数理解析研究所講究録
第 554 巻 1985 年 122-146

Interactng Korteweg-de Vries Equations and Interacting Toda Equations - An Interacting Soliton Picture -

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§0 Introduction and Summary

My topic is Interacting korteweg-de Vries (Int KdV) equations and Interacting Toda (Int Toda) equations.

Subject: Nonlinear Classical Wave Solitons
Picture: an Interacting Soliton Picture
Method: New Operators $\hat{\partial}_i$
Equation: an Extension or a Decoupling
Solution: a Simple Sum
Identity: Without Exchange
Interaction: Attractive

Table 1 Summary of this talk

We treat nonlinear classical waves. The central idea is an interacting soliton picture[1),2)]; it is very simple and easy to understand. By the picture the original soliton equation such as the KdV, the sine-Gordon and the Toda equations are extended to obtain coupled nonlinear differential equations which we call interacting (Int) soliton equations. They can also be regarded as results of a decoupling of the original soliton equation. By introducing new operators, solutions of an Int soliton equations are obtained starting with the exact N-soliton solution. The N-soliton solution is decomposed into a simple sum of the solutions of the Int soliton equations, each of which is regarded as a soliton suffering much deformation when another soliton (other solitons) comes near in space. These single solitons as

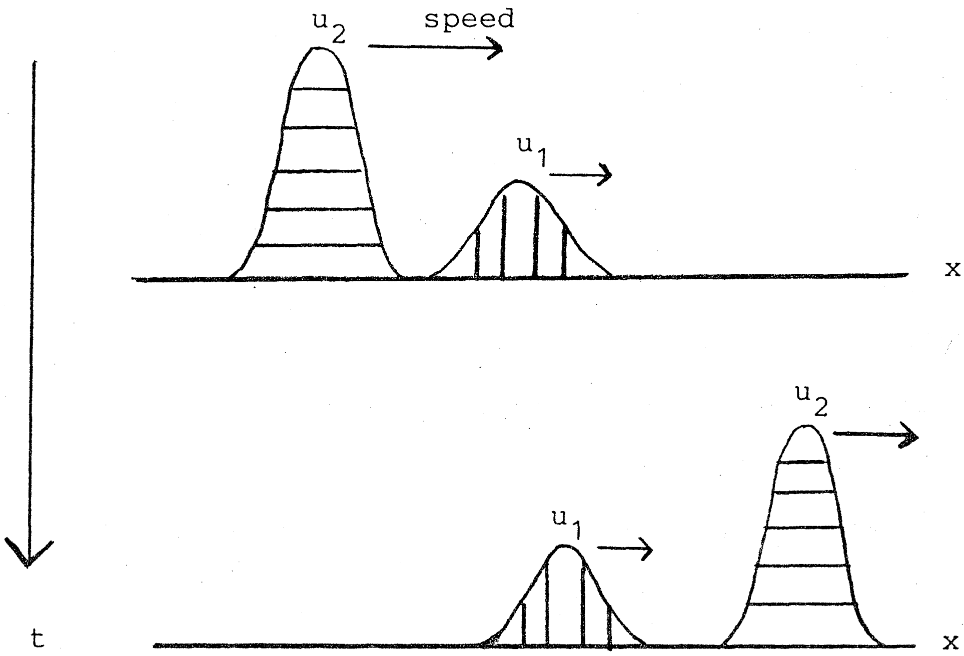


Fig.1 Without exchanging their identities

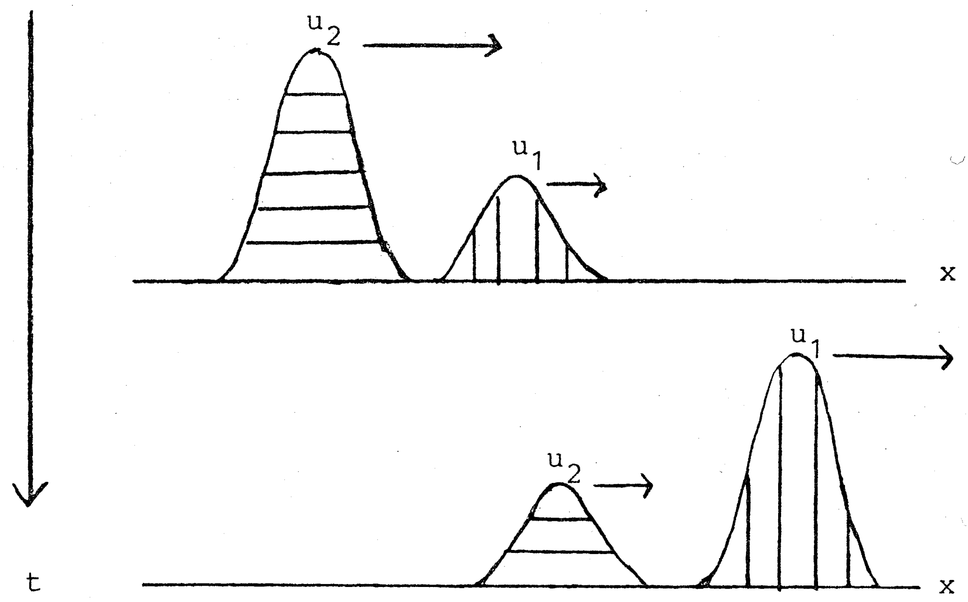


Fig.2 Exchanging their identities

classical waves interact attractively and eventually become apart in space without exchanging their identities. We mean by "without exchanging their identities" such a situation shown in Fig.1 in the N=2 case, but not the one in Fig.2.

The relation to the inverse scattering method is also discussed in detail in several cases. Further, "partial" Lax forms corresponding to the Int KdV equations are shown.

To summarize: We shall present an interacting soliton picture, in which several single solitons interact attractively with each other. Attractiveness is obvious in the KdV case from the phase shift analysis[3)] already done if we accept the fact that the solitons interact without losing their identities during collisions.

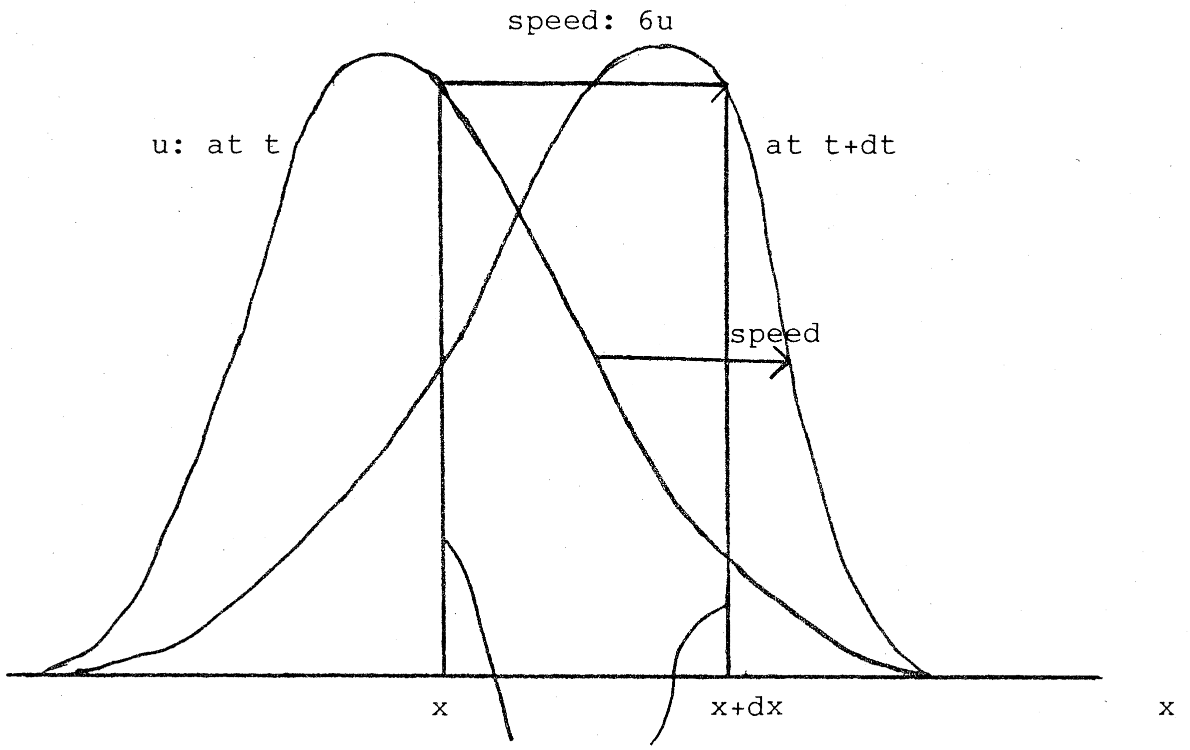
Linear classical waves do interfere linearly, but interact neither attractively nor repulsively. On the other hand, the classical wave solitons interfere nonlinearly (i.e. u_i is deformed when other solitons come near in space) and interact attractively.

§1 An Interacting Soliton Picture

We take the KdV equation in the following form as an example of a soliton equation.

$$(\partial/\partial t)u+6u(\partial/\partial x)u+(\partial/\partial x)^3 u=0 \tag{1-1}$$

nonlinear term   dispersive term



$u(t,x)=u(t+dt,x+dx)=u(t,x)+u_t dt+u_x dx$

nonlinear term only: $(\partial/\partial t)u+6u(\partial/\partial x)u = 0$, $dx/dt = -u_t/u_x= 6u$

Fig.3 Nonlinear effect

When there is only one single soliton u in space, its speed originated from nonlinear effect is 6u according to the KdV equation (Fig.3). If there is another soliton u' near u, we expect the total wave becomes u+u'. Then the speed of u becomes 6(u+u'), which is also the one of u'. Of course u and u' greatly affect each other when they come across, then eventually become apart without exchanging their identities. The following coupled equations are expected to hold.

$$du_i+6u^{(2)}\partial u_i+\partial^3 u_i = 0 \qquad (1\text{-}2)$$

and

$$u^{(2)} = u_1+ u_2; \qquad u_1=u,\ u_2=u' \qquad (1\text{-}3)$$

where

$$d \equiv \partial/\partial t,\ \partial \equiv \partial/\partial x. \qquad (1\text{-}4)$$

Thus, when there are N single solitons initially apart in space, we expect that they interact with each other satisfying the following coupled nonlinear differential equations which we call the interacting KdV equations (Int KdV equations)

$$du_i + 6u^{(N)}\partial u_i + \partial^3 u_i = 0, \quad (i=1,2,\cdots,N) \qquad (1\text{-}5)$$

and that the total wave $u^{(N)}$ is a simple sum of each wave u_i;

$$u^{(N)}= \textstyle\sum_{i=1}^{N} u_i. \qquad (1\text{-}6)$$

One aspect of equations (1-5) is, in this way, a natural extension of the original KdV equation (1-1). Of course, each equation of (1-5) may have its own solution with any soliton number, but here each u_i is taken for a single soliton as $t\to\pm\infty$.

There is another aspect of the Int KdV equations, that is, they are results of a decoupling of the KdV equation. Summing up equations (1-5) from i=1 to i=N, we get the KdV equation using Eq.(1-6). Note that decoupling of the KdV equation is not unique.

In this section we have used only a knowledge of the form of the KdV equation.

An example of the simplest N=2 case is shown in Fig.4.[1)]

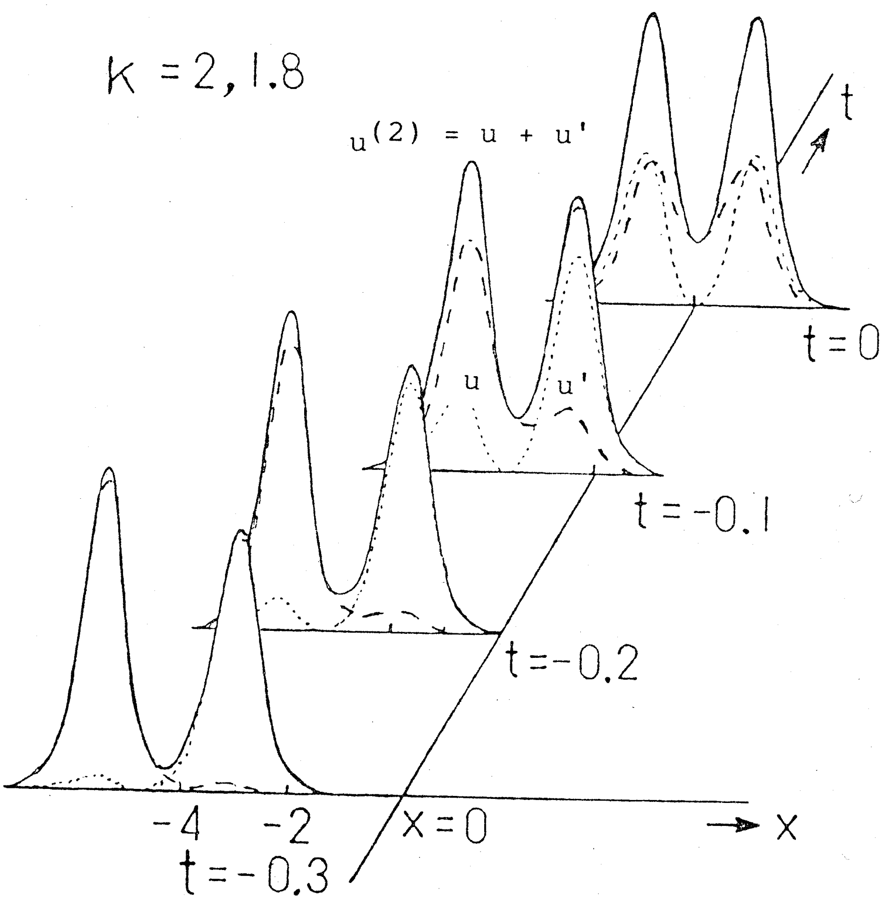


Fig.4

§2 Interacting KdV Equations

2-1 The Form of Equations

As was explained in the preceding section, the form of the Int KdV equations are able to be derived using only the knowledge of the form of the original KdV equation. The form of eqs.(1-5) has already been obtained in Ref.4)

an extension = a decoupling

the original KdV eq. $\Longrightarrow$ Interacting KdV eqs.

$du+6u\partial u+\partial^3 u = 0$	$du_i+6u\partial u_i+\partial^3 u_i = 0$

simple sum

the exact N-soliton solution $u^{(N)} \Rightarrow u_i$ (a single soliton as $t\to\pm\infty$)

$$u^{(N)} = \sum_{i=1}^{N} u_i$$

$u^{(N)} = 2\partial^2 \ln\det(I+B)$	$u_i = 2\partial\hat{\partial}_i \ln\det(I+\underset{\sim}{B})$
$\partial\equiv\partial/\partial x$	$\hat{\partial}_i F(x_1,x_2,\cdots,x_N)=(\partial/\partial x_i)F\vert_{x_1=x_2=\cdots=x}$

$$\partial = \sum_{i=1}^{N}\hat{\partial}_i$$

where	
I,B: N x N matrices	$I,\underset{\sim}{B}$: N x N matrices
$I_{ij} = \delta_{ij}$	$I_{ij} = \delta_{ij}$
$B_{ij} = \phi_i\phi_j/(\kappa_i+\kappa_j)$	$\underset{\sim}{B}_{ij} = \Phi_i\Phi_j/(\kappa_i+\kappa_j)$
$\phi_i\equiv C_i\exp\gamma_i\equiv C_i\exp(\kappa_i x-2^2\kappa_i{}^3 t)$	$\Phi_i\equiv C_i\exp\Gamma_i=C_i\exp(\kappa_i x_i-2^2\kappa_i{}^3 t)$

$$0<\kappa_1<\kappa_2<\cdots<\kappa_N$$

Table 2 Summary of 2-1 and 2-2

2-2 Solution u_i of the Int KdV Equation

We have obtained Int KdV equations in the previous section by a physically natural consideration. Their exact solutions can be obtained in the following three steps.

a) First step: The exact N-soliton solution of the KdV equation has already been obtained in several ways [3)-8)] and its asymptotic behavior was fully examined.[3)] We shall take these results as our starting point. It is known that the solution $u^{(N)}$ has the form

$$u^{(N)} = 2(\partial/\partial x)^2 \ln f. \qquad (2\text{-}1)$$

The function f is defined as

$$f(\gamma_1,\gamma_2,\cdots,\gamma_N) \equiv \det(I+B) \qquad (2\text{-}2)$$

where I and B are N x N matrices whose elements are

$$I_{kl} = \delta_{kl}$$

and

$$B_{kl} = \frac{C_k C_l}{\kappa_k+\kappa_l}\exp(\gamma_k+\gamma_l), \quad (1 \leqq k,l \leqq N), \qquad (2\text{-}3)$$

and here

$$\gamma_i \equiv \kappa_i x - 2^2\kappa_i{}^3 t \qquad (i=1,2,\cdots,N) \qquad (2\text{-}4)$$

with Kronecker's δ_{ij} and arbitrary positive constants κ_i and C_i. We assume

$$\kappa_1<\kappa_2<\cdots<\kappa_N \qquad (2\text{-}5)$$

without losing generality. Note that the function f is a rational function of $\exp\gamma_i$'s.

Here we remember that there is a notice in Ref.3) that f and $f\cdot\exp\{\alpha(t)x+\beta(t)\}$ give the same solution u(x,t) for any functions $\alpha(t)$ and $\beta(t)$.

b) Second step: Before obtaining solutions of Eqs.(1-5), we introduce new variables, a function and operators. First, let us introduce N independent space variables x_i $(i=1,2,\cdots,N)$ and following the definition (2-4) define Γ_i as

$$\Gamma_i \equiv \kappa_i x_i - 2^2\kappa_i{}^3 t. \qquad (i=1,2,\cdots,N) \qquad (2\text{-}6)$$

We can define N independent time variables t_i as well. In dealing with the Toda equation in §3, we will use them. Secondly, we define a function $F^{(N)}$ of Γ_i following (2-2):

$$F^{(N)}(\Gamma_1,\Gamma_2,\cdots,\Gamma_N) \equiv A\{\Gamma_i\}\ \det(I+\underset{\sim}{B}), \qquad (2\text{-}7)$$

where $\underset{\sim}{B}$ is a N x N matrix whose elements are

$$\underset{\sim}{B}_{kl} \equiv \frac{C_k C_l}{\kappa_k+\kappa_l}\exp(\Gamma_k+\Gamma_l) \qquad (1\leq k,l\leq N) \qquad (2\text{-}8)$$

and $A\{\Gamma_i\}$ is $\exp\{\sum_{i=1}^{N}\alpha_i(t)\Gamma_i+\beta(t)\}$ with arbitrary functions $\alpha_i(t)$ and $\beta(t)$. Lastly, new operators $\hat{\partial}_i$ $(i=1,2,\cdots,N)$ are introduced. They operate as follows;

$$\hat{\partial}_i F(\Gamma_1,\Gamma_2,\cdots,\Gamma_N) \equiv [(\partial/\partial x_i)F(\Gamma_1,\Gamma_2,\cdots,\Gamma_N)]\big|_{x_1=x_2=\cdots=x_N=x}. \qquad (2\text{-}9)$$

Using these operators, the following equations hold;

$$(\partial/\partial x)f \equiv \partial f = \sum_{l=1}^{N}\hat{\partial}_l F \qquad (2\text{-}10)$$

and

$$\partial\hat{\partial}_i = \hat{\partial}_i \sum_l \partial_l . \qquad (2\text{-}11)$$

Note that these operators $\hat{\partial}_i$ and Hirota's bilinear differential operator D_x are much alike.

c) Last step: With these preparations, we are now able to obtain the solution of Eq.(1-5). Using Eqs.(2-10) and (2-11), we get the expression

$$u^{(N)} = 2\partial\sum_k \hat{\partial}_k \ln F = 2\sum_k \hat{\partial}_k \sum_l \partial_l \ln F. \qquad (2\text{-}12)$$

Substituting Eq.(2-12) to Eq.(1-1), we have

$$\sum_i \hat{\partial}_i [(\partial/\partial t)(2\sum_k \partial_k \ln F) + 3\{\sum_l \partial_l (2\sum_k \partial_k \ln F)\}^2 + (\sum_l \partial_l)^3 (2\sum_k \partial_k \ln F)] = 0. \qquad (2\text{-}13)$$

In this equation, [ ]$\equiv$G is a rational function of exp Γ_i's and κ_i $(i=1,2,\cdots,N)$, for F is such a function. N.B. that $\hat{\partial}_j \exp\Gamma_i = 0$ $(i\neq j)$ and $\hat{\partial}_i \exp\Gamma_i = \kappa_i \exp\gamma_i$. Further, the values κ_i's are arbitrary. Therefore, to satisfy Eq.(2-13), $\sum_i \hat{\partial}_i G = 0$, G must be identically zero as a function of space variables $x_1, x_2, \cdots, x_N$ and of time variable t;

$$G \equiv 0. \qquad (2\text{-}14)$$

So, for each i we reach

$$\hat{\partial}_i G = 0. \qquad (2\text{-}15)$$

Using Eqs.(2-10) and (2-11), we get from Eq.(2-15)

$$(\partial/\partial t)(2\partial\hat{\partial}_i \ln F) + 6(2\partial^2 \ln f)\partial(2\partial\hat{\partial}_i \ln F)$$
$$+\partial^3(2\partial\hat{\partial}_i \ln F) = 0. \qquad (2\text{-}16)$$

Thus we obtain the solution u_i of the Int KdV equations (1-5);

$$u_i = 2\partial\hat{\partial}_i \ln F. \qquad (i=1,2,\cdots,N) \qquad (2\text{-}17)$$

It is obvious from Eqs. (2-10),(2-1) and (2-17) that

$$u^{(N)} = 2\partial\sum_{i=1}^{N}\hat{\partial}_i \ln F = \sum_{i=1}^{N} u_i. \qquad (2\text{-}18)$$

Equation (2-18) is Eq.(1-6) itself.

It is easy and straightforward to derive certain properties of the u_i's. For example, we can get

$$\int_{-\infty}^{\infty} u_i dx = 4\kappa_i, \qquad (2\text{-}19)$$

therefore the total area of u_i is invariant in time whatever deformation it suffers. We can also show that each u_i becomes a single soliton as $t\to\pm\infty$. The simplest N=2 case is very interesting and important.[1] We will see in the next subsection that there is a relation of our u_i's to the eigenfunctions which appear in the IM and they have many well-known properties.

In this subsection we have only made use of the fact that the solution $u^{(N)}$ is of the form ∂X (here $X=2\partial \ln f$) and that the function f is a rational function of $\exp\gamma_i$'s. In the next subsection we shall use the full information about $u^{(N)}$.

2-3 A Relation to the IM in the KdV case

def. $$\psi_k \equiv \sum\left(\frac{I}{I+B}\right)_{km}\phi_m$$

$$\partial B_{kl} = (\kappa_k+\kappa_l)B_{kl}$$

$$\begin{aligned}\partial \ln \det(I+B) &= \partial \mathrm{Tr}\ \ln(I+B)\\ &= \sum_{k,l}\left(\frac{I}{I+B}\right)_{kl}\partial B_{lk}\\ &= 2\sum\kappa_k\left(\frac{B}{I+B}\right)_{kk}\\ &= 2\sum\kappa_k - 2\sum\kappa_k\left(\frac{I}{I+B}\right)_{kk}\end{aligned}$$

$$\hat{\partial}_i \underset{\sim}{B}_{kl} = \kappa_i(\delta_{ik}+\delta_{il})B_{kl}$$

$$\begin{aligned}\mathrm{Tr}\hat{\partial}_i \ln(I+\underset{\sim}{B}) &= 2\kappa_i\left(\frac{B}{I+B}\right)_{ii}\\ &= 2\kappa_i - 2\kappa_i\left(\frac{I}{I+B}\right)_{ii}\end{aligned}$$

Using $$\partial A^{-1} = -A^{-1}(\partial A)A^{-1}$$

$$\begin{aligned}u &= 2\partial\{\partial \ln \det(I+B)\}\\ &= 4\sum\kappa\left(\frac{I}{I+B}\right)_{ik}\phi_k\phi_l\left(\frac{I}{I+B}\right)_{li}\\ &= 4\sum_i\kappa_i\psi_i^{\,2}\end{aligned}$$

$$\begin{aligned}u_i &= 2\partial\hat{\partial}_i \ln \det(I+\underset{\sim}{B})\\ &= -4\kappa_i\partial\left(\frac{I}{I+B}\right)_{ii}\\ &= 4\kappa_i\psi_i^{\,2}\end{aligned}$$

Table 3 Summary of 2-3

Here we shall show that the following relations hold.

$$u_i = 4\kappa_i\psi_i^{\,2} \quad (i = 1, 2, \cdots, N) \tag{2-20}$$

In ref. 8), Wadati and Sawada showed that: Using a formula

$$\ln \det(I+B) = \mathrm{Tr}\ \ln(I+B) \tag{2-21}$$

for a square matrix I+B (2-3) and with the properties

$$B_{kl} = B_{lk} \tag{2-22}$$

and
$$\partial B_{kl} = (\kappa_k+\kappa_l)B_{kl}, \tag{2-23}$$

they derived

$$\partial \ln \det(I+B) = \sum_{k,l}\left(\frac{I}{I+B}\right)_{kl}\partial B_{kl}$$

$$= 2\sum_k \kappa_k \left(\frac{B}{I+B}\right)_{kk} = 2\sum_k \kappa_k - 2\sum_k \kappa_k \left(\frac{I}{I+B}\right)_{kk}. \qquad (2\text{-}24)$$

Then they defined ϕ_m and ψ_l for $1 \leq m, l \leq N$ as follows;

$$\phi_m \equiv C_m \exp \gamma_m \qquad (2\text{-}25)$$

$$\psi_l \equiv \sum_{m=1}^{N}\left(\frac{I}{I+B}\right)_{lm}\phi_m. \qquad (2\text{-}26)$$

For any normal matrix A, we have

$$\partial A^{-1} = -A^{-1}(\partial A)A^{-1}. \qquad (2\text{-}27)$$

From this and eq.(2-24) follows

$$u = 2\partial\{\partial \ln \det(I+B)\}$$

$$= \sum_{k,l,m} 4\kappa_k \left(\frac{I}{I+B}\right)_{kl}\phi_l\phi_m\left(\frac{I}{I+B}\right)_{mk} = \sum_l 4\kappa_l \psi_l^{\,2}. \qquad (2\text{-}28)$$

By making the best use of their results with our new operators $\hat{\partial}_i$ introduced in subsection 2-2, we can show explicitly that Eq. (2-20) holds. We have introduced $\underset{\sim}{B}$ in Eq.(2-8),

$$\underset{\sim}{B}_{kl} \equiv \frac{C_k C_l}{\kappa_k + \kappa_l}\exp(\Gamma_k + \Gamma_l) \qquad (2\text{-}29)$$

for which equations

$$\hat{\partial}_i \underset{\sim}{B}_{kl} = \kappa_i(\delta_{ik} + \delta_{il})B_{kl} \tag{2-30}$$

hold, and so we get

$$\mathrm{Tr}\ \hat{\partial}_i \ln(I+\underset{\sim}{B}) = 2\kappa_i\left(\frac{B}{I+B}\right)_{ii} = 2\kappa_i - 2\kappa_i\left(\frac{I}{I+B}\right)_{ii}. \tag{2-31}$$

Hence Eq.(2-20) is proved to hold:

$$\begin{aligned} u_i &= 2\partial\hat{\partial}_i \ln\det(I+\underset{\sim}{B}) = -4\kappa_i\partial\left(\frac{I}{I+B}\right)_{ii} \\ &= 4\kappa_i \textstyle\sum_{k,l}\left(\frac{I}{I+B}\right)_{ik}\phi_k\phi_l\left(\frac{I}{I+B}\right)_{li} = 4\kappa_i\psi_i^{\ 2} \end{aligned} \tag{2-32}$$

This means that there exists an explicit relation between our method and the IM. The function ψ_i which appear in the IM is not an auxiliary quantity to the solution u of the KdV equation, but relates directly to it.

In this subsection we have used the full knowledge of the form of the KdV N-soliton solution.

2-4 "Partial" Lax Forms

It is well known that for the KdV equation (1-1) there exists the Lax form[9])

$$(\partial/\partial t)L = [A,L] = AL - LA, \tag{2-33}$$

where

$$L = -\partial^2 - u \tag{2-34}$$

and

$$A = -4\partial^3 - 6u\partial - 3u_x. \tag{2-35}$$

Here u_x is a function obtained by differentiating u with respect to x. With these operators L and A, two equations

$$L\psi_i = -\kappa_i^2\psi_i \qquad (2\text{-}36)$$

and

$$(\partial/\partial t)\psi_i = A\psi_i \qquad (2\text{-}37)$$

hold.

We only point out a fact, using a relation

$$\{\partial_i \underset{\sim}{u}\} = \textstyle\sum_j\{\partial_j \underset{\sim}{u}_i\} \qquad (2\text{-}38)$$

with

$$\underset{\sim}{u} = 2\textstyle\sum_{k,l}\{\partial_k\partial_l \ln(I+\underset{\sim}{B})\}, \qquad (2\text{-}39)$$

that "partial" Lax forms

$$(\partial/\partial t)\underset{\sim}{L}_i = [\underset{\sim}{A}_i, \underset{\sim}{L}] \quad (i=1,2,\cdots,N) \qquad (2\text{-}40)$$

hold, where

$$\underset{\sim}{L} = -(\textstyle\sum_j\partial_j)^2 - \underset{\sim}{u}, \qquad (2\text{-}41)$$

$$\underset{\sim}{L}_i = -\partial_i\textstyle\sum_j\partial_j - \underset{\sim}{u}_i \qquad (2\text{-}42)$$

and

$$\underset{\sim}{A}_i = -4(\textstyle\sum_j\partial_j)^2\partial_i - 4\underset{\sim}{u}\partial_i - 2\underset{\sim}{u}_i\textstyle\sum_j\partial_j - 3\underset{\sim}{u}_{i,x}. \qquad (2\text{-}43)$$

In Eq.(2-43) $\underset{\sim}{u}_{i,x}$ stands for a function $\sum_j\{\partial_j\underset{\sim}{u}_i\}$. Here $\{\partial_j\underset{\sim}{u}_i\}$ means a function obtained by differentiating $\underset{\sim}{u}_i$ with respect to x_j. So, for example, $\partial_j\underset{\sim}{u}_i = \{\partial_j\underset{\sim}{u}_i\} + \underset{\sim}{u}_i\partial_j$. It is obvious that summing up Eqs.(2-40) from i=1 to i=N and equating all x_i with x, we get equation (2-33).

We can not give a discussion based on the unitary equivalence here. It is left for a future study.

§3 Interacting Toda Equations

In this section, we only present tables without discussion; they speak for themselves. Detailed discussion will soon be published elsewhere.

3-1 Int Toda Equations and their Solutions $V_{n,i}$

an extension = a decoupling

the original Toda eq. $\Longrightarrow$ Interacting Toda eqs.

the original Toda eq.	Interacting Toda eqs.
$d^2\ln(1+V_n)=V_{n+1}+V_{n-1}-2V_n$	
$d\{d\,V_n/(1+V_n)\} = \Delta^2 V_n$	$d\{dV_{n,i}/(1+V_n)\} = \Delta^2 V_{n,i}$
$(\Delta V_n \equiv V_{n+\frac{1}{2}} - V_{n-\frac{1}{2}})$	

simple sum

the exact N-soliton solution $V_n \Rightarrow V_{n,i}$ (a single soliton as $t \to \pm\infty$)

$$V_n = \sum_{i=1}^{N} V_{n,i}$$

$V_n = d^2\ln\det(I+B)$	$V_{n,i} = d\hat{d}_i \ln\det(I+\underset{\sim}{B})$
$d \equiv \partial/\partial t$	$\hat{d}_i F(t_1,t_2,\cdots,t_N) = (\partial/\partial t_i)F\vert_{t_1=t_2=\cdots=t}$

$$d = \sum_{i=1}^{N} \hat{d}_i$$

I, B : $N \times N$ matrices	$I, \underset{\sim}{B}$: $N \times N$ matrices
$I_{ij} = \delta_{ij}$	$I_{ij} = \delta_{ij}$
$B_{ij} = \phi_i\phi_j/(1-z_iz_j)$	$\underset{\sim}{B}_{ij} = \Phi_i\Phi_j/(1-z_iz_j)$
$\phi_i \equiv C_i\exp\{\beta_i t-\alpha_i(n+1)\}$	$\Phi_i \equiv C_i\exp\{\beta_i t_i-\alpha_i(n+1)\}$

$$\beta_i = \sinh\alpha_i = (z_i^{-1}-z_i)/2, \quad z_i = \exp(-\alpha_i)$$

Table 5 Int Toda equations and their solutions $V_{n,i}$

3-2 The Form of Solutions $V_{n,i}$ of Int Toda Equations

def. $\psi_k(n) \equiv \sum\left(\frac{I}{I+B}\right)_{km}\phi_m(n), \qquad \chi_k(n) \equiv \sum\left(\frac{I}{I+B}\right)_{km}\phi_m(n-1)$

$dB_{kl} = (\beta_k+\beta_l)\phi_k\phi_l/(1-z_k z_l)$ $= (z^{-\alpha_k}+z^{-\alpha_\ell})\phi_k\phi_l/2$ $=\{\phi_k(n-1)\phi_l(n)+\phi_k(n)\phi_l(n-1)\}/2$	$d_i \underset{\sim}{B}_{kl}=\beta_i(\delta_{ik}+\delta_{il})\Phi_k\Phi_l/(1-z_k z_l)$
$d\ln\det(I+B) = \sum 2\beta_m\left(\frac{B}{I+B}\right)_{mm}$ $V_n = d^2\ln\det(I+B)$ $=\sum\beta_m\left(\frac{I}{I+B}\right)_{mk}\{\phi_k(n-1)\phi_l(n)+\phi_k(n)\phi_l(n-1)\}\left(\frac{I}{I+B}\right)_{lm}$ $=\sum_m 2\beta_m\chi_m(n)\psi_m(n)$	$d_i\ln\det(I+\underset{\sim}{B}) = 2\beta_i\left(\frac{\underset{\sim}{B}}{I+\underset{\sim}{B}}\right)_{ii}$ $V_{n,i} = \hat{d}d_i\ln(I+\underset{\sim}{B})$ $=2\beta_i\chi_i(n)\psi_i(n)$

Table 6 The form of solutions $V_{n,i}$ of Int Toda equations

3-3 N. Saitoh transformation[10]; the Toda Eq. to the KdV Eq.

$$\begin{array}{ccc} \text{Toda eq.} & \xrightarrow[(0<h\leqq 1)]{h\to 0} & \text{KdV} \\ V_n(t) & & u(x,t) \end{array}$$

$V_n = h^2 u_n(\tau)$

$n=(x/h)+(h^{-2}-h^2)\tau/h \to x=hn-(h^{-2}-h^2)\tau$

$(\partial/\partial\tau)f\{V_n(\tau)\} = \{(\partial/\partial t)-(h^{-2}-h^2)(\partial/\partial x)\}f(u(x,\tau))$

the Toda eq.: $d^2\ln(1+V_n) \quad = \quad h^{-4}\{u(x+h,\tau)+u(x-h,\tau)-2u(x,\tau)\}$

$=-2\partial^2 u/\partial x\partial\tau\cancel{+h^{-2}\partial^2 u/\partial x^2}-2^{-2}\partial^2(u^2)/\partial x^2 \qquad \cancel{h^{-2}\partial^2 u/\partial x^2}+12^{-1}\partial^4 u/\partial x^4$

$\partial\{\text{KdV}\}: (\partial/\partial x)\{-2\partial u/\partial\tau-2^{-1}\partial(u^2)/\partial x\} \qquad (\partial/\partial x)\{12^{-1}\partial^3 u/\partial x^3\}$

similarly, Int Toda eqs. → Int KdV eqs.

Table 7 N. Saitoh transformation; the Toda eq. to the KdV eq.

§4 Direct Relationships to the IM; the KdV, the sine-Gordon and the Modified KdV cases[11)]

It is shown that the solutions of three kinds of the interacting soliton equations can be expressed explicitly by the squared eigenfunctions of the corresponding two-component equations which appear in the IM. The KdV, the sine-Gordon and the modified KdV (MKdV) cases are shown.

We consider the KdV, the sine-Gordon and the MKdV equations in the following forms;

(a) The KdV equation:

$$du + 6u\partial u + \partial^3 u = 0. \qquad (4\text{-}1)$$

The operators d and ∂ are,

$$d \equiv \partial/\partial t \qquad (4\text{-}2)$$

and

$$\partial \equiv \partial/\partial x. \qquad (4\text{-}3)$$

(b) The sine-Gordon equation:

$$d\partial\sigma = \sin\sigma,$$

but here we use $u \equiv \partial\sigma/2$ instead of σ, so the form is

$$d\partial u = u\cos\sigma. \qquad (4\text{-}4)$$

(c) The MKdV equation:

$$du + 6u^2\partial u + \partial^3 u = 0. \qquad (4\text{-}5)$$

The Int soliton equations in the three cases are then given as follows. Detailed discussions in the cases of (b) and (c) will soon be published elsewhere.

(a) The Int KdV equations:

$$du_i + 6u(\partial u_i) + \partial^3 u_i = 0. \qquad (4\text{-}6)$$

(b) The Int sine-Gordon equations:

$$d\partial u_i = u_i \cos\sigma. \qquad (4\text{-}7)$$

(c) The Int MKdV equations:

$$du_i + 6u^2(\partial u_i) + \partial^3 u_i = 0. \qquad (4\text{-}8)$$

Of course, each solution u_i of Eqs.(4-6)∿(4-8) may have its own solution with any soliton number, but here each u_i is taken for a single soliton as $t\to\pm\infty$.[2] Summing up each solution u_i from $i = 1$ to $i = N$ gives the exact N-soliton solution $u^{(N)}$;

$$\sum_{i=1}^{N} u_i = u^{(N)}. \qquad (4\text{-}9)$$

Here, we adopt the two-component equations in the following forms:[12),13)]

$$(\partial - \kappa)\psi_1 = u\psi_2 \qquad (4\text{-}10a)$$

$$(\partial + \kappa)\psi_2 = r\psi_1 \tag{4-10b}$$

and

$$d\psi_1 = A(t,x,\kappa)\psi_1 + B(t,x,\kappa)\psi_2 \tag{4-11a}$$

$$d\psi_2 = -A(t,x,\kappa)\psi_2 + C(t,x,\kappa)\psi_1 \tag{4-11b}$$

where κ is an eigenvalue, ψ_1 and ψ_2 are corresponding eigenfunctions and functions A, B and C are so chosen that κ is time invariant.

Now, we shall give proofs that the solution of the Int soliton equations are expressed explicitly by the squared eigenfunctions of the corresponding two-component equations.

(a) The KdV case: In this case,

$$u_i = c_i\psi_2{}^2 \tag{4-12}$$

where c_i is a constant,

$$r = -1 \tag{4-13}$$

and

$$\begin{aligned} A &= -(4\kappa^3 + 2\kappa u + u_x) \\ B &= -(4\kappa^2 u + 2\kappa u_x + 2u^2 + u_{xx}) \\ C &= 4\kappa^2 + 2u. \end{aligned} \tag{4-14}$$

(Subscripts denote partial differentiation.) From Eqs.(4-10) and (4-13), we get

$$\partial\psi_1 = \kappa\psi_1 + u\psi_2 \tag{4-15a}$$

and
$$\partial\psi_2 = -\kappa\psi_2 - \psi_1, \tag{4-15b}$$
whence
$$\partial^2\psi_1 = (-u + \kappa^2)\psi_1 + u_x\psi_2, \tag{4-16a}$$
$$\partial^2\psi_2 = (-u + \kappa^2)\psi_2 \tag{4-16b}$$
and
$$\partial^3\psi_2 = (-u + \kappa^2)(\partial\psi_2) - u_x\psi_2. \tag{4-17}$$

From Eqs.(4-16),(4-14) and (4-15b),

$$d\psi_2 = -(4\kappa^2 + 2u)(\partial\psi_2) + u_x\psi_2. \tag{4-18}$$

The proof is very simple and straightforward. Substituting (4-12) to the l.h.s. of Eq.(4-6) divided by $2c_i$, we have

$$\psi_2 d\psi_2 + 6u\psi_2\partial\psi_2 + \psi_2(\partial^3\psi_2) + 3(\partial\psi_2)(\partial^2\psi_2)$$

$$= \psi_2\{-(4\kappa^2 + 2u)(\partial\psi_2) + u_x\psi_2\} + 6u\psi_2\partial\psi_2$$

$$+ \psi_2\{(-u + \kappa^2)(\partial\psi_2) - u_x\psi_2\} + 3(\partial\psi_2)\{(-u + \kappa^2)\psi_2\} = 0. \tag{4-19}$$

(b) The sine-Gordon case: In this case

$$u_i = c_i(\psi_1{}^2 + \psi_2{}^2) \tag{4-20}$$

where c_i is a constant,

$$r = -u \tag{4-21}$$

and

$$A = \cos\sigma/(4\kappa)$$

$$B = C = \sin\sigma/(4\kappa). \tag{4-22}$$

Using the following relations

$$\psi_1\partial\psi_1 - \psi_2\partial\psi_2 = \kappa(\psi_1{}^2 + \psi_2{}^2) - 2u\psi_1\psi_2$$

and

$$\psi_1\partial\psi_2 + \psi_2\partial\psi_1 = u(\psi_1{}^2 - \psi_2{}^2), \tag{4-23}$$

l.h.s. of Eq.(4-7) becomes

$$d\partial u_i = 2c_i\partial(\psi_1 d\psi_1 + \psi_2 d\psi_2) = 2c_i\partial\{A(\psi_1{}^2 - \psi_2{}^2) + 2B\psi_1\psi_2\}$$

$$= c_i\partial\{\cos\sigma(\psi_1{}^2 - \psi_2{}^2) + 2\sin\sigma\psi_1\psi_2\}/(2\kappa)$$

$$= c_i\{-2u\sin\sigma(\psi_1{}^2 - \psi_2{}^2) + 2\cos\sigma(\psi_1\partial\psi_1 - \psi_2\partial\psi_2)$$

$$+ 4u\cos\sigma\psi_1\psi_2 + 2\sin\sigma(\psi_1\partial\psi_2 + \psi_2\partial\psi_1\}/(2\kappa)$$

$$= c_i(\psi_1{}^2 + \psi_2{}^2)\cos\sigma = u_i\cos\sigma = \text{r.h.s. of Eq.(4-7)}.$$

(c) The MKdV case: Similar to the sine-Gordon case, the proof is simple and straightforward. In this case we use the following relations obtained from Eqs.(4-10) and (4-21):

$$\psi_1\partial\psi_1 + \psi_2\partial\psi_2 = \kappa(\psi_1{}^2 - \psi_2{}^2)$$

$$\psi_1 \partial \psi_1 - \psi_2 \partial \psi_2 = \kappa(\psi_1{}^2 + \psi_2{}^2) + 2u\psi_1\psi_2$$

$$\psi_1 \partial \psi_2 - \psi_2 \partial \psi_1 = -u(\psi_1{}^2 + \psi_2{}^2) - 2\kappa\psi_1\psi_2. \qquad (4\text{-}24)$$

We have shown in this section that the solutions u_i of the Int KdV, the Int sine-Gordon and the Int MKdV equations are proportional to one of or a sum of the squared eigenfunctions of the corresponding two-component equations, using only the knowledge of the form of the equations. If we use the full knowledge of the functional form of the exact N-soliton solution, we can determine the values of c_i in each cases: e.g. the functional form of the MKdV solution is found in Ref.14). A detailed discussion will soon be published elsewhere.

Similar treatment of the Toda equation is left for a future study. There is also a problem left about the relation between the dependence of u_i on ψ and the type of the equations under consideration.

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