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Classical Boussinesq Equation

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Abstract.

The classical Boussinesq equation

$$u_t = [(1 + u)v + v_{xx}]_x$$

$$v_t = (u + \frac{1}{2} v^2)_x$$

describing shallow water wave, is a pq = c reduction of a first-modified KP equation

$$(D_1^{\,2} + D_2)\, \tau \cdot \tau' = 0 \ ,$$

$$(D_1^{\,3} - 3D_1 D_2 - 4D_3)\, \tau \cdot \tau' = 0 \ .$$

We find a coupled equation in the text book, (Whitham: Linear and Non-linear Waves Equation (13. 101) p.465)

$$\eta_t + (1 + \alpha\eta)w_x - \frac{1}{6}\beta w_{xxx} + O(\alpha\beta,\beta^2) = 0 ,$$

$$w_t + \alpha w w_x + \eta_x - \frac{1}{2}\beta w_{xxt} + O(\alpha\beta,\beta^2) = 0.$$

Let

$$w = u + \frac{1}{2}\beta u_{xx} + O(\alpha\beta,\beta^2) .$$

Then

$$\eta_t + [(1 + \alpha\eta)u]_x + \frac{1}{3}\beta u_{xxx} + O(\alpha\beta,\beta^2) = 0 ,$$

$$u_t + [\eta + \frac{\alpha}{2}u^2]_x + O(\alpha\beta,\beta^2) = 0 .$$

cf. S. Kawamoto: J. Phys. Soc. Jpn.53 (1984) 2922.

E.V. Krishnan: J. Phys. Soc. Jpn. 51 (1982) 2391.

Let $u = w_x$. Then we have

$$w_{tt} + (1 + \alpha)w_x w_{xt} + \alpha w_t w_{xx} - w_{xx}$$

$$+ \frac{3\alpha}{2}(w_x)^2 w_{xx} - \frac{1}{3}\beta w_{xxxx} = O(\alpha\beta,\beta^2) ,$$

which is Kaup's higher order water wave equation.

cf. D.J. Kaup: Prog. Theor. Phys. 54 (1975) 396.

On the other hand, we have the Jaulent-Miodek eq.

$$u_t = \frac{1}{4}v_{3x} - uv_x - \frac{1}{2}u_x v ,$$

$$v_t = -u_x - \frac{3}{2} vv_x ,$$

which is the integrabilinty conditions of the following spectral problem

$$[\frac{\partial^2}{\partial x^2} + (\lambda^2 - \lambda v - u]\psi = 0 ,$$

$$\psi_t = [\frac{1}{4} v_x - \frac{1}{2} (u^2 + 2\lambda) \frac{\partial}{\partial x}]\psi .$$

M. Joulent and I. Miodek: Lett. Math. Phys. 1 (1976) 243.

The Joulent-Miodek equation is transformed into the classical Boussinesq equation

$$\phi_\tau = [(1 + \phi)v + v_{\xi\xi}]_\xi ,$$

$$v_\tau = (\phi + \frac{1}{2} v^2)_\xi ,$$

through the transformation

$$\phi = u + \frac{1}{2} v^2 - 1 ,$$

$$\tau = - \frac{1}{2} t , \quad \xi = \frac{1}{2} x .$$

Sato and Mori have studied a hierarchy of the classical Boussinesq equations

$$w_{t1} = (v_{xx} + 2vw)_x ,$$

$$v_{t1} = (w + v^2)_x ,$$

$$w_{t2} = (w_{xx} + 3vv_{xx} + \frac{3}{2} v_x^2 + \frac{3}{2} v^2 + 3v^2 w)_x ,$$

$$v_{t2} = (v_{xx} + 3vw + v^3)_x .$$

They called it Higher order Nonlinear Schrödinger equations.

M. Sato and Y. Mori: RIMS Kôkyûroku 388 (1980) 183.

Ito has studied a symmetries and conservation laws of the classical Boussinesq eq. and found a recursion operator for symmetries.

M. Ito: Physics Lett. 104 A (1984) 248.

The classical Boussinesq eq.

$$u_t = [(1 + u)v + v_{xx}]_x ,$$

$$v_t = (u + \frac{1}{2} v^2)_x ,$$

is transformed into the bilinear form

$$(D_\tau - D_x^2)\, f\cdot g = 0 ,$$

$$[D_x D_\tau - (1 + u_0)D_x - D_x^3]\, f\cdot g = 0 ,$$

where $D_\tau = D_t - v_0 D_x$ through the transformation

$$v = v_c + 2\phi_x ,$$

$$u = u_c + 2\rho_{xx} ,$$

$$\phi = \log(f/g) ,$$

$$\rho = \log(fg) .$$

Hierarchies of K-P equation and first modified K-P equation are obtained by Jimbo and Miwa:

K-P equation

$$(D_1^4 + 3D_2^2 - 4D_1D_3)\, \tau\cdot\tau = 0 ,$$

$$(D_1^3 D_2 + 2D_2D_3 - 3D_1D_4)\, \tau\cdot\tau = 0 ,$$

$$\cdots .$$

First modified K-P equation

$$(D_1^2 + D_2)\, \tau\cdot\tau' = 0 ,$$

$$(D_1^3 - 3D_1D_2 - 4D_3)\, \tau\cdot\tau' = 0 ,$$

$$\cdots .$$

M. Jimbo and T. Miwa: "Solitons and Infinite Dimensional Lie Algebras", RIMS - 439, March (1983).

N-Soliton solution to the first modified K-P equation is well-known. We write it for N = 2

$$\tau = 1 + q_1 e^{\eta_1} + q_2 e^{\eta_2} + q_1 q_2 a_{12} e^{\eta_1+\eta_2} ,$$

$$\tau' = 1 + p_1 e^{\eta_1} + p_2 e^{\eta_2} + p_1 p_2 a_{12} e^{\eta_1+\eta_2} ,$$

where $\eta_i = (p_i - q_i)x_1 + (p_i^2 - q_i^2)x_2 + (p_i^3 - q_i^3)x_3 + \cdots ,$

for $i = 1, 2, \cdots$

$$a_{12} = \frac{(p_1 - q_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)} .$$

where p_i and q_i for $i = 1, 2$, are free parameters.

Now we impose the following conditions on p_i and q_i

$$p_i q_i = c \qquad \text{for all } i ,$$

which we call "pq = c reduction" of first modified K-P equation. Then, we find that the following equation holds

$$(D_1^3 + 3cD_1 - D_3)\ \tau\cdot\tau' = 0 .$$

Substituting the above relation into the first modified K-P equation, we have

$$(D_1^2 + D_2)\ \tau\cdot\tau' = 0 ,$$

$$(D_1^3 + D_1 D_2 + 4cD_1)\ \tau\cdot\tau' = 0 ,$$

which becomes the bilinear form of the classical Boussinesq equation by the following identification

$$x_1 = x \,, \qquad x_2 = -\tau \,, \qquad 1 + u_0 = 4c \,.$$

Hence, two-soliton solution to the classical Boussinesq equation is readily constructed by the two-soliton solution of the first modified K-P equation.

All we do is to calculate the factor a_{12} which describes a phase shift of a soliton after colliding another soliton, using the condition $p_i q_i = c$,

$$\begin{aligned} a_{12} &= \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)} \\ &= \frac{c(p_1 - p_2)^2}{(p_1 p_2 - c)^2} \end{aligned}$$

and the dispersion relation which relate the imaginary frequency $\Omega_i \equiv p_i^2 - q_1^2$ to the imaginary wave number $\mathcal{P}_i \equiv p_i - q_i$:

$$\begin{aligned} \Omega_1^2 &= (p_i^2 - q_i^2)^2 \\ &= (p_i - q_i)^2[(p_i - q_i)^2 + 4p_i q_i] \\ &= \mathcal{P}_i^2[\mathcal{P}_i^2 + 4c] \,. \end{aligned}$$

We rewrite "pq = c reduction" of the modified K-P equation

$$(iD_t + D_x^2)\, g \cdot f = 0 \,,$$

$$(iD_xD_t + D_x^3 - 4\rho_0^2 D_x)g\cdot f = 0 ,$$

where we put

$$x_1 = x , \quad x_2 = -it , \quad c = -\rho_0^2 , \quad \tau = g , \quad \tau' = f .$$

We also rewrite the two-soliton solution

$$g = 1 + (q_1/p_1)e^{\eta_1} + (q_2/p_2)e^{\eta_2} + (q_1q_2/p_1p_2)a_{12}e^{\eta_1+\eta_2} ,$$

$$f = 1 + e^{\eta_1} + e^{\eta_2} + a_{12}e^{\eta_1+\eta_2} ,$$

where $\eta_i = P_i x - \Omega_i t$,

$$q_i/p_i = - (p_i^2 + i\Omega_i)/(p_i^2 - i\Omega_i) ,$$

$$\Omega_i^2 = P_i^2(4\rho_o^2 - P_i^2) .$$

We note that f and g constitute two-dark solitons of the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} - 2|\psi|^2\Psi = 0$$

$$\psi = \rho_0 \exp(-2i\rho_0^2 t)(g/f) .$$

In fact, the bilinear equation for real f

$$(iD_t + D_x^2)g\cdot f = 0 ,$$

$$(iD_tD_x + D_x^3 - 4\rho_0^2 D_x)g\cdot f = 0$$

is transformed into

$$(iD_t + D_x^2)g\cdot f = 0 ,$$

$$D_x^2 f\, f - 2\rho_0^2 f^2 = c_0 g^* g$$

where c_0 is determined by the boundary condition on f and g at $|x| = \infty$. Now we consider the case $\rho_0 = 0$ or "pq = 0 reduction" of the modified K-P equation. We have for $q_1 = c/p_1$, $p_2 = c/q_2$

$$f = 1 + \frac{c}{p_1} e^{\eta_1} + q_2 e^{\eta_2} + \frac{c}{p_1} q_2 \frac{(c - p_1 q_2)^2}{c(p_1 - q_2)^2} e^{\eta_1+\eta_2} ,$$

$$g = 1 + p_1 e^{\eta_1} + \frac{c}{q_2} e^{\eta_2} + \frac{c}{q_2} \frac{(c - p_1 q_2)^2}{c(p_1 - q_2)^2} e^{\eta_1+\eta_2} ,$$

where

$$\eta_1 = (p_1 - \frac{c}{p_1})x - i(p_1^2 - \frac{c^2}{p_1^2})t + \eta_1^0 ,$$

$$\eta_2 = (\frac{c}{q_2} - q_2)x - i(\frac{c^2}{q_2^2} - q_2^2)t + \eta_2^0 ,$$

which becomes, in the limit c = 0,

$$f = 1 + q_2 e^{\eta_2} + \frac{p_1 q_2^3}{(p_1 - q_2)^2} e^{\eta_1+\eta_2} ,$$

$$g = 1 + p_1 e^{\eta_1} + \frac{p_1^3 q_2}{(p_1 - q_2)^2} e^{\eta_1+\eta_2} ,$$

where

$$\eta_1 = p_1 x - i p_1^2 t + \eta_1^0 ,$$

$$\eta_2 = -q_2 x + iq_2^2 t + \eta_2^0 .$$

Let $q_2 = -p_1^*$, $\eta_2^0 = \eta_1^{0*} + i\pi$. Then

$$f = 1 + p_1^* e^{\eta_1^*} + \frac{p_1 p_1^{*3}}{(p_1 + p_1^*)^2} e^{\eta_1 + \eta_1^*} , \quad g = f^* .$$

Accordingly f and g are solutions to the bilinear equation

$$(iD_t + D_x^2) g \cdot f = 0 ,$$

$$(iD_x D_t + D_x^3) g \cdot f = 0 ,$$

but they cannot be a solution to the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0$$

any more because f is no longer real.

Polynomial solutions to

$$(iD_t + D_x^2) f^* f = 0$$

$$(iD_x D_t + D_x^3) f^* f = 0$$

have been found (A. Nakamura and R. Hirota, submitted to J. phys. Soc. Jpn. (1984)). They are

$$f_0 = 1 , \quad g_0 = 1 ,$$

$$f_1 = x^2 + 2it , \quad g_1 = f_1^* ,$$

$f_2 = (x^6 - 36x^2t^2) + i(6x^4t + 72t^3)$, $g_2 = f_2^*$,

... .

Rational solutions to the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0$$

are constrcted using f_n and f_n^* as follows.

$$\psi = \frac{D_x f_n^* \cdot f_{n+1}^*}{f_n f_{n+1^*} + f_n^* f_{n+1}} .$$

We have found the following reductions.

Reductions of K-P equation

$p^2 - q^2 = c(p - q)$, ... KdV equation

$p^3 - q^3 = c(p - q)$, ... Boussinesq equation

$p^4 - q^4 = 0$ . ... Coupled KdV equation

Reductions of mokified K-P equation

$p^2 - q^2 = c(p - q)$, ... modified KdV equation

$p^3 - q^3 = c(p - q)$, ... modified Boussinesq equation

$p \cdot q = c$ . ... Classical Boussinesq equation