde{Y}$. In the case $\widetilde{X} = \widetilde{Y}$, the existance theorem of tubular neighborhoods of strata shows that the pair (X, Y) satisfies condition a_f (see [8]).
There are three possibilities of dimensions of a pair of strata $(\widetilde{X}, \widetilde{Y})$ when $\widetilde{X} \neq \widetilde{Y}$ and $\overline{\widetilde{X}} \supset \widetilde{Y}$ as follows, where $\overline{\widetilde{X}}$ denotes the closure of $\widetilde{X}$ in $Z\times\mathbb{R}^2$.

$\dim\widetilde{X}$	$\dim\widetilde{Y}$
2 + dimZ	1 + dimZ
2 + dimZ	0 + dimZ
1 + dimZ	0 + dimZ

(I) The case $(\dim\widetilde{X}, \dim\widetilde{Y}) = (2 + \dim Z, 0 + \dim Z)$ or $(1 + \dim Z, 0 + \dim Z)$.

In this case, by geometric characterization and $\mathrm{Sing}(F) = \{(f, \mathrm{Sing}(f)) \mid f \in J^k_{\mathbb{R}}(n,p)\}$, there exists a semi-algebraic neighborhood U_Z of $Z\times\{\mathbb{0}\}$ in $Z\times\mathbb{R}^n$ such that the pair $(X\cap U_Z, Y\cap U_Z)$ is a non-singular pair. Hence the pair $(X\cap U_Z, Y\cap U_Z)$ satisfies condition a_f.

(II) The case $(\dim\widetilde{X}, \dim\widetilde{Y}) = (2 + \dim Z, 1 + \dim Z)$.

It is sufficient to consider only the case $Y \subset \mathrm{Sing}(F)$. In this case there exists a semi-algebraic neighborhood U_Z of $Z\times\{\mathbb{0}\}$ in $Z\times\mathbb{R}^n$ such that for each point $(f^0, \mathbf{x}^0) \in Y\cap U_Z$, $\mathrm{rank} F$ at $(f^0, \mathbf{x}^0)$ is $1 + \dim J^k_{\mathbb{R}}(n,2)$. By suitable analytic coordinate transformations we can assume that $F(f, \mathbf{x}) = (f, x_1, g(f, x_1, \cdots, x_n))$ in a sufficiently small neighborhood $V_{(f^0,\mathbf{x}^0)}$ of $(f^0, \mathbf{x}^0)$ in U_Z, where $\mathbf{x} = (x_1, \cdots, x_n)$ and $g : V_{(f^0,\mathbf{x}^0)} \to \mathbb{R}$ is an analytic function.

We set $\mathbf{x}^0 = (x^0_1, \cdots, x^0_n)$ under this coordinate chart.

We also set

$$D = \{(f, \mathbf{x}) \in V_{(f^0, \mathbf{x}^0)} \mid x_1 = x^0_1,\ f = f^0\},$$

$$D' = \{(f, \mathbf{x}) \in V_{(f^0, \mathbf{x}^0)} \mid x_1 = x^0_1\}$$ and

$$\widetilde{D} = \{(f, y_1, y_2) \in Z\times\mathbb{R}^2 \mid (f, y_1, y_2) \in F(V_{(f^0, \mathbf{x}^0)}),\ y_1 = x^0_1\}.$$

We may assume that $g^{-1}(g(f^0, \mathbf{x}^0)) \cap V_{(f^0, \mathbf{x}^0)} = Y\cap D'$.

In the sufficiently small neighborhood $V_{(f^0, x^0)}$ of (f^0, x^0) in U_Z, we can assume that the stratum Y is transversal to the submanifold D. Since the mapping $g : V_{(f^0, x^0)} \to \mathbb{R}$ is a function, the existance theorem of a good stratification implies that there exists a stratification $S(Y \cap D)$ such that the restricted function

$$g|_{(X \cup Y) \cap D} : (X \cup Y) \cap D \to (\tilde{X} \cup \tilde{Y}) \cap \tilde{D}$$

is a Thom mapping with respect to the stratifications $\{X \cap D, S(Y \cap D)\}$ and $\{\tilde{X} \cap \tilde{D}, \tilde{Y} \cap \tilde{D}\}$ (see [5] or [3]).

We also see that in the sufficiently small neighborhood $V_{(f^0, x^0)}$ of (f^0, x^0) in U_Z, the restricted mapping

$$F|_{X \cup Y} : X \cup Y \to \tilde{X} \cup \tilde{Y}$$

is considered as an analytically trivial unfolding of the restricted function

$$g|_{(X \cup Y) \cap D} : (X \cup Y) \cap D \to (\tilde{X} \cup \tilde{Y}) \cap \tilde{D}.$$

Therefore the restricted mapping

$$F|_{(X \cup Y) \cap V_{(f^0, x^0)}} : (X \cup Y) \cap V_{(f^0, x^0)} \to (\tilde{X} \cup \tilde{Y})$$

is a Thom mapping with respect to the canonically extended stratifications from $\{X \cap D, S(Y \cap D)\}$ and $\{\tilde{X} \cap \tilde{D}, \tilde{Y} \cap \tilde{D}\}$.

By the above (I) and (II), we see that for each stratum Z of $S(J^k_{\mathbb{R}}(n,2)_{C^0}-\mathcal{K})$ there exist a neighborhood U_Z of $\mathbb{0}$ in $Z\times\mathbb{R}^n$ and stratifications $S''(Z\times\mathbb{R}^n)$, $S''(Z\times\mathbb{R}^2)$ such that the restricted mapping

$$F|_{U_Z} : U_Z \to Z\times\mathbb{R}^2$$

is a Thom mapping with respect to the canonically induced stratifications $S''((Z\times\mathbb{R}^n)\cap U_Z)$, $S''(Z\times\mathbb{R}^2)$.

Now the proof of theorem 3(2) follows from proposition 1. □

§ 5. Thom's example

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $f, g : \mathbb{K}^n \to \mathbb{K}^p$ be C^∞ (for $\mathbb{K} = \mathbb{R}$) or holomorphic (for $\mathbb{K} = \mathbb{C}$) mappings. We say f and g are <u>topologically equivalent</u> if there are homeomorphisms $h : \mathbb{K}^n \to \mathbb{K}^n$ and $h' : \mathbb{K}^p \to \mathbb{K}^p$ such that $f = (h')^{-1} \circ g \circ h$.

In [11], Thom considered the following one-parameter real polynomial mapping family $P(k) : \mathbb{R}^3 \to \mathbb{R}^3$, where k is a real parameter, and he proved that if any two fixed real numbers k_1, k_2 are not equal then $P(k_1)$ and $P(k_2)$ are not topologically equivalent.

$$P(k): \begin{cases} X = [x(x^2+y^2-a^2)-2ayz]^2[(x+ky)(x^2+y^2-a^2)-2a(y-kx)z]^2 \\ Y = x^2+y^2-a^2 \\ Z = z \end{cases}$$

where (x, y, z), (X, Y, Z) are coordinates of the source space and the target space respectively, a is a non-zero fixed real number and k is a real parameter.

In this section, we recall quickly Thom's idea of proof, which is used in the proof of theorem 3(2).

<u>Thom's idea of proof</u>.

Let k_0 be a fixed real number. We consider the following surface $H(k_0)$ and circle $C(k_0)$.

$$H(k_0) = \{(x, y, z) \in \mathbb{R}^3 \mid [x(x^2+y^2-a^2)-2ayz]^2[(x+k_0y)(x^2+y^2-a^2)-2a(y-k_0x)z]^2 = 0\}$$

$$C(k_0) = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2+y^2-a^2 = 0\}$$

Then $C(k_0) \subset H(k_0)$ and $C(k_0) \subset \mathrm{Sing}(P(k_0))$.

We also consider the following two surfaces $H_1(k_0)$ and $H_2(k_0)$.

$$H_1(k_0) = \{(x, y, z) \in \mathbb{R}^3 \mid x(x^2+y^2-a^2)-2ayz = 0\}$$

$$H_2(k_0) = \{(x, y, z) \in \mathbb{R}^3 \mid (x+k_0y)(x^2+y^2-a^2)-2a(y-k_0x)z = 0\}$$

Then $H(k_0) = H_1(k_0) \cup H_2(k_0)$ and $H_1(k_0) \cap H_2(k_0) = C(k_0) \cup \{(0, 0, z) \in \mathbb{R}^3\}$.

Furthermore we have

$$P(k_0)(H_1(k_0) \cap \{(x, y, z) \in \mathbb{R}^3 \mid \ell x+my = 0\})$$
$$= \{(0, Y, Z) \in \mathbb{R}^3 \mid mY+2a\ell Z = 0\}$$

$$P(k_0)(H_2(k_0) \cap \{(x, y, z) \in \mathbb{R}^3 \mid \ell x+my = 0\})$$
$$= \{(0, Y, Z) \in \mathbb{R}^3 \mid (m-k_0\ell)Y+2a(\ell+k_0m)Z = 0\}$$

for any two real numbers ℓ, m such that $\ell^2+m^2 \neq 0$.

Now if there exist homeomorphisms $h, h' : \mathbb{R}^3 \to \mathbb{R}^3$ such that $P(k_0) = (h')^{-1} \circ P(k_1) \circ h$ for any two fixed non-zero real numbers k_0, k_1 ($k_0 \neq k_1$), then we have the following.

Lemma 5. (1) $h(H(k_0)) = H(k_1)$.

(2) $h(C(k_0)) = C(k_1)$.

(3) For any germ of continyoys curve $\varphi(t)$ at any point $p \in C(k_0)$ (resp. $C(k_1)$) in $H(k_0)$ (resp. $H(k_1)$), $P(k_0)$ (resp. $P(k_1)$) maps $\varphi(t)$ to a germ of continuous curve at $(0, 0, 0) \in \{(0, Y, Z) \in \mathbb{R}^3\}$ in $\{(0, Y, Z) \in \mathbb{R}^3\}$ and this germ of curve has a tangent line at $(0, 0, 0)$.

By this fact, if k_0, k_1 are both non-zero, then the restricted homeomorphism $h|_{C(k_0)} : C(k_0) \to C(k_1)$ must have the property that for any two points $x, y \in C(k_0)$ such that angle $\angle\widehat{xy} = \mathrm{Tan}^{-1}(k_0)$ angle $\angle\widehat{h(x)h(y)} = \mathrm{Tan}^{-1}(k_1)$. But this contradicts to Van Kampen's theorem in [13].

Remark. It is easily seen that if we change the one-parameter real polynomial mapping family $P(k) : \mathbb{R}^3 \to \mathbb{R}^3$ to $\widetilde{P}(k) : \mathbb{R}^3 \to \mathbb{R}^3$ as follows, then we also have the property that if $k_0 \neq k_1$ then $P(k_0)$ and $P(k_1)$ are not topologically equivalent.

$$\widetilde{P}(k): \begin{cases} X = [x(x^2+y^2-a^2)-yz]^2[(x+ky)(x^2+y^2-a^2)-(y-kx)z]^2 \\ Y = x^2+y^2-a^2 \\ Z = z \end{cases}$$

§6. Proof of theorem 3(2)

Theorem 3(2). $J^k_{\mathbb{R}}(n,p)_{C^0-\mathcal{K}}/C^0-\mathcal{A}$ is as infinite set if $n \geqq 4$, $p \geqq 4$, $k \geqq 12$. In fact they have topological moduli.

Proof. We divide the conditions on dimensions into the following three cases.

(Case I) $n = 4$, $p \geqq 4$.

(Case II) $n \leqq p$, $n > 4$.

(Case III) $n > p$, $p \geqq 4$.

Proof in case I. Let $\tilde{Q}(k) : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ be a one-parameter polynomial map-germ family defined as follows;

$$\tilde{Q}(k): \begin{cases} X = [x(x^2+y^2-u^2)-yz]^2[(x+ky)(x^2+y^2-u^2)-(y-kx)z]^2 \\ Y = x^2+y^2-u^2 \\ Z = z \\ U = u^2 \end{cases}$$

where (x, y, z, u), (X, Y, Z, U) are coordinates of the source and the target spaces respectively and k is a real porameter. Let $P'(k)$ be a one-parameter polynomial map-germ family defined as follows;

$$P'(k) : (\mathbb{R}^4, 0) \to (\mathbb{R}^p, 0)$$
$$P'(k)(x, y, z, u) = (\tilde{Q}(k), 0).$$

For any fixed k_0, $P'(k_0)^{-1}(\{0\}) = \{0\}$. So $P'(k_0)$ is a C^0-$\mathcal{K}$-finite polynomial map-germ by geometric characterization of C^0-$\mathcal{K}$-finiteness.

Let $H_1'(k_0)$, $H_2'(k_0)$, $H'(k_0)$ and $C'(k_0)$ be as follows;

$$H_1'(k_0) = \{(x, y, z, u) \in \mathbb{R}^4 \mid x(x^2+y^2-u^2)-yz = 0\},$$
$$H_2'(k_0) = \{(x, y, z, u) \in \mathbb{R}^4 \mid (x+k_0y)(x^2+y^2-u^2)-(y-k_0x)z = 0\},$$
$$H'(k_0) = H_1'(k_0) \cup H_2'(k_0),$$
$$C'(k_0) = \{(x, y, 0, u) \in \mathbb{R}^4 \mid x^2+y^2-u^2 = 0\}.$$

Then we have

$$P'(k_0)(H_1'(k_0) \cap \{(x, y, z, u) \in \mathbb{R}^4 \mid \ell x+my = 0\})$$
$$= \{(0, Y, Z, U, \mathbb{0}) \in \mathbb{R}^p \mid mY+\ell Z = 0\},$$

$$P'(k_0)(H_2'(k_0) \cap \{(x, y, z, u) \in \mathbb{R}^4 \mid \ell x+my = 0\})$$
$$= \{(0, Y, Z, U, \mathbb{0}) \in \mathbb{R}^p \mid (m-k_0\ell)Y+(\ell+k_0m)Z = 0\},$$

for any two real numbers ℓ, m such that $\ell^2+m^2 \neq 0$.

If there are germs of homeomorphisms $h : (\mathbb{R}^4, \mathbb{0}) \to (\mathbb{R}^4, \mathbb{0})$, $h' : (\mathbb{R}^p, \mathbb{0}) \to (\mathbb{R}^p, \mathbb{0})$ such that $P'(k_0) = (h')^{-1} \circ P'(k_1) \circ h$ as germs at $\mathbb{0}$ for any two fixed non-zero real numbers k_0, k_1 $(k_0 \neq k_1)$, then we have the following lemma like lemma 5 in §6.

Lemma 6. (1) $h(H'(k_0)) = H'(k_1)$ *as germs at* $\mathbb{0}$.

(2) $h(C'(k_0)) = C'(k_1)$ *as germs at* $\mathbb{0}$.

(3) $h(C'(k_0) \cap P'(k_0)^{-1}((0, 0, 0, u_0, \mathbb{0}))) =$
$C'(k_1) \cap P'(k_1)^{-1}((0, 0, 0, h'_4((0, 0, 0, u_0, \mathbb{0})), \mathbb{0}))$ *as germs at* $\mathbb{0}$ *for any real number* u_0 *close to zero and* h'_4 *is the forth component function of* h' (*see Figure I*).

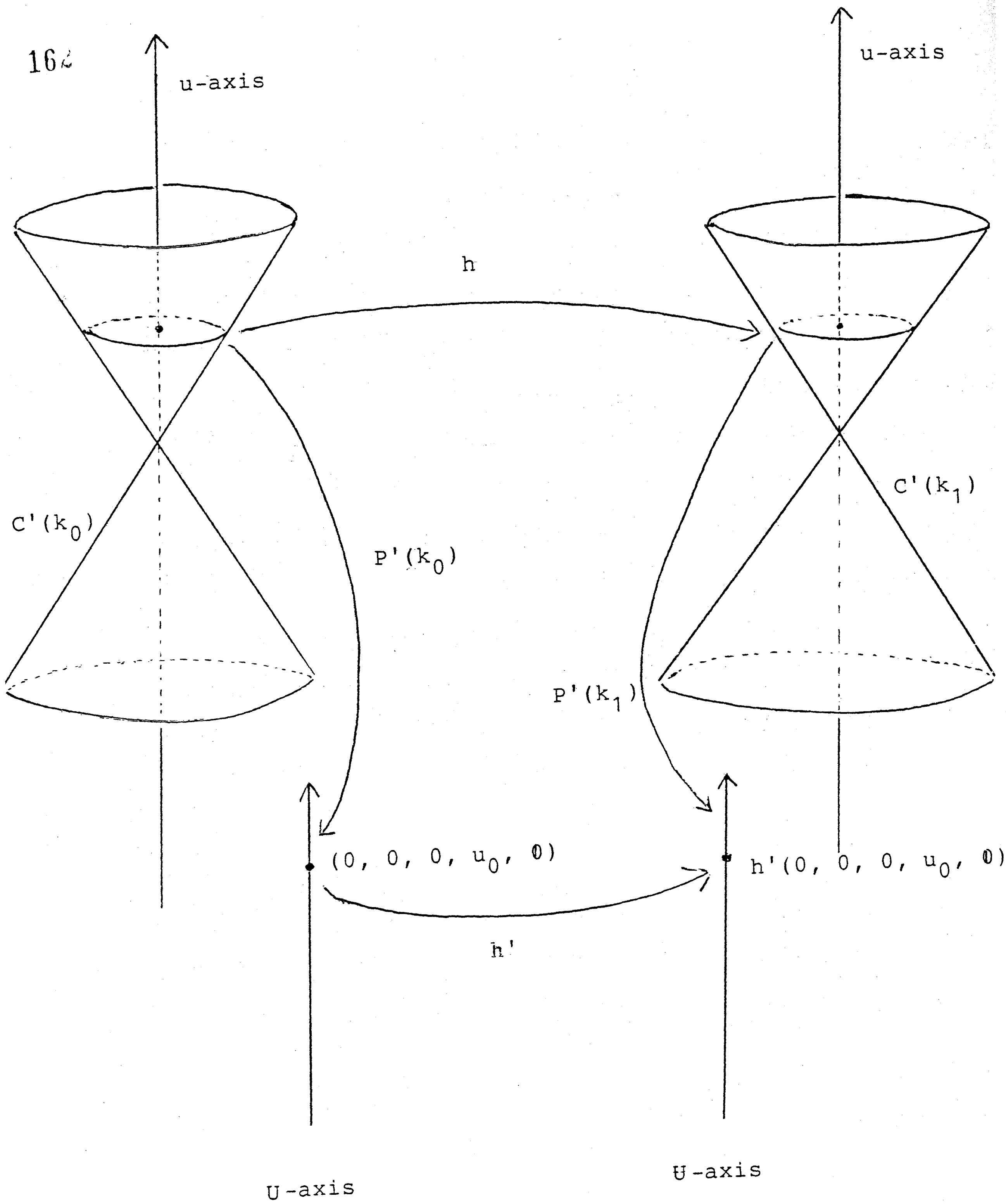
(4) *For any germ of continuous curve* $\mathbb{q}(t)$ *at any point* $\mathbb{p} = (x, y, 0, u) \in C'(k_0)$ (*resp.* $C'(k_1)$) *in* $H'(k_0)$ (*resp.* $H'(k_1)$), $P'(k_0)$ (*resp.* $P'(K_1)$) *maps* $\mathbb{q}(t)$ *to a germ of continuous curve at* $(0, 0, 0, u^2, \mathbb{0}) \in \mathbb{R}^p$ *in* $\{(0, Y, Z, U, \mathbb{0}) \in \mathbb{R}^p\}$ *and* $\pi \circ P'(k_0)(\mathbb{q}(t))$ (*resp.* $\pi \circ P'(k_1)(\mathbb{q}(t))$) *is a germ of continuous curve at* $(0, 0, 0) \in \mathbb{R}^3$ *in* $\{(0, Y, Z) \in \mathbb{R}^3\}$ *and this germ of curve has a tangent line at* $(0, 0, 0)$, *where* $\pi : \mathbb{R}^p \to \mathbb{R}^3$ *is a natural projection*

$$(X, Y, Z, U, V_1, \cdots, V_{p-4}) \mapsto (X, Y, Z).$$

The proof of lemma 6 is analogous to lemma 5 and we omit it.

By this lemma, we have a contradiction to Van Kampen's theorem as same as Thom's proof. □

162



(FIGURE I)

Proof in case II. Let $P''(k)$ be a one-parameter polynomial map-germ family as follows;

$$P''(k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$$
$$P''(k)(x, y, z, u, v_1, \cdots, v_{n-4}) = (\tilde{Q}(k)(x, y, z, u), v_1, \cdots, v_{n-4}, 0)$$

where $(x, y, z, u, v_1, \cdots, v_{n-4})$ is a coordinate of the source space and $\tilde{Q}(k)$ is as before.

For any fixed k_0, $P''(k_0)^{-1}(\{0\}) = \{0\}$. So $P''(k_0)$ is a C^0-$\mathcal{K}$-finite polynomial map-germ.

Let $H''(k_0)$ and $C''(k_0)$ be as follows;

$$H''(k_0) = \{(x, y, z, u, v_1, \cdots, v_{n-4}) \in \mathbb{R}^n \mid [x(x^2+y^2-u^2)-yz][(x+k_0y)(x^2+y^2-u^2)-(y-k_0x)z] = 0\},$$
$$C''(k_0) = \{(x, y, 0, u, v_1, \cdots, v_{n-4}) \in \mathbb{R}^n \mid x^2+y^2-u^2 = 0\}.$$

If there are germs of homeomorphisms $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ $h' : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ such that $P''(k_0) = (h')^{-1} \circ P''(k_0) \circ h$ as germs at 0 for any two fixed real numbers k_0, k_1 ($k_0 \neq k_1$), then we have the following lemma, which is analogous to lemma 6.

Lemma 7. (1) $h(H''(k_0)) = H''(k_1)$ as germs at 0.
(2) $h(C''(k_0)) = C''(k_1)$ as germs at 0.

(3) $h(C''(k_0) \cap P''(k_0)^{-1}((0, 0, 0, u^0, v_1^0, \cdots, v_{n-4}^0, \mathbb{0}))) =$
$C''(k_1) \cap P''(k_1)^{-1}(h'((0, 0, 0, u^0, v_1^0, \cdots, v_{n-4}^0, \mathbb{0})))$ *as germs at* $\mathbb{0}$ *for any real numbers* u^0 (≥ 0), $v_1^0, \cdots, v_{n-4}^0$ *sufficiently close to zero*.

(4) *For any germ of continuous curve* $\mathbb{q}(t)$ *at any point* $\mathbb{p} = (x, y, 0, u, v_1, \cdots, v_{n-4}) \in C''(k_0)$ (*resp.* $C''(k_1)$) *in* $H''(k_0)$ (*resp.* $H''(k_1)$), $P''(k_0)$ (*resp.* $P''(k_1)$) *maps* $\mathbb{q}(t)$ *to a germ of continuous curve at* $(0, 0, 0, u^2, v_1, \cdots, v_{n-4}, \mathbb{0}) \in \mathbb{R}^p$ *in* $\{(0, Y, Z, U, V_1, \cdots, V_{n-4}, \mathbb{0}) \in \mathbb{R}^p\}$ *and* $\pi \circ P''(k_0)(\mathbb{q}(t))$ (*resp.* $\pi \circ P''(k_0)(\mathbb{q}(t))$) *is a germ of continuous curve at* $(0, 0, 0) \in \mathbb{R}^3$ *in* $\{(0, Y, Z) \in \mathbb{R}^3\}$ *and this germ of curve has a tangent line at* $(0, 0, 0)$, *where* $\pi : \mathbb{R}^p \to \mathbb{R}^3$ *is a natural projection*

$$(X, Y, Z, U, V_1, \cdots, V_{p-4}) \longmapsto (X, Y, Z).$$

The proof of this lemma 7 is almost as same as one of lemma 6 and we omit it.

This lemma 7 yields a contradiction to Van Kampen's theorem. □

Proof in case III. Let $\hat{Q}(k)$ be a one-parameter polynomial map-germ family as follows;

$$\hat{Q}(k) : (\mathbb{R}^{n-p+4}, 0) \to (\mathbb{R}^4, 0)$$

$$\hat{Q}(k): \begin{cases} X = [x(x^2+y^2-u^2-v_1^2-\cdots-v_{n-p}^2)-yz]^2 \times \\ \quad [(x+ky)(x^2+y^2-u^2-v_1^2-\cdots-v_{n-p}^2)-(y-kx)z]^2 \\ Y = x^2+y^2-u^2-v_1^2-\cdots-v_{n-p}^2 \\ Z = z \\ U = u^2+v_1^2+\cdots+v_{n-p}^2 \end{cases}$$

where $(x, y, z, u, v_1, \cdots, v_{n-p})$, (X, Y, Z, U) are coordinates of the source and the target spaces respectively and k is a parameter. Let $P'''(k)$ be a one-parameter polynomial map-germ as follows;

$$P'''(k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$$

$$P'''(k)(x, y, z, u, v_1, \cdots, v_{n-p}, w_1, \cdots, w_{p-4})$$
$$= (\hat{Q}(k)(x, y, z, u, v_1, \cdots, v_{n-p}), w_1, \cdots, w_{p-4}).$$

For any fixed k_0, $P'''(k_0)^{-1}(\{0\}) = \{0\}$. So $P'''(k_0)$ is C^0-$\mathcal{K}$-finite.

Let $H'''(k_0)$ and $C'''(k_0)$ be as follows;

$$H'''(k_0) = \{(x, y, z, u, v_1, \cdots, v_{n-p}, w_1, \cdots, w_{p-4}) \in \mathbb{R}^n \mid$$
$$[x(x^2+y^2-u^2-v_1^2-\cdots-v_{n-p}^2)-yz] \times$$
$$[(x+k_0y)(x^2+y^2-u^2-v_1^2-\cdots-v_{n-p}^2)-(y-k_0x)z] = 0\}.$$

$$C'''(k_0) = \{(x, y, 0, u, v_1, \cdots, v_{n-p}, w_1, \cdots, w_{p-4}) \in \mathbb{R}^n \mid x^2+y^2-u^2-v_1^2-\cdots-v_{n-p}^2 = 0\}.$$

If there are germs of homeomorphisms $h : (\mathbb{R}^n, \mathbb{0}) \to (\mathbb{R}^n, \mathbb{0})$, $h' : (\mathbb{R}^p, \mathbb{0}) \to (\mathbb{R}^p, \mathbb{0})$ such that $P'''(k_0) = (h')^{-1} \circ P'''(k_1) \circ h$ as germs at $\mathbb{0}$ for any two fixed non-zero real numbers k_0, k_1 ($k_0 \neq k_1$), then we have the following lemma, which is analogous to lemma 6 and lemma 7, hence we give no proof of it.

<u>Lemma</u> 8. (1) $h(H'''(k_0)) = H'''(k_1)$ <u>as germs at</u> $\mathbb{0}$.

(2) $h(C'''(k_0)) = C'''(k_1)$ <u>as germs at</u> $\mathbb{0}$.

(3) <u>For any real numbers</u> u^0 (≥ 0), $w_1^0, \cdots, w_{p-4}^0$ sufficiently close <u>to zero</u>, $h(C'''(k_0) \cap P'''(k_0)^{-1}((0, 0, 0, u^0, w_1^0, \cdots, w_{p-4}^0))) = C'''(k_1) \cap P'''(k_1)^{-1}(h'(0, 0, 0, u^0, w_1^0, \cdots, w_{p-4}^0))$ <u>as germs at</u> $\mathbb{0}$.

(4) <u>For any germ of continuous curve</u> $q(t)$ <u>at any point</u> $p = (x, y, 0, u, v_1, \cdots, v_{n-p}, w_1, \cdots, w_{p-4}) \in C'''(k_0)$ (<u>resp</u>. $C'''(k_1)$) <u>in</u> $H'''(k_0)$ (<u>resp</u>. $H'''(k_1)$), $P'''(k_0)$ (<u>resp</u>. $P'''(k_1)$) <u>maps</u> $q(t)$ <u>to a germ of continuous curve at</u> $(0, 0, 0, u^2+ v_i^2, w_1, \cdots, w_{p-4}) \in \mathbb{R}^p$ <u>in</u> $\{(0, Y, Z, U, W_1, \cdots, W_{p-4})\} \in \mathbb{R}^p$ <u>and</u> $\pi \circ P'''(k_0)(q(t))$ (<u>resp</u>. $\pi \circ P'''(k_1)(q(t))$) <u>is a germ of continuous curve at</u> $(0, 0, 0) \in \mathbb{R}^3$ <u>in</u> $\{(0, Y, Z) \in \mathbb{R}^3\}$ <u>and this germ of curve has a tangent line at</u> $(0, 0, 0)$, <u>where</u> $\pi : \mathbb{R}^p \to \mathbb{R}^3$ <u>is a natural projection</u>

$$(X, Y, Z, U, W_1, \cdots, W_{p-4}) \mapsto (X, Y, Z).$$

As $C'''(k_0) \cap P'''(k_0)^{-1}((0, 0, 0, u^0, w_1^0, \cdots, w_{p-4}^0)$ is a space $\sqrt{u^0} \cdot S^1 \times \sqrt{u^0} \cdot S^{n-p}$ if $u^0 > 0$, the restriction of the homeomorphism h to $\sqrt{u^0} \cdot S^1 \times \sqrt{u^0} \cdot S^{n-p}$ maps $\sqrt{u^0} \cdot S^1 \times \sqrt{u^0} \cdot S^{n-p}$ to $\sqrt{\widetilde{u}^0} \cdot S^1 \times \sqrt{\widetilde{u}^0} \cdot S^{n-p}$, where $\widetilde{u}^0 = h_4'(0, 0, 0, u^0, w_1^0, \cdots, w_{p-4}^0)$.

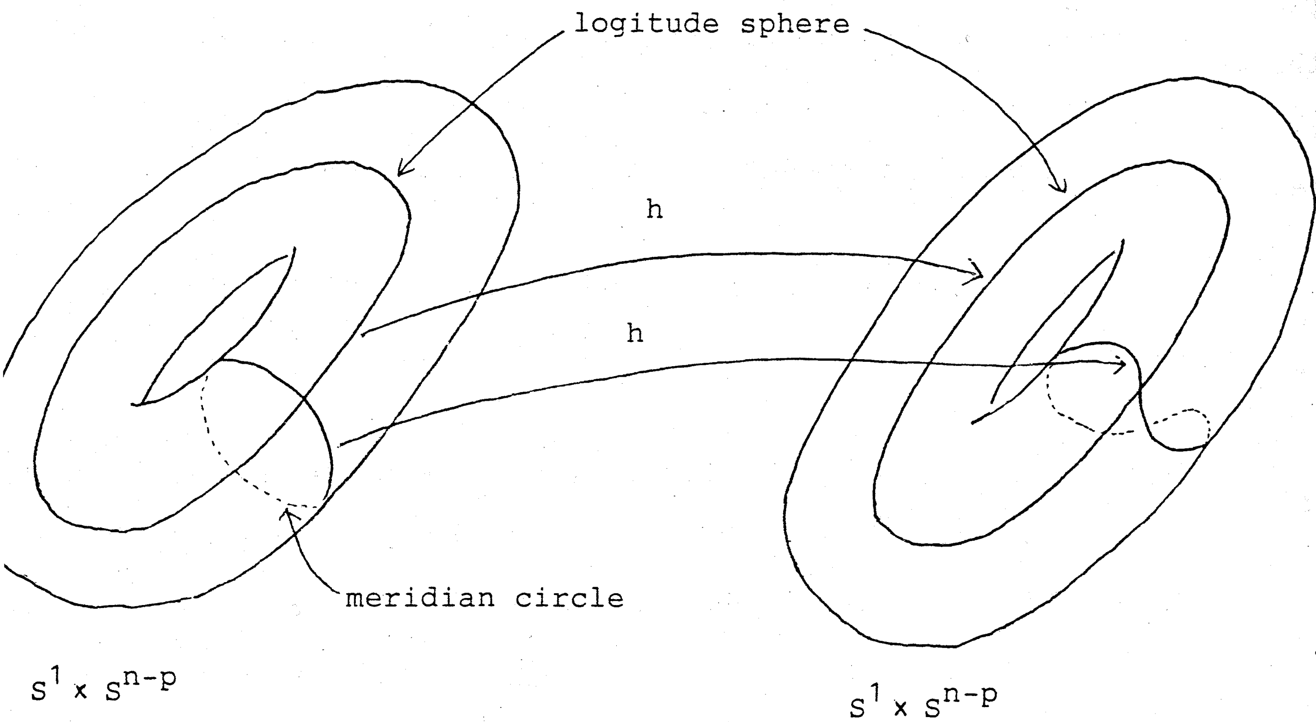
Definition. In the space $\sqrt{c} \cdot S^1 \times \sqrt{c} \cdot S^{n-p} = \{(x, y, u, v_1, \cdots, v_{n-p}) \in \mathbb{R}^{n-p+3} \mid x^2+y^2 = u^2+ \ v_i^2 = \text{constant } c\ (>0)\}$, the spaces $\{(x, y, u, v_1, \cdots, v_{n-p}) \in \mathbb{R}^{n-p+3} \mid x = \text{const.}, y = \text{const.}\}$, $\{(x, y, u, v_1, \cdots, v_{n-p}) \in \mathbb{R}^{n-p+3} \mid u, v_1, \cdots, v_{n-p} : \text{const.}\}$ are called longitude spheres, meridian circles respectively.

To conclude the proof in case (III), we need the following lemma.

Lemma 9. In each $C'''(k_0) \cap P'''(k_0)^{-1}((0, 0, 0, u^0, w_1^0, \cdots, w_{p-4}^0)) = \sqrt{u^0} \cdot S^1 \times \sqrt{u^0} \cdot S^{n-p}$ for $u^0 (>0)$ close to zero, each longitude sphere is mapped to a longitude sphere in $C'''(k_1) \cap P'''(k_1)^{-1}(h'((0, 0, 0, u^0, w_1^0, \cdots, w_{p-4}^0)) = \sqrt{\widetilde{u}^0} \cdot S^1 \times \sqrt{\widetilde{u}^0} \cdot S^{n-p}$ by the restriction of the homeomorphism h of the source space.

Proof of lemma 9. We take any germ of continuous curve $q(t)$ at $(0, 0, 0, u^0, w_1^0, \cdots, w_{p-4}^0)$ in $\{(0, Y, Z, u_1^0, w_1^0, \cdots, w_{p-4}^0) \in \mathbb{R}^p\}$ which has a tangent line at $(0, 0, 0, u^0, w_1^0, \cdots, w_{p-4}^0)$. Then $P'''(k_0)^{-1}(q(t))$ is homeomorphic to $S^{n-p} \times I$ with a certain longitude sphere in $H'''(k_0)$ as its center, where I is an open interval.

If the inverse image of this longitude sphere by the homeomorphism h of the source space is not a longitude sphere, then $P'''(k_1)(h^{-1}(P'''(k_0)^{-1}(q(t))$ is not a germ of continuous curve at $h'(0, 0, 0, u^0. w_1^0, \cdots, w_{p-4}^0)$. This is a contradiction to the commutativity $P'''(k_0) = (h')^{-1} \circ P'''(k_1) \circ h$ with homeomorphisms h, h'. □



(FIGURE II)

By this lemma 9, we have the following.

Lemma 10. *For any* $C'''(k_0)\cap P'''(k_0)^{-1}((0, 0, 0, u^0, w_1^0, \cdots, w_{p-4}^0)) = \sqrt{u^0}\cdot S^1 \times \sqrt{u^0}\cdot S^{n-p}$ *for any positive number* u^0 *close to zero, the image of any meridian circle by the restriction of homeomorphism* h *is isotopic to any meridian circle in* $C'''(\bar{k}_1)\cap P'''(k_1)^{-1}(h'(0, 0, 0, u^0, w_1^0, \cdots, w_{p-4}^0))$ *by an isotopy with* (x, y)*-coordinates preserving*.

Now lemma 8 and lemma 10 yield a contradiction to Van Kampen's theorem as same as we see in case (I) and (II). □

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