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Hypergeometric solutions of Toda equation

愛媛大, 工 亀高 惟倫 (Yoshinori Kametaka)

§1. Introduction

Hypergeometric solutions of Toda equation can be obtained by Bäcklund transformation from separated solutions. We have three kinds of separated solutions. So totality of hypergeometric solutions forms three kinds of linear spaces T_1, T_2 and T_3. We found the structure of these linear spaces of solutions. We found several types of linear transformations which carry a hypergeometric solution of Toda equation to a new one. Operations of these linear transformations are compatible with operations of some group or algebra. More accurately Lie groups $SL(2,\mathbb{C})=G(1,1)$, $G(0,1)$, $G(0,0)$ and corresponding Lie algebra $sl(2,\mathbb{C})=g(1,1)$, $g(0,1)$, $g(0,0)$ act at the same time on T_1, T_2 and T_3 respectively.

Let us solve Toda equation

$$(1.1) \qquad (\log \tau_n)'' = \tau_{n+1}\tau_{n-1}/\tau_n^{\,2} \qquad (\tau_n = \tau_n(t), \quad ' = \frac{\partial}{\partial t})$$

using Gauss hypergeometric functions

$$(1.2) \qquad F(a,b,c;z) = \sum_{j=0} c_j(a,b,c) z^j \qquad c_j(a,b,c) = \frac{(a)_j (b)_j}{(c)_j \, j!}$$

where usual convention $(a)_j = \Gamma(j+a)/\Gamma(a)$ is used. Put

$$(1.3) \qquad f_n(z) = F(n+a,b,c;z)$$

then we have relations

$$(1.4) \qquad f_{n+1}/f_n = (zf_n'/f_n + n+a)/(n+a),$$

$$f_{n-1}/f_n = -(\, z(1-z)f_n'/f_n - (n+a-c+bz))/(n+a-c),$$

$$z(1-z)f_n'' + \{c-(n+a+b+1)z\}f_n' - (n+a)bf_n = 0.$$

Using (1.4) it is easy to show that

$$(1.5) \qquad t_n(z) = A(n)((1-z)/z^2)^{(n+a)(n+a-c)/2},$$

where $A(n)$ is determined by $A(n+1)A(n-1)/A(n)^2 = (n+a)(n+a-c)$ with suitable initial conditions $A(0)$ and $A(1)$,

$$(1.6) \qquad \tau_n(z) = t_n(z)f_n(z)$$

satisfy Toda equation

$$(1.7) \qquad ((1-z)\frac{d}{dz})^2 \log \tau_n = \tau_{n+1}\tau_{n-1}/\tau_n^2$$

$$(1.8) \qquad ((1-z)\frac{d}{dz})^2 \log t_n = t_{n+1}t_{n-1}/t_n^2 = (n+a)(n+a-c)(1-z)/z^2.$$

Confluent hypergeometric function

$$(1.9) \qquad F(a,c;z) = \sum_{j=0} c_j(a,c)z^j \qquad c_j(a,c) = \frac{(a)_j}{(c)_j\ j!}$$

is obtained as a limit of Gauss hypergeometric function.

$$(1.10) \qquad F(a,c;z) = \lim_{b\to\infty} F(a,b,c;z/b).$$

Putting b tends to infinity after replacing z by z/b in (1.7) and (1.8) we can show that

$$(1.11) \qquad \tilde{t}_n(z) = \lim_{b\to\infty} b^{-(n+a)(n+a-c)} t_n(z/b) = A(n) z^{-(n+a)(n+a-c)},$$

$$(1.12) \qquad \tilde{\tau}_n(z) = \lim_{b\to\infty} b^{-(n+a)(n+a-c)} \tau_n(z/b) = \tilde{t}_n(z) F(n+a,c;z)$$

satisfy also Toda equation

$$(1.13) \qquad \left(\frac{d}{dz}\right)^2 \log \tilde{\tau}_n = \tilde{\tau}_{n+1}\tilde{\tau}_{n-1}/\tilde{\tau}_n^{\,2},$$

$$(1.14) \qquad \left(\frac{d}{dz}\right)^2 \log \tilde{t}_n = \tilde{t}_{n+1}\tilde{t}_{n-1}/\tilde{t}_n^{\,2} = (n+a)(n+a-c)z^{-2}.$$

Thus we obtained a confluent hypergeometric solution of Toda equation.

Another hypergeometric solution is also possible. Put

$$(1.15) \qquad f_n(z) = F(a,b,n+c;z).$$

We have relations

(1.16) $$f_{n+1}/f_n = \frac{n+c}{(n+c-a)(n+c-b)}((1-z)f_n'/f_n + n+c-a-b),$$

$$f_{n-1}/f_n = \frac{1}{n+c-1}(zf_n'/f_n + n+c-1),$$

$$z(1-z)f_n'' + \{n+c-(a+b+1)z\}f_n' - abf_n = 0.$$

Define A(n) and B(n) by the following relations.

(1.17) $$A(n+1)A(n-1)/A(n)^2 = b(n+c-a-1) - (n+c-1)(n+c-a),$$

$$A(0) = b^{(c-1)(c-a)/2}, \quad A(1) = b^{c(1+c-a)/2},$$

(1.18) $$B(n+1)B(n-1)/B(n)^2 = \frac{b(n+c-a) - (n+c)(n+c-a)}{b(n+c-a-1) - (n+c-1)(n+c-a)},$$

$$B(0) = 1, \quad B(1) = c-a.$$

By direct calculation using (1.16) it is easy to show that

(1.19) $$t_n(z) = A(n)(1-z)^{-b(n+c-a-1)}(z(1-z))^{(n+c-1)(n+c-a)/2},$$

(1.20) $$\tau_n(z) = t_n(z)\frac{B(n)}{(c)_n} f_n(z)$$

satisfy Toda equation

(1.21) $$(z(1-z)\frac{d}{dz})^2 \log \tau_n = \tau_{n+1}\tau_{n-1}/\tau_n^2,$$

$$(1.22)\qquad (z(1-z)\frac{d}{dz})^2 \log t_n = t_{n+1}t_{n-1}/t_n{}^2$$

$$= \{b(n+c-a-1) - (n+c-1)(n+c-a)\}z(1-z).$$

Since we have

$$(1.23)\qquad \lim_{b\to\infty} b^{-(n+c-1)(n+c-a)/2}A(n) = \tilde{A}(n),$$

where $\tilde{A}(n)$ is given by $\tilde{A}(n+1)\tilde{A}(n-1)/\tilde{A}(n)^2 = n+c-a-1$, $\tilde{A}(0) = \tilde{A}(1) = 1$,

$$(1.24)\qquad \lim_{b\to\infty} B(n) = (c-a)_n$$

then it follows that

$$(1.25)\qquad \tilde{t}_n(z) = \lim_{b\to\infty} t_n(z/b) = \tilde{A}(n)e^{(n+c-a-1)z}z^{(n+c-1)(n+c-a)/2}$$

$$(1.26)\qquad \tilde{\tau}_n(z) = \lim_{b\to\infty} \tau_n(z/b) = \tilde{t}_n(z)\frac{(c-a)_n}{(c)_n}F(a,n+c;z)$$

satisfy Toda equation

$$(1.27)\qquad (z\frac{d}{dz})^2 \log \tilde{\tau}_n = \tilde{\tau}_{n+1}\tilde{\tau}_{n-1}/\tilde{\tau}_n{}^2,$$

$$(1.28)\qquad (z\frac{d}{dz})^2 \log \tilde{t}_n = \tilde{t}_{n+1}\tilde{t}_{n-1}/\tilde{t}_n{}^2 = (n+c-a-1)z.$$

Thus we obtained another confluent hypergeometric solution .

Now we introduce a new independent variable x by

(1.29) $z = -x^2/4, \qquad z\frac{d}{dz} = \frac{x}{2}\frac{d}{dx}.$

After slight modification we can show that

(1.30) $t_n(x) = A(n)e^{-(n+c-a-1)x^2/4}(-x^2/4)^{n^2/2},$

(1.31) $\tau_n(x) = t_n(x)\frac{(c-a)_n}{(c)_n}F(a,n+c;-x^2/4)$

satisfy Toda equation

(1.32) $(\frac{x}{2}\frac{d}{dx})^2\log\tau_n = \tau_{n+1}\tau_{n-1}/\tau_n{}^2,$

(1.33) $(\frac{x}{2}\frac{d}{dx})^2\log t_n = t_{n+1}t_{n-1}/t_n{}^2 = -(n+c-a-1)x^2/4.$

Here $A(n)$ is defined by $A(n+1)A(n-1)/A(n)^2 = n+c-a-1$, $A(0) = 1$, $A(1) = -a$. Since $(-a)^{-n^2/2}A(n) \longrightarrow 1$ as $a \longrightarrow \infty$ then we have

(1.34) $\tilde{t}_n(x) = \lim_{a\to\infty} t_n(x/\sqrt{a}) = e^{x^2/4}(x/2)^{n^2},$

(1.35) $\tau_n(x) = \lim_{a\to\infty} (c-a)^{-n}\tau_n(x/\sqrt{a}) = \tilde{t}_n(x)J_{n+c-1}(x).$

$J_\nu(x)$ is Bessel function. These functions satisfy also Toda equation

(1.36) $(\frac{x}{2}\frac{d}{dx})^2\log\tilde{\tau}_n = \tilde{\tau}_{n+1}\tilde{\tau}_{n-1}/\tilde{\tau}_n{}^2,$

(1.37) $(\frac{x}{2}\frac{d}{dx})^2\log\tilde{t}_n = \tilde{t}_{n+1}\tilde{t}_{n-1}/\tilde{t}_n{}^2 = x^2/4.$

As we observed above we can easily construct various types of hypergeometric solutions of Toda equation by direct calculation. As a conclusion we see that hypergeometric solutions have always the structure $\tau_n(t) = t_n(t)u_n(t)$ where $t_n(t)$ is a "simple" solution of Toda equation and $u_n(t)$ is given by hypergeometric function. "Simple" means that $(\log t_n)'' = f(n)g(t)$ is a product of two functions: the one is a function of only n and the other is a function of only t. That is t_n is a "separated" solution of Toda equation.

§2. Bäcklund transformation of separated solutions.

Hereafter we consider Toda equation with two time variables so-called 2-dimensinal Toda equation.

(2.1) $$XY \log t_n = t_{n+1}t_{n-1}/t_n^2 \qquad (X = \partial/\partial x, \quad Y = \partial/\partial y).$$

Introducing new dependent variables r_n and s_n by

(2.2) $$r_n = XY \log t_n, \qquad s_n = Y \log t_{n-1}/t_n$$

(2.1) is equivalent to the following

(2.3) $$Y\, r_n = r_n(s_n - s_{n+1}), \qquad X\, s_n = r_{n-1} - r_n.$$

Eliminating s_n (2.3) is equivalent to

(2.4) $$XY \log r_n = r_{n+1} - 2r_n + r_{n-1}.$$

Toda equation (1.1) was discovered by Toda in 1966 [1]. But, to our great surprise, Toda equation (2.4) can be seen in the famous book of G.Darboux [2]. If $t_n(t)$ is a solution of 1-dimensional Toda equation (1.1) then $t_n(x+y)$ is a solution of 2-dimensional Toda equation (2.1).

Observation in paragraph 1 indicates that once we have a solution t_n of Toda equation then we can find a suitable multiplier u_n so that $\tau_n = t_n u_n$ is a new solution of the same Toda equation. That is to say Toda equation, even though it is a nonlinear equation, can be solved by d'Alembert's method of reduction of order. But this method is nothing but the Bäcklund transformation. Suppose that t_n is a solution of Toda equation (2.1). r_n and s_n are associated by the relation (2.2). We consider a triple of partial differential operators

$$\text{(2.5)} \qquad M_n = XY + s_{n+1}X + r_n,$$

$$X_n = -r_n^{-1}X, \qquad Y_n = Y + s_{n+1}.$$

Using these operators we introduce a linear space T of infinit-dimensional column vectors ${}^t(\ldots,u_n,\ldots)$ (n-th component $u_n(x,y)$ is a function of x and y).

$$\text{(2.6)} \quad T = \{u_n;\ M_0u_0 = 0,\ u_{n+1} = Y_nu_n\ (n \geqslant 0),\ u_{n-1} = X_nu_n\ (n \leqslant 0)\}.$$

We can show the following theorem.

Theorem 2.1 If $u_n \in T$ then we have $M_n u_n = 0$, $u_{n+1} = Y_n u_n$, $u_{n-1} = X_n u_n$ for $n = 0, \pm 1, \pm 2, \ldots$. $\tau_n = t_n u_n$ is a solution of Toda equation (2.1).

In this sense we regard T as a linear space of solutions of Toda equation.

Next let us determine separated solutions of Toda equation. Assume $r_n = f(n)g(x,y)$ then we can derive Liouville equation satisfied by $g(x,y)$ from Toda equation (2.3) and also some difference equation satisfied by $f(n)$. Solving these equations we have

Theorem 2.2 Separated solution $r_n = f(n)g(x,y)$ of Toda equation has one of the following form

1. $r_n = (n-a)(n-b)h'(x)k'(y)(h(x)+k(y))^{-2}$,

2. $r_n = (n-a)h(x)k(y)$,

3. $r_n = h(x)k(y)$,

where a and b are arbitrary constants, $h(x)$ and $k(y)$ are arbitrary functions.

Following theorems show properties of Toda equation under the change of independent variables.

Theorem 2.3 If $t_n(x,y)$ is a solution of Toda equation then $t_n(h(x),k(y))(h'(x)k'(y))^{n(n+1)/2}(h_1(x)k_1(y))^n h_2(x)k_2(y)$ is also a solution of Toda equation for any functions $h(x)$, $h_1(x)$, $h_2(x)$, $k(y)$, $k_1(y)$ and $k_2(y)$.

Theorem 2.4 If $t_n(x,y)$ is a solution of Toda equation and $u_n(x,y) \in T[t_n(x,y)]$ then $\tilde{t}_n(x,y) = t_n(h(x),k(y))(h'(x)k'(y))^{n(n+1)/2}$ is also a solution of Toda equation and $\tilde{u}_n(x,y) = u_n(h(x),k(y))k'(y)^{n+1} \in T[\tilde{t}_n(x,y)]$ for any functions $h(x)$ and $k(y)$.

Since we have above theorems when we treat Bäcklund transformations of separated solutions we can assume with no loss of generality the following simple form for separated solutions.

Fundamental separated solutions

1. $t_n = A(n)(x-y)^{-(n-a)(n-b)}$, $r_n = -(n-a)(n-b)(x-y)^{-2}$,

   $s_n = (a+b+1-2n)(x-y)^{-1}$,

where $A(n)$ is defined by $A(n+1)A(n-1)/A(n)^2 = -(n-a)(n-b)$, $A(0) = A(1) = 1$. This most important separated solution was found by G.Darboux.

2. $t_n = A(n)\exp((a-n)xy)$, $r_n = -(n-a)$, $s_n = x$,

where $A(n)$ is defined by $A(n+1)A(n-1)/A(n)^2 = -(n-a)$, $A(0) = A(1) = 1$.

3. $t_n = \exp(xy)$, $r_n = 1$, $s_n = 0$.

a and b are arbitrary constants.

§3. Structure of linear space T_1

As starting solution t_n we choose the first fundamental separated solution $t_n = A(n)(x-y)^{-(n-a)(n-b)}$. The triple of differential operators given by (2.5) takes the form of

$$(3.1) \quad M_n = XY + (a+b-1-2n)(x-y)^{-1}X - (n-a)(n-b)(x-y)^{-2},$$

$$X_n = ((n-a)(n-b))^{-1}(x-y)^2 X, \quad Y_n = Y + (a+b-1-2n)(x-y)^{-1}.$$

M_n is a Euler-Poisson-Darboux operator. The linear space given by (2.6) using this triple of operators is denoted by T_1. The structure of the linear space T_1 is clarified if we know linear operators which keep T_1 invariant. Linear partial differential operators which keep T_1 invariant form 3-dimensional vector space. As standard bases we can choose the following three operators.

$$(3.2) \quad E_n = - X - Y, \quad F_n = x^2X + y^2Y + (2n+1-a-b)y,$$

$$H_n = - (2xX + 2yY + 2n + 1 - a - b).$$

We have the following commutation relations.

$$(3.3) \quad [E_n, F_n] = H_n, \quad [H_n, E_n] = 2E_n, \quad [H_n, F_n] = -2F_n.$$

Casimir operator

$$(3.4) \quad L_n = H_n^2/8 + (E_nF_n + F_nE_n)/4$$

$$= ((a-b)^2 - 1)/8 - (x-y)^2 M_n/2$$

commutes with E_n, F_n and H_n. So E_n, F_n and H_n commute with M_n modulo M_n. One-parameter groups $\tilde{E}_n(\lambda)$, $\tilde{F}_n(\mu)$ and $\tilde{H}_n(\nu)$ of linear transformations with generators E_n, F_n and H_n are given by

(3.5) $$\tilde{E}_n(\lambda)u_n(x,y) = u_n(x-\lambda, y-\lambda),$$

$$\tilde{F}_n(\mu)u_n(x,y) = (1-\mu y)^{a+b-1-2n}u_n(x/(1-\mu x), y/(1-\mu y)),$$

$$\tilde{H}_n(\nu)u_n(x,y) = e^{(a+b-1-2n)\nu}u_n(e^{-2\nu}x, e^{-2\nu}y).$$

We have the following intertwining properties.

(3.6) $$X_n(\lambda E_n + \mu F_n + \nu H_n) = (\lambda E_{n-1} + \mu F_{n-1} + \nu H_{n-1})X_n,$$

$$Y_n(\lambda E_n + \mu F_n + \nu H_n) = (\lambda E_{n+1} + \mu F_{n+1} + \nu H_{n+1})Y_n,$$

$$X_n L_n = L_{n-1}X_n, \quad Y_n L_n = L_{n+1}Y_n,$$

$$X_n\tilde{E}_n(\lambda)\tilde{F}_n(\mu)\tilde{H}_n(\nu) = \tilde{E}_{n-1}(\lambda)\tilde{F}_{n-1}(\mu)\tilde{H}_{n-1}(\nu)X_n,$$

$$Y_n\tilde{E}_n(\lambda)\tilde{F}_n(\mu)\tilde{H}_n(\nu) = \tilde{E}_{n+1}(\lambda)\tilde{F}_{n+1}(\mu)\tilde{H}_{n+1}(\nu)Y_n$$

for any λ, μ and ν.

As a conclusion we have

Theorem 3.1 If $u_n \in T_1$ then
$(\lambda E_n + \mu F_n + \nu H_n)u_n$, $\tilde{E}_n(\lambda)\tilde{F}_n(\mu)\tilde{H}_n(\nu)u_n \in T_1$
for any λ, μ and ν.

Theorem 3.2 If $u_n \in T_1$ then

$$\rho_n(g)u_n(x,y) = (\alpha-\gamma y)^{a+b-1-2n}u_n\left(\frac{\delta x-\beta}{-\gamma x+\alpha}, \frac{\delta y-\beta}{-\gamma y+\alpha}\right) \in T_1$$

for any $g = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in SL(2,\mathbb{C})$. Moreover we have
$\rho_n(g_1)\rho_n(g_2) = \rho_n(g_1 g_2)$ for any $g_1, g_2 \in SL(2,\mathbb{C})$.

4. Structure of linear space T_2

In this paragraph $t_n = A(n)\exp((a-n)xy)$ is the second fundamental separated solution. By (2.5) we have

$$(4.1) \qquad M_n = XY + xX + a - n, \quad X_n = (n-a)^{-1}X, \quad Y_n = Y + x.$$

The linear space given by (2.6) using this triple of operators is denoted by T_2. Linear partial differential operators which keep T_2 invariant form 4-dimensional Lie algebra. Its standard bases are

$$(4.2) \qquad K_n = 1, \quad E_n = X + y, \quad F_n = Y, \quad H_n = yY - xX + n.$$

Commutation relations are

$$(4.3) \qquad [E_n, F_n] = -K_n, \quad [H_n, E_n] = E_n, \quad [H_n, F_n] = -F_n,$$

$$[K_n, E_n] = [K_n, F_n] = [K_n, H_n] = 0.$$

$M_n = E_n - H_n + a$ commutes with K_n, E_n, F_n and H_n. One-parameter groups $\widetilde{K}_n(\kappa)$, $\widetilde{E}_n(\lambda)$, $\widetilde{F}_n(\mu)$ and $\widetilde{H}_n(\nu)$ of linear transformations with generators K_n, E_n, F_n and H_n are given by

$$(4.4) \quad \widetilde{K}_n(\kappa)u_n(x,y) = e^{\kappa}u_n(x,y), \quad \widetilde{E}_n(\lambda)u_n(x,y) = e^{\lambda y}u_n(x+\lambda,y),$$

$$\widetilde{F}_n(\mu)u_n(x,y) = u_n(x,y+\mu), \quad \widetilde{H}_n(\nu)u_n(x,y) = e^{\nu n}u_n(e^{-\nu}x, e^{\nu}y).$$

We have necessary intertwining properties which are similar to (3.6). Conclusion is

Theorem 4.1 If $u_n \in T_2$ then

$(\kappa K_n + \lambda E_n + \mu F_n + \nu H_n)u_n$, $\widetilde{K}_n(\kappa)\widetilde{E}_n(\lambda)\widetilde{F}_n(\mu)\widetilde{H}_n(\nu)u_n \in T_2$

for any κ, λ, μ and ν.

Theorem 4.2 If $u_n \in T_2$ then

$$\rho_n(g)u_n(x,y) = \exp(\kappa+\lambda y+\nu n)u_n(e^{-\nu}(x+\lambda), e^{\nu}(y+\mu)) \in T_2$$

for any $g = g(\kappa,\lambda,\mu,\nu) = \begin{pmatrix} 1 & \mu e^{\nu} & \kappa & \nu \\ & e^{\nu} & \lambda & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in G(0,1)$. Moreover

$\rho_n(g_1)\rho_n(g_2) = \rho_n(g_1 g_2)$ for any g_1, $g_2 \in G(0,1)$.

Group operation in $G(0,1)$ is given by

$$(4.5) \quad g(\kappa_1,\lambda_1,\mu_1,\nu_1)g(\kappa_2,\lambda_2,\mu_2,\nu_2) =$$

$$g(\kappa_1+\kappa_2+\lambda_2\mu_1 e^{\nu_1}, \lambda_1+\lambda_2 e^{\nu_1}, \mu_1+\mu_2 e^{-\nu_1}, \nu_1+\nu_2).$$

Corresponding Lie algebra $g(0,1) = \{\kappa k+\lambda e+\mu f+\nu h;\ \kappa,\lambda,\mu,\nu \in \mathbb{C}\}$ has the standard bases

$$(4.6) \qquad k = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & & & \\ & & & \\ & & & \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & & \\ & & & \end{pmatrix},$$

$$f = \begin{pmatrix} 0 & 1 & 0 & 0 \\ & & & \\ & & & \\ & & & \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ & & & \\ & & & \end{pmatrix}.$$

Commutation relations among these bases

$$(4.7) \qquad [e,f] = -k, \quad [h,e] = e, \quad [h,f] = -f,$$

$$[k,e] = [k,f] = [k,h] = 0$$

agree with (4.3). By theorem 4.2 we have

$$(4.8) \qquad \tilde{E}_n(\lambda) = \rho_n(g(0,\lambda,0,0)), \quad \tilde{F}_n(\mu) = \rho_n(g(0,0,\mu,0)),$$

$$\tilde{H}_n(\nu) = \rho_n(g(0,0,0,\nu)).$$

Moreover we have

$$(4.9) \qquad E_n = d\rho_n(e), \quad F_n = d\rho_n(f), \quad H_n = d\rho_n(h)$$

where $d\rho_n$ is a differential representation of ρ_n.

§5. Structure of linear space T_3

Here $t_n = \exp(xy)$ is the third fundamental separated solution. By (2.5) we have

$$(5.1) \quad M_n = XY + 1 \text{ (telegraph operator)}, \quad X_n = -X, \quad Y_n = Y.$$

The linear space given by (2.6) using this triple is denoted by T_3. Linear partial differential operators which keep T_3 invariant form 3-dimensional Lie algebra. Its bases are

$$(5.2) \quad E_n = X, \quad F_n = Y, \quad H_n = yY - xX + n.$$

Commutation relations are

$$(5.3) \quad [E_n, F_n] = 0, \quad [H_n, E_n] = E_n, \quad [H_n, F_n] = -F_n.$$

$M_n = E_nF_n + 1 = XY + 1$ commutes with E_n, F_n and H_n. One-parameter groups $\tilde{E}_n(\lambda)$, $\tilde{F}_n(\mu)$ and $\tilde{H}_n(\nu)$ of linear transformations with generators E_n, F_n and H_n are given by

$$(5.4) \quad \tilde{E}_n(\lambda)u_n(x,y) = u_n(x+\lambda, y), \quad \tilde{F}_n(\mu)u_n(x,y) = u_n(x, y+\mu),$$

$$\tilde{H}_n(\nu)u_n(x,y) = e^{n\nu}u_n(e^{-\nu}x, e^{\nu}y).$$

We have also necessary intertwining properties which are similar to (3.6). Conclusion is

Theorem 5.1 If $u_n \in T_3$ then

$(\lambda E_n + \mu F_n + \nu H_n)u_n,\ \tilde{E}_n(\lambda)\tilde{F}_n(\mu)\tilde{H}_n(\nu)u_n \in T_3$

for any λ, μ and ν.

Theorem 5.2 If $u_n \in T_3$ then

$$\rho_n(g)u_n(x,y) = e^{n\nu}u_n(e^{-\nu}(x+\lambda), e^{\nu}(y+\mu)) \in T_3$$

for any $g = g(\lambda,\mu,\nu) = \begin{pmatrix} 1 & & & \nu \\ & e^{-\nu} & & \mu \\ & & e^{\nu} & \lambda \\ & & & 1 \end{pmatrix} \in G(0,0)$. Moreover

$$\rho_n(g_1)\rho_n(g_2) = \rho_n(g_1g_2) \quad \text{for any } g_1,\ g_2 \in G(0,0).$$

Group operation in G(0,0) is given by

$$(5.5) \qquad g(\lambda_1,\mu_1,\nu_1)g(\lambda_2,\mu_2,\nu_2) = g(\lambda_1+e^{\nu_1}\lambda_2, \mu_1+e^{-\nu_1}\mu_2, \nu_1+\nu_2).$$

Corresponding Lie algebra $g(0,0) = \{\lambda e + \mu f + \nu h; \lambda,\mu,\nu \in \mathbb{C}\}$ has the standard bases

$$(5.6) \qquad e = \begin{pmatrix} & & & \\ & & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix},\quad f = \begin{pmatrix} & & 0 & 0 \\ & & 0 & 1 \\ & & & \\ & & & \end{pmatrix},\quad h = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ & & 1 & 0 \\ & & 0 & 0 \end{pmatrix}.$$

Commutation relations among these bases

$$(5.7) \qquad [e,f] = 0, \quad [h,e] = e, \quad [h,f] = -f$$

agree with (5.3). By theorem 5.2 we have

(5.8) $\tilde{E}_n(\lambda) = \rho_n(g(\lambda,0,0))$, $\tilde{F}_n(\mu) = \rho_n(g(0,\mu,0))$,

$\tilde{H}_n(\nu) = \rho_n(g(0,0,\nu))$.

Moreover we have

(5.9) $E_n = d\rho_n(e)$, $F_n = d\rho_n(f)$, $H_n = d\rho_n(h)$

where $d\rho_n$ is a differential representation of ρ_n.

§ 6. Eigenfunction expansion and hypergeometric functions with two variables

When we have a linear space and a linear operator which keep the linear space invariant the structure of the linear space is completely described if we can choose special bases which are all eigenvectors of the linear operator.

Theorem 6.1 Dimension of the vector space $T_1 \cap \{u_n \in \ker(H_n + a+b+1-2c)\}$ is 2. Its bases are given by

(6.1) $f_n(a,b,c;x,y) = (1-c)_n (y-x)^{b-n} y^{a-c} F(c-a,b-n,c-n;x/y)$,

$$g_n(a,b,c;x,y) = \frac{(1-a)_n(1-b)_n}{(2-c)_n}(y-x)^{b-n} x^{n+1-c} y^{a-1-n}$$

$$F(n+1-a,b+1-c,n+2-c;x/y).$$

Further we have

$$(6.2)\quad E_n^{\,k} f_n(a,b,c;x,y) = \frac{(c-b)_k (c-a)_k}{(c)_k} f_n(a,b,c+k;x,y),$$

$$F_n^{\,k} f_n(a,b,c;x,y) = (1-c)_k f_n(a,b,c-k;x,y),$$

$$E_n^{\,k} g_n(a,b,c;x,y) = (c-1)_k g_n(a,b,c+k;x,y),$$

$$F_n^{\,k} g_n(a,b,c;x,y) = \frac{(a+1-c)_k (b+1-c)_k}{(2-c)_k} g_n(a,b,c-k;x,y).$$

By theorem 3.1 each $E_n^{\,k} f_n(a,b,c;x,y)$ belongs to T_1. Since T_1 is a linear space linear combination of these functions with suitable coefficients also belongs to T_1 if it converges. As a special case we can show that

$$(6.3)\quad F(b',c-b;E_n) f_n(a,b,c;x,y) =$$

$$(1-c)_n (y-x)^{b-n} y^{a-c} F_1(c-a,b-n,b',c-n;x/y,1/y)$$

belongs to T_1. Here $F(a,c;z)$ is a confluent hypergeometric function and

$$(6.4)\quad F_1(a,b,b',c;x,y) = \sum_{j,k \geqslant 0} \frac{(a)_{j+k} (b)_j (b')_k}{(c)_{j+k} j!k!} x^j y^k$$

is the first one of Apell's hypergeometric functions with two variables. Thus we can also construct solutions of Toda equation by F_2 and F_3 the second and the third one of Apell's hypergeometric functions. But it seems difficult to construct a solution of Toda equation by the fourth one F_4. That is to say among the Apell's hypergeometric functions with two variables the first three $F_1(a,b,b',c;x,y)$, $F_2(a,b,b',c,c';x,y)$ and $F_3(a,a',b,b',c;x,y)$ belong to "Toda family" but the last one $F_4(a,b,c,c';x,y)$ does not. According to Horn's list [3] we have 34 hypergeometric functions with two variables "of order 2". We confirmed that 21 functions among those belong to "Toda family". We only show a list of "Toda family" without further explanation.

1. $$\begin{array}{ccccc} F_3 & \longrightarrow & \Xi_1 & \longrightarrow & \Xi_2, \\ | & & | & & \\ F_1 & \longrightarrow & \Phi_1 & & \end{array} \qquad \begin{array}{ccc} H_2 & \longrightarrow & \mathcal{H}_{11}, \\ | & & | \\ G_2 & \longrightarrow & \Gamma_1 \end{array} \qquad F_2 \longrightarrow \Psi_1,$$

   $$H_2 \longrightarrow \mathcal{H}_2 \longrightarrow \mathcal{H}_3, \quad H_4.$$

2. $\Xi_1 \longrightarrow \Phi_2 \longrightarrow \Phi_3$, $\mathcal{H}_2 \longrightarrow \mathcal{H}_4$, $\Gamma_1 \longrightarrow \Gamma_2$, Φ_1, $\mathcal{H}_9$.

3. $\Xi_2 \longrightarrow \Phi_3$, $\mathcal{H}_3 \longrightarrow \mathcal{H}_5$, $\mathcal{H}_{10}$.

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Department of Applied Mathematics,
Faculty of Engineering,
Ehime University