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Integrability of the integrable systems with higher order perturbations

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Abstract

The intergable Hamiltonian systems with certain classes of perturbations such as higher order corrections are discussed. It is shown especially that the integrable systems with first order corrections in the dispersive nonlinear wave equations can be transformed into the higher order integrable systems at the same order.

1. Introduction

In many cases, integrable systems appearing in the physical problems are just the approximated equations of the original physical systems in an appropriate asymptotic sense. Among these systems there are linearized equations for certain nonlinear problems (of finite or infinite degrees of freedom), the Korteweg-de Vries (KdV) equation and the nonlinear Schrödinger (NLS) equation describing weakly and strongly dispersive nonlinear wave phenomena, respectively, in the leading order approximations [1]. (Both KdV and NLS equations are known as the completely integrable Hamiltonian systems by means of the method of inverse scattering transformation [2]). Those integrable systems are valid for certain time length determined from the physical settings (e.g. the order of nonlinearity and/or the smoothness of the initial conditions). For the problem related to large time behavior of the solutions (such as the stability of the critical points), one needs to study the effects of the higher order terms which are neglected in the derivation of the integrable systems. For finite dimensional systems, this problem has been studied extensively [3], and the several methods for analyzing the problems have been developed. A most successful one is the Birkhoff normal form theory in which the perturbed Hamiltonian systems can be transformed into the integrable normal forms by successive canonical transformations under the non-resonant condition in the sense of formal power series [3].

In recent years, there has been several discussions related to the problem for infinite dimensional systems, such as the higher order corrections of the KdV and the NLS equations [4,5,6]. Especially, ref.[5,6] conjectured that the KdV and NLS equations with first order corrections as the perturbations can be approximated by the integrable systems in the same order.

The main purpose in this paper is to study the integrability of such perturbed Hamiltonian systems (i.e. the integrable systems with certain classes of perturbations such as the higher order corrections). In the sections 2 and 3, we describe the type of perturbations considered here, and give several physical examples which will be discussed throughout this paper. The examples are a) N-uncoupled harmonic oscillators (a well-known classical example [3]), b) the linearized KdV equation (a example of linear dispersive wave equation), and c) the KdV equation. In the section 4, we try to find the integrals for the perturbed systems for studying their integrability. Particularly, in the case of the linearized KdV equation, we show by use of an algebra of the differential polynomials that the perturbed equation actually possesses an infinitely many integrals up to second order. It is however difficult to find all of the integrals of the perturbed equation, in general. But for certain classes of perturbations such as our case, the perturbed equations for given order can be characterized by only few integrals. So that if we found several (not necessary to be all) integrals of the perturbed equation, one can show that the equation is actually integrable up to the same order. This is the main result in this paper. In order to show this, we first define, in the section 5, a normal form corresponding to the perturbed equation on a constant surface determined by some integrals of the unperturbed system. The normal form can be given in an integrable form, if we found enough integrals to characterize the perturbed equation. Then, in the section 6, we construct a map (a canonical transformation) between the perturbed equation and the corresponding normal form, and show that our examples a)-c) are integrable up to first order correction. In this paper, we mainly consider the first order problem, but the higher order problems can be studied in the same way discussed here.

2. Integrable systems with perturbations

We consider the following form of evolution equation as a perturbed integrable system of dynamical variable $u(t)$ on certain smooth manifold M, with the small parameter ε,

$$\dot{u}(t) := \frac{d}{dt} u(t) = X(u(t);\varepsilon) = X^{(0)}(u) + \varepsilon X^{(1)}(u) + \cdots , \qquad (1)$$

which is also written in a Hamiltonian form, i.e. there exist a skew-symmetric Hamiltonian operator $\mathcal{J}$ and a Hamiltonian $H \in C^{\infty}(M)$ such that $X(u;\varepsilon)$ is the Hamiltonian vector field given by

$$X(u;\varepsilon) = \mathcal{J}(u;\varepsilon)\nabla H[u;\varepsilon] \, . \qquad (2)$$

Here, $\mathcal{J}$ and H are also given in the power series of ε,

$$\left.\begin{aligned} \mathcal{J}(u;\varepsilon) &= \mathcal{J}^{(0)}(u) + \varepsilon\, \mathcal{J}^{(1)}(u) + \cdots , \\ H[u;\varepsilon] &= H^{(0)}[u] + \varepsilon H^{(1)}[u] + \cdots , \end{aligned}\right\} \qquad (3)$$

and, ∇H, the gradient of H is defined in the usual way, i.e. for any vector field Y,

$$(Y\cdot\nabla H)[u] := \lim_{\delta\to 0} \frac{d}{d\delta} H[u+\delta Y] \, . \qquad (4)$$

The poisson bracket generated by $\mathcal{J}$ for functions $F, G \in C^{\infty}(M)$ is given by

$$\{F,G\}[u] := \lim_{\delta\to 0} \frac{d}{d\delta} F[u+\delta\,\mathcal{J}\,\nabla G] = (\nabla F\cdot \mathcal{J}\,\nabla G)[u] \, . \qquad (5)$$

Recall that $\mathcal{J}$ is a Hamiltonian operator if (5) forms a Lie algebra on $C^{\infty}(M)$ (i.e. the skew-symmetric bilinear form $\{\cdot,\cdot\}$ satisfies the Jacobi identity). The unperturbed equation $\dot{u}=X^{(0)} = \mathcal{J}^{(0)}\nabla H^{(0)}$ (i.e. (1) with $\varepsilon=0$) is an integrable Hamiltonian system in the usual sense: $\exists$ a set of $I_{\nu}^{(0)}[u] \in C^{\infty}(M)$ for $\nu \in \Gamma_0$ (where the number of elements in the index set Γ_0 is equal to the degree of freedom of the system), such that

$$\left.\begin{array}{lll} \text{Ia)} & X^{(0)}\cdot\nabla I_\nu^{(0)} = 0 \quad \text{for} & \forall \nu \in \Gamma_0 \,, \\ \text{Ib)} & \{I_\nu^{(0)}, I_\mu^{(0)}\}^{(0)} = \nabla I_\nu^{(0)}\cdot \mathcal{J}^{(0)}\nabla I_\mu^{(0)} = 0 \quad \text{for} & \forall \nu,\mu \in \Gamma_0 \,, \\ \text{Ic)} & \{\nabla I_\nu^{(0)}\}_{\nu\in\Gamma_0} & \text{are independent .} \end{array}\right\} \quad \text{(I)}$$

(In the case of infinite dimensional system, one needs more careful definition. See examples below).

For the perturbations $X^{(\ell)}(u)$, consisting of the polynomials or the differential polynomials in u, we define the degree of $X^{(\ell)}(u)$, say Deg $(X^{(\ell)}(u))$, in the form,

$$\begin{aligned} \mathrm{Deg}(X^{(\ell)}(u)) := & \ (\#(u) \text{ in } X^{(\ell)})\cdot\mathrm{Deg}(u) \\ & + (\#(\partial_x) \text{ in } X^{(\ell)})\cdot\mathrm{Deg}(\partial_x) \ , \end{aligned} \tag{6}$$

where $\#(u)$ and $\#(\partial_x)$ denote the number of u's and the number of derivatives $\partial/\partial x$, respectively. Namely, $\mathrm{Deg}(X^{(\ell)})$ indicates the scales of nonlinearity and smoothness of the vector field. Here $\mathrm{Deg}(u)$ and $\mathrm{Deg}(\partial_x)$ are determined from the self-similar property of the unperturbed equation based on the scaling of the physical setting. In this lecture, we consider the perturbations satisfying,

$$\left.\begin{array}{ll} \text{Xa)} & \text{for each } \ell,\ \mathrm{Deg}(X^{(\ell)}(u)) \text{ is fixed,} \\ \text{Xb)} & \mathrm{Deg}(X^{(\ell+1)}(u)) = \mathrm{Deg}(X^{(\ell)}(u)) + \mathrm{Deg}(u) \ , \\ \text{Xc)} & \#(u) \text{ in } X^{(\ell+1)} \le \#(u) \text{ in } X^{(\ell)}+1 \ . \end{array}\right\} \quad \text{(X)}$$

It should be noted that these conditions (X) (i.e. the ordering of $X^{(\ell)}$'s) are naturally appeared in the method of asymptotic expansions used in the derivation of the equation (1) from the original physical problem [5,6].

3. Examples of the perturbed systems

Here we give three examples of the perturbed equations which will be studied throughout this paper.

3a. N-uncoupled harmonic oscillators [3].

Our first example is a classical example of the weakly coupled nonlinear oscillators on $M=\mathbb{R}^{2N}$. The unperturbed system $X^{(0)}=\mathcal{J}^{(0)}\nabla H^{(0)}$ is the N-uncoupled harmonic oscillators, i.e. in terms of canonical coordinates $(x_1,\cdots,x_N,y_1,\cdots,y_N)=u\in\mathbb{R}^{2N}$,

$$H^{(0)}[u] = \frac{1}{2}\sum_{\nu=1}^{N} \omega_\nu^{(0)}(x_\nu^2+y_\nu^2) \quad . \tag{7}$$

Here the Hamiltonian operator $\mathcal{J}^{(0)}$ is given by 2N×2N antisymmetric matrix

$$\mathcal{J}^{(0)} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} , \quad I = N\times N \text{ identity matrix.} \tag{8}$$

The perturbations are given by

$$X^{(\ell)} = \mathcal{J}^{(0)}\nabla H^{(\ell)} \tag{9}$$

where $H^{(\ell)}$ is a homogeneous polynomial of degree $\ell+2$,

$$H^{(\ell)}[u] = \sum_{|m|+|n|=\ell+2} C_{mn}x^m y^n \ , \text{ with } x^m y^n = \prod_{\nu=1}^{N} x_\nu^{m_\nu} y_\nu^{n_\nu} \ ,$$

$$|m| = m_1+\cdots+m_N \ , \quad m_\nu\in\mathbb{Z}^+ \ . \tag{10}$$

Here $\mathrm{Deg}(X^{(\ell)})$ is given by #(u) in $X^{(\ell)}$ i.e. Deg(u)=1, so that $\mathrm{Deg}(X^{(\ell)})=\ell+1$. (We sometimes say $\mathrm{Deg}(H^{(\ell)}[u])=\ell+2$).

3b. A linear dispersive wave (the linearized KdV equation).

The second example is a dispersive wave equation where the unperturbed equation is the linearized KdV equation, i.e.

$$\dot{u} = X^{(0)}(u) = u_{3x} := u_{xxx} \ . \tag{11}$$

This example may be considered as an infinite dimensional analogue of the previous example. The space M considered here for u is $M=C^{\infty}_{\downarrow}(\mathbb{R})$ defined by

$$C^{\infty}_{\downarrow}(\mathbb{R}) := \{u(x,\cdot)\,|\,u(x) \in C^{\infty}(\mathbb{R}) \quad \text{and,} \quad \text{for any } m,n \in \mathbb{Z}^{+} ,$$
$$|x|^{m}\cdot|\partial_{x}^{n}u| \to 0 \ , \quad \text{as} \quad |x| \to \infty\} \ .$$

The first order perturbation satisfying the conditions (X) with the choice of degrees $\mathrm{Deg}(u)=2$ and $\mathrm{Deg}(\partial_x)=1$ (same as the case of the KdV equation) is given by

$$X^{(1)}(u) = a_1^{(1)} u_{5x} + a_2^{(1)} uu_{3x} + a_3^{(1)} u_x u_{2x} \ , \tag{12}$$

which can be put into a Hamiltonian form,

$$X^{(1)}(u) = \mathcal{J}^{(0)}\nabla H^{(1)} + \mathcal{J}^{(1)}\nabla H^{(0)} \ , \tag{13}$$

where the Hamiltonian structure is given by

$$\left.\begin{aligned} &\mathcal{J}^{(0)} = \partial_x \ , \quad \mathcal{J}^{(1)} = b_1^{(1)}\partial_x^3 + b_2^{(1)}(\partial_x u + u\partial_x) \ , \\ &H^{(0)}[u] = -\frac{1}{2}\int_{-\infty}^{\infty} u_x^2 \, dx \ , \quad H^{(1)}[u] = b_3^{(1)}\int_{-\infty}^{\infty} uu_x^2 \, dx \ . \end{aligned}\right\} \tag{14}$$

Here the sets of constants $\{a_\ell^{(1)}\}_{\ell=1}^{3}$ and $\{b_\ell^{(1)}\}_{\ell=1}^{3}$ are isomorphic, and $b_1^{(1)}=a_1^{(1)}$, $b_2^{(1)}=(2a_2^{(1)}-a_3^{(1)})/3$, $b_3^{(1)}=(a_2^{(1)}-2a_3^{(1)})/6$. Note that the Hamiltonian structure is not unique, and in fact there is another choice given by

$$\left.\begin{aligned} \tilde{\mathcal{J}}^{(0)} &= \partial_x^3 \ , \quad \tilde{\mathcal{J}}^{(1)} = c_1^{(1)}\partial_x^5 + c_2^{(1)}(\partial_x^3 u + u\partial_x^3) \ , \\ \tilde{H}^{(0)}[u] &= \frac{1}{2}\int u^2\, dx \ , \quad \tilde{H}^{(1)}[u] = c_3^{(1)}\int u^3\, dx \ , \end{aligned}\right\} \qquad (14')$$

where $c_1^{(1)}=a_1^{(1)}$, $c_2^{(1)}=(3a_2^{(1)}-a_3^{(1)})/3$, $c_3^{(1)}=(-2a_2^{(1)}+a_3^{(1)})/6$. It should be also noted that the Hamiltonian operator for the infinite dimensional non-canonical systems (e.g. this example) generally depend on the coordinates on M unlike the finite dimensional case where the Darboux theorem holds (i.e. the symplectic structure is locally constant).

3c. The KdV equation.

Let the KdV equation be in the following form,

$$\dot{u} = X^{(0)}(u) = 6uu_x + u_{3x} \ , \qquad \text{for} \quad u \in C_{\downarrow}^{\infty}(R) \ , \qquad (15)$$

where $X^{(0)} = \mathcal{J}^{(0)}\nabla H^{(0)}$, and

$$\mathcal{J}^{(0)} = \partial_x \ , \quad H^{(0)}[u] = \int_{-\infty}^{\infty} (u^3 - \frac{1}{2}u_x^2)\, dx \ . \qquad (16)$$

In this case, the degrees of the perturbations are $\mathrm{Deg}(X^{(\ell)})=2\ell+5$ with the choice of $\mathrm{Deg}(u)=2$, $\mathrm{Deg}(\partial_x)=1$, based on the self-similarity of the KdV equation (i.e. if $u(x,t)$ is a solution of (15), then $v(x,t)=\delta^2 u(\delta x, \delta^3 t)$ is also a solution). Thus, $X^{(1)}(u)$ satisfying $\mathrm{Deg}(X^{(1)})=7$ with the conditions (X) is

$$\begin{aligned} X^{(1)}(u) &= a_1^{(1)}u_{5x} + a_2^{(1)}uu_{3x} + a_3^{(1)}u_x u_{2x} + a_4^{(1)}u^2 u_x \\ &= \mathcal{J}^{(0)}\nabla H^{(1)}[u] + \mathcal{J}^{(1)}(u)\nabla H^{(0)}[u] \ , \end{aligned} \qquad (17)$$

where the Hamiltonian structure is given by

$$\mathcal{J}^{(1)}(u) = b_1^{(1)}\partial_x^3 + b_2^{(1)}(\partial_x u + u\partial_x) \ ,$$

$$H^{(1)}[u] = \int_{-\infty}^{\infty} (b_3^{(1)} u u_x^2 + b_4^{(1)} u^4)\, dx \;. \tag{18}$$

Here the relations between the two sets of constants $\{a_\ell^{(1)}\}_{\ell=1}^4$ and $\{b_\ell^{(1)}\}_{\ell=1}^4$ are given by $b_1^{(1)}=a_1^{(1)}$, $b_2^{(1)}=(6a_1^{(1)}+2a_2^{(1)}-a_3^{(1)})/3$, $b_3^{(1)}=(30a_1^{(1)}+a_2^{(1)}-2a_3^{(1)})/6$, $b_4^{(1)}=(-30a_1^{(1)}-10a_2^{(1)}+5a_3^{(1)}+a_4^{(1)})/12$.

4. Integrals of the perturbed equations

The existence of the integrals for given equation is a key to its integrability. In this section, we look for the integrals for the perturbed equation (1) in the following formal power series,

$$I_\nu[u;\varepsilon] = I_\nu^{(0)}[u] + \varepsilon I_\nu^{(1)}[u] + \cdots \;, \text{ for } \underline{\text{some}}\ \nu \in \Gamma_0 \;, \tag{19}$$

where $\{I_\nu^{(0)}[u]\}_{\nu\in\Gamma_0}$ are the integrals for the unperturbed equation satisfying (I). Let us define Γ_n, a subset of the index set Γ_0; If for each $\nu \in \Gamma_n \subset \Gamma_0$ there exist $I_\nu^{(\ell)}[u]$ in (19), for $1\le\ell\le n$, satisfying $X\cdot\nabla I_\nu = O(\varepsilon^{n+1})$, or equivalently,

$$\sum_{m=0}^{\ell} X^{(\ell-m)}\cdot\nabla I_\nu^{(m)} = 0 \;, \quad \text{for} \quad \nu\in\Gamma_n \quad \text{and} \quad 1 \le \ell \le n \;, \tag{20}$$

(i.e. $I_\nu[u;\varepsilon]$ for $\nu\in\Gamma_n$ is the integrals of (1) up to order ε^n). Note that $\Gamma_0 \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_n \supseteq \cdots \supseteq \Gamma_\infty$, and if $\Gamma_n=\Gamma_0$, then the perturbed system (1) is integrable up to ε^n. We call the system "nearly integrable". The main purpose in this section is to find Γ_n (i.e. find $I_\nu^{(\ell)}[u]$ by solving (20)). With this purpose, we study three examples presented in the previous section.

4a. <u>N-uncoupled harmonic oscillators</u>.

The equation for $I_\nu^{(\ell)}[u]$ in this case can be expressed by

$$L_{X^{(0)}} I_\nu^{(\ell)} := -X^{(0)}\cdot\nabla I_\nu^{(\ell)} = \sum_{m=0}^{\ell-1} X^{(\ell-m)}\cdot\nabla I_\nu^{(m)} , \tag{21}$$

where $L_{X^{(0)}}$ is the Lie derivative with respect to $X^{(0)} = \mathcal{J}^{(0)}\nabla H^{(0)}$, and given by

$$L_{X^{(0)}} = \sum_{\ell=1}^{N} \omega_\ell^{(0)} \left(x_\ell \frac{\partial}{\partial y_\ell} - y_\ell \frac{\partial}{\partial x_\ell}\right) . \tag{22}$$

In terms of the action-angle variables defined by $\rho_\ell = (x_\ell^2 + y_\ell^2)/2$, $\theta_\ell = \arctan\,(x_\ell/y_\ell)$, $\ell = 1,\cdots,N$, eq.(21) can be written in

$$L_{X^{(0)}} I_\nu^{(\ell)} = \sum_{\ell=1}^{N} \omega_\ell^{(0)} \frac{\partial}{\partial\theta_\ell} I_\nu^{(\ell)} = G_\nu^{(\ell)} , \tag{23}$$

where $G_\nu^{(\ell)}$ is of the right hand side of (21). For the case $\ell=1$, from $X^{(1)} = \mathcal{J}^{(0)}\nabla H^{(1)}$, and choosing $I_\nu^{(0)} = \rho_\nu$, we have

$$L_{X^{(0)}} I^{(1)} = -\frac{\partial}{\partial\theta_\nu} H^{(1)} . \tag{24}$$

Here $H^{(1)}$ in terms of (ρ,θ) is given by

$$H^{(1)}[u] = \sum_{|r|+|s|=3} C_{rs} x^r y^s = \sum_{|r|+|s|=3} \hat{C}_{rs}(\rho) e^{i(r-s)\cdot\theta} , \tag{25}$$

where $\hat{C}_{rs} = \hat{C}^*_{sr}$ (complex conjugate), and $(r-s)\cdot\theta = \sum_{\ell=1}^{N} (r_\ell - s_\ell)\theta_\ell$. Thus, if $(r-s)\cdot\omega^{(0)} = \sum_{\ell=1}^{N} (r_\ell - s_\ell)\omega_\ell^{(0)} \neq 0$ for $|r|+|s|=3$ (<u>non-resonant</u> case), the particular solution of $I_\nu^{(1)}$ can be obtained by

$$I_\nu^{(1)}[u] = \sum_{|r|+|s|=3} \hat{C}_{rs}(\rho) \frac{r_\nu - s_\nu}{(r-s)\cdot\omega^{(0)}} e^{i(r-s)\cdot\theta} . \tag{26}$$

Moreover, one can prove that if $\{\omega_\ell^{(0)}\}_{\ell=1}^{N}$ are <u>rationally independent</u> (i.e. for $r \in \mathbb{Z}^N$, if $r\cdot\omega^{(0)} = 0$ then $r=0$), then the solution $I_\nu^{(\ell)}$ of (23) exists for any $\ell > 1$ and $\nu \in \Gamma_0$ (i.e. $\Gamma_\infty = \Gamma_0$). (In the resonant case, $\{I_\nu[u;\varepsilon]\}_{\nu=1}^{N}$ do not exist in general). This is a consequence of the Birkhoff theorem for the normal form expansion of (1) (see [3] and the following sections).

4b. The linearized KdV equation.

The equation for $I_\nu^{(\ell)}$ in this case is given by the following form similar to (23),

$$L_{X^{(0)}} I_\nu^{(\ell)} = \int_0^\infty dk\ \omega_k^{(0)} \frac{\partial}{\partial\theta_k} I_\nu^{(\ell)} = G_\nu^{(\ell)} , \tag{27}$$

where $\omega_k^{(0)}=k^3$ (the linear dispersion relation), and the action-angle variables (ρ,θ) may be defined by

$$\rho_k = \frac{|\hat{u}(k)|^2}{k} , \ \theta_k = \arg \hat{u}(k) , \quad \text{for} \quad k \geq 0 . \tag{28}$$

Here $\hat{u}(k)$ is the Fourier component of $u(x)$,

$$\left.\begin{aligned} \hat{u}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) e^{-ikx}\, dx , \qquad \text{for} \quad k \geq 0 , \\ u(x) &= \frac{\overline{2}}{\pi} \int_0^\infty \sqrt{k\rho(k)} \cos(kx+\theta_k)\, dk . \end{aligned}\right\} \tag{29}$$

Let $I_\nu^{(0)}[u]$ be the integrals for (11) given by

$$I_{\nu+1}^{(0)}[u] = \frac{(-1)^\nu}{2} \int_{-\infty}^{\infty} u_{\nu x}^2\, dx , \qquad \text{for} \quad \nu=0,1,\cdots . \tag{30}$$

Then $G_\nu^{(1)}(\rho,\theta)$ in (27) can be written by

$$G_\nu^{(1)}(\rho,\theta) = \int_0^\infty\!\!\int_0^\infty \Phi_\nu^{(1)}(k,\rho) \sin(\theta_{k_1}+\theta_{k_2}-\theta_{k_1+k_2})\, dk_1 dk_2 , \tag{31}$$

where $\Phi_\nu^{(1)}(k,\rho)=\Phi_\nu^{(1)}(k_1,k_2,\rho_{k_1},\rho_{k_2})$ does not depend on θ, and $|\Phi_\nu^{(1)}(k,\rho)| \propto k^{2\nu+1}$ for small k. (See (33) below for the explicit form of $G^{(1)}$ in terms of u). By virtue of the fact that $\omega_{k_1}^{(0)} + \omega_{k_2}^{(0)} \neq \omega_{k_1+k_2}^{(0)}$ for $k_1,k_2>0$ (non-resonant condition), one can find a solution of (31) for $\ell=1$ in the similar form of (26),

$$I^{(1)}[u] = -\int_0^\infty\!\!\int_0^\infty \frac{\Phi_\nu^{(1)}(k,\rho)}{\omega_{k_1}^{(0)}+\omega_{k_2}^{(0)}-\omega_{k_1+k_2}^{(0)}} \cos(\theta_{k_1}+\theta_{k_2}-\theta_{k_1+k_2})\, dk_1 dk_2 . \tag{32}$$

Thus, if there is no-resonance, one can find $I_\nu^{(\ell)}$. However, the higher order equations ($\ell \geq 2$) generally contain the resonant terms (e.g. $\omega_{k_1}^{(0)}+\omega_{k_2}^{(0)}-\omega_{k_3}^{(0)}-\omega_{k_1+k_2-k_3}^{(0)}$), and in these cases, we need to study the singular integral equations having the poles at the resonant points, and give the analytical estimates for the existence of the solution $I_\nu^{(\ell)}$.

Fortunately, there is another way to find the solution for (27) based on an algebra of the functionals consisting of the differential polynomials (FDP), instead of the analytical way discussed above. This algebraic method is particularly useful for the case of the KdV equation where the higher order equation does <u>not</u> have simple forms like those in the previous case (e.g. (31)) [8]. Noticing that the linear part of $X^{(1)}$, i.e. u_{5x}, is perpendicular to $\nabla I_\nu^{(0)}$ (this implies #(u) in $(X^{(1)}\cdot\nabla I_\nu^{(0)})=3$), eq.(27) for $\ell=1$ can be written in

$$(X^{(0)}\cdot\nabla I_\nu^{(1)})[u] = \sum_{\substack{\ell_1+2\ell_2=2\nu+1 \\ 1\leq\ell_1\leq\ell_2}} a_{\ell_1\ell_2}^{(1)} \int_{-\infty}^{\infty} u_{\ell_1 x} u_{\ell_2 x}^2 \, dx \,, \tag{33}$$

where the constants $a_{\ell_1\ell_2}^{(1)}$ are determined from $\{a_\ell^{(1)}\}_{\ell=1}^3$ in $X^{(1)}$. From the relation $\mathrm{Deg}(I_\nu^{(1)}[u])=2\nu+4$, one can find the solution $I_\nu^{(1)}$ in the following form,

$$I_\nu^{(1)}[u] = \sum b_{\ell_1\ell_2}^{(1)} \int_{-\infty}^{\infty} u_{(\ell_1-1)x} u_{(\ell_2-1)x}^2 \, dx + c_\nu^{(1)} I_{\nu+1}^{(0)}[u] \,, \tag{34}$$

where the sum is taken over the same way as (33), $c_\nu^{(1)}$ is an arbitrary constant, and $b_{\ell_1\ell_2}^{(1)}$ are determined uniquely from $a_{\ell_1\ell_2}^{(1)}$ by comparing the coefficients of the independent FDPs, i.e. $\int u_{\ell_1 x} u_{\ell_2 x}^2 \, dx$ for each ℓ_1, ℓ_2, with $\ell_1+2\ell_2=2\nu+1$ on both sides of (33). The explicit forms of $I_\nu^{(1)}$ for $\nu=1,2$ are given by

$$\left.\begin{aligned} I_1^{(1)}[u] &= b_{11}^{(1)} \int u^3 \, dx + c_1^{(1)} I_2^{(0)}[u] \ , \\ I_2^{(1)}[u] &= b_{12}^{(1)} \int u u_x^2 \, dx + c_2^{(1)} I_3^{(0)}[u] \ , \end{aligned}\right\} \tag{35}$$

where $b_{11}^{(1)}=(-2a_2^{(1)}+a_3^{(1)})/6$, $b_{12}^{(1)}=(-a_2^{(1)}+2a_3^{(1)})/6$. It is important to note that the first order vector field $X^{(1)}$ can be determined conversely by giving the set of integrals (35), except the linear term $a_1^{(1)}u_{5x}$ which can be determined by fixing the linear dispersion relation i.e. $\omega_k=k^3-\varepsilon a_1^{(1)}k^5$. Namely, the perturbed equation up to order ε can be characterized by two integrals (35) and the linear dispersion relation. This fact will be important in the following sections.

One can also construct the next order solution $I_\nu^{(2)}$ easily by using the algebraic method. Eq.(27) for $\ell=2$ can be expressed by

$$\begin{aligned} X^{(0)}\cdot\nabla I_\nu^{(2)} = &\sum_{\substack{\ell_1+\ell_2+2\ell_3=2\nu+1 \\ 0\le\ell_1<\ell_2\le\ell_3}} a_{\ell_1\ell_2\ell_3}^{(2)} \int u_{\ell_1 x}u_{\ell_2 x}u_{\ell_3 x}^2 \, dx \\ &+ \sum_{\substack{m_1+2m_2=2\nu+3 \\ 1\le m_1\le m_2}} a_{m_1 m_2}^{(2)} \int u_{m_1 x}u_{m_2 x}^2 \, dx \ . \end{aligned} \tag{36}$$

It is easy to see that the solution of (36) is given by

$$\begin{aligned} I_\nu^{(2)}[u] = &\sum b_{\ell_1\ell_2\ell_3}^{(2)} \int u_{\ell_1 x}u_{(\ell_2-1)x}u_{(\ell_3-1)x}^2 \, dx \\ &+ \sum b_{m_1 m_2}^{(2)} \int u_{(m_1-1)x}u_{(m_2-1)x}^2 \, dx \\ &+ c_\nu^{(2)} I_{\nu+2}^{(0)}[u] \ , \end{aligned} \tag{37}$$

where $c_\nu^{(2)}$ are arbitrary constants, and the constants $b_{\ell_1\ell_2\ell_3}^{(2)}$, $b_{m_1m_2}^{(2)}$ are determined by the same way as before. Thus, even the case $\ell=2$ which has the resonant terms, this method can give the solution explicitly in terms of the FDPs. However, at the next order $\ell=3$, the solution may not be obtained directly from the form of the FDP on the right hand side of (27), because of the increasing nonlinearity, and the method should be modified.

4c. The KdV equation [7].

The set of integrals $\{I_\nu^{(0)}[u]\}_{\nu=0}^{\infty}$ of the KdV equation is given by the following recurrent formula [2],

$$\mathcal{J}_0^{(0)}\nabla I_{\nu+1}^{(0)} = \mathcal{J}_1^{(0)}\nabla I_\nu^{(0)} \quad \text{with } I_0^{(0)}[u] = \frac{1}{2}\int_{-\infty}^{\infty} u\, dx \,, \tag{38}$$

where the Hamiltonian operators $\mathcal{J}_0^{(0)}$ and $\mathcal{J}_1^{(0)}$ are defined by

$$\mathcal{J}_0^{(0)} = \mathcal{J}^{(0)} = \partial_x \,, \quad \mathcal{J}_1^{(0)} := \mathcal{J}_0^{(1)} = \partial_x^3 + 2(\partial_x u + u\partial_x) \,. \tag{39}$$

The first three integrals are as follows:

$$\left.\begin{aligned} I_1^{(0)}[u] &= \frac{1}{2}\int u^2\, dx \,, \\ I_2^{(0)}[u] &= H^{(0)}[u] = \int (u^3 - \frac{1}{2}u_x^2)\, dx \,, \\ I_3^{(0)}[u] &= \int (\frac{5}{2}u^4 - 5uu_x^2 + \frac{1}{2}u_{2x}^2)\, dx \,. \end{aligned}\right\} \tag{40}$$

In this case, the equation for $I_\nu^{(1)}$ has the following form,

$$X^{(0)}\cdot\nabla I_\nu^{(1)} = \sum a^{(1)}_{\ell_0\cdots\ell_N}\int (u^{\ell_0}u_x^{\ell_1}\cdots u_{Nx}^{\ell_N})\, dx \,, \tag{41}$$

where the sum is taken over $(\ell_0,\cdots,\ell_N)$ with the constraints $2\ell_0+3\ell_1+\cdots+(N+2)\ell_N=2\nu+7$, and $\ell_N\geq 2$, and the constants $a^{(1)}_{\ell_0\cdots\ell_N}$ are determined from $\{a_\ell^{(1)}\}_{\ell=1}^{4}$ in $X^{(1)}$. The general solution of (41) may not be obtained directly. However, for the first three cases $(\nu=1,2,3)$ where the degree of nonlinearity $(\ell_0+\ell_1+\cdots+\ell_n\leq\nu+2)$ is 5 or less, the solutions can be found explicitly by the same way as the previous case. Namely,

$$\left.\begin{aligned}(X^{(0)}\cdot\nabla I_1^{(1)})[u] &= a_{03}^{(1)}\int u_x^3\,dx\ ,\\ (X^{(0)}\cdot\nabla I_2^{(1)})[u] &= \int (a_{13}^{(1)}uu_x^3 + a_{012}^{(1)}u_xu_{2x}^2)\,dx\ ,\\ (X^{(0)}\cdot\nabla I_3^{(1)})[u] &= \int (a_{23}^{(1)}u^2u_x^3 + a_{112}^{(1)}uu_xu_{2x}^2 + a_{0102}^{(1)}u_xu_{3x}^2)\,dx\ ,\end{aligned}\right\}\quad (42)$$

which lead to the solutions,

$$\left.\begin{aligned}I_1^{(1)}[u] &= b_{01}^{(1)}\int u^3dx + c_1^{(1)}I_2^{(0)}[u]\ ,\\ I_2^{(1)}[u] &= H^{(1)}[u] + c_2^{(1)}I_3^{(0)}[u]\ ,\\ I_3^{(1)}[u] &= \int (b_5^{(1)}uu_{2x}^2 + b_6^{(1)}u^2u_x^2 + b_7^{(1)}u^5)\,dx + c_3^{(1)}I_4^{(0)}[u]\ ,\end{aligned}\right\}\quad (43)$$

where $H^{(1)}[u]$ is the higher order Hamiltonian given in (18). While in the cases $\nu\geq 4$, it is not obvious that the forms $I_\nu^{(1)}$ assumed in the similar manner as above satisfy (41), since the possible number of dimension for $X^{(1)}\cdot\nabla I_\nu^{(0)}\geq$the maximum dimension of $I_\nu^{(1)}$, (the dimension of FDP with fixed degree is defined by the number of independent FDPs there). However, one can actually show that the solutions $I_\nu^{(1)}$ exist for all $\nu\in\Gamma_0$ $(=\{0,1,2,\cdots\infty\})$, i.e. $\Gamma_1=\Gamma_0$. We will discuss this in the last section where we construct a transformation between the perturbed equation (1) with (17) and an integrable system, from the fact that the perturbed equation has three integrals (43). While at the next order, one can show that there exist $I_\nu^{(2)}$ for $\nu=1,2$, but not for $\nu=3$, in general. This fact will be important for the integrability of the second order equation.

5. Normal forms

If the perturbed equation has several integrals, then by changing the coordinates on M, one can expect to transform the equation into a simpler equation having the same number of integrals. The transformed equation may help us to find more integrals of the perturbed equation (which we could not find easily), because of its (simple) form (it is sometimes linear or integrable). We call the resulting simple equation "a normal form" of the perturbed equation. In the next section, we will construct such transformations.

The normal forms may be defined as follows: Let $\dot{v}=X_0^{(0)}(v) = \mathcal{J}_0^{(0)}\nabla H_0^{(0)}[v]$ be an integrable Hamiltonian system in the sense of (I), and $X_0^{(\ell)}(v)$, $\ell\geq 1$, be the vector fields satisfying the Conditions (X). A perturbed Hamiltonian equation

$$\dot{v} = \sum_{\ell=0}^{n} \varepsilon^{\ell} X_0^{(\ell)}(v) + O(\varepsilon^{n+1})$$

$$= X_0(v;\varepsilon) = \mathcal{J}_0(v;\varepsilon)\nabla H_0[v;\varepsilon] \quad , \tag{44}$$

where $\mathcal{J}_0 = \mathcal{J}_0^{(0)} + \varepsilon\,\mathcal{J}_0^{(1)} + \cdots$, $H_0 = H_0^{(0)} + \varepsilon H_0^{(1)} + \cdots$, is said to be a normal form on $\{I_\nu^{(0)}[v]=\text{const.}\}_{\nu\in\Gamma_n}$ of the perturbed equation (1) (sometimes called a normal form on Γ_n), if the vector fields $X_0^{(\ell)}(v)$ satisfy

$$X_0^{(\ell)}\cdot\nabla I_\nu^{(0)} = 0 \ , \ \text{for } 1 \leq \ell \leq n \text{ and } \nu\in\Gamma_n \ . \tag{45}$$

(i.e. $\{I_\nu^{(0)}[v]\}_{\nu\in\Gamma_n}$ are integrals of (44) up to order ε^n). Here, Γ_n is determined from the integrals of the perturbed equation (1) by (20), i.e. $\Gamma_n=\{\nu\in\Gamma_0 \mid \exists\, I_\nu[u;\varepsilon] \text{ such that } X\cdot\nabla I_\nu = O(\varepsilon^{n+1})\}$. Namely, the normal form on Γ_n is a perturbed equation in which $X_0^{(\ell)}$, $1\leq\ell\leq n$, are on the tangent space of the integral surface given by $I_\nu^{(0)}=\text{const.}$ for $\nu\in\Gamma_n$. Note that if $\Gamma_n=\Gamma_0$ then $X_0^{(\ell)}$, $1\leq\ell\leq n$, are the Hamiltonian vector fields given by the integrals $I_\nu^{(0)}$, and (44) is integrable up to ε^n. In the rest of this section, we give the normal forms for the examples in the previous section.

5a. N-uncoupled harmonic oscillators.

As an example of the normal forms of (1) with (9), we consider the classical one which is defined on the constant energy surface, i.e.

$$X_0 \cdot \nabla H^{(0)} = 0 \ . \qquad (46)$$

Here if $X_0 = \mathcal{J}^{(0)} \nabla K_0$ with $\mathcal{J}^{(0)}$ defined in (8) and a Hamiltonian K_0, then (46) can be written in

$$L_{X_0^{(0)}} K_0 = \sum_{\mu=1}^{N} \omega_\mu^{(0)} \frac{\partial}{\partial \theta_\mu} K_0 = 0 \ . \qquad (47)$$

In the case where $\{\omega_\mu^{(0)}\}_{\mu=1}^{N}$ are rationally independent (non-resonant case) the solution K_0 can be expressed as a function of the action variables only, i.e. $K_0 = K_0[\rho_1, \cdots, \rho_N]$, and the normal form is an integrable system given by

$$\left.\begin{aligned} \dot{\theta}_\mu &= \frac{\partial K_0}{\partial \rho_\mu} = \omega_\mu^{(0)} + \varepsilon \omega_\mu^{(1)}(\rho) + \cdots \ , \\ \dot{\rho}_\mu &= - \frac{\partial K_0}{\partial \theta_\mu} = 0 \ , \end{aligned}\right\} \qquad (48)$$

where $\omega_\mu^{(\ell)} = \partial K_0^{(\ell)} / \partial \rho_\mu$ with the formal series $K_0 = K_0^{(0)} + \varepsilon K_0^{(1)} + \cdots$. In the resonant case, i.e. there exists a non-empty set of integers $\Delta = \{ s \in \mathbb{Z}^N \,|\, \omega^{(0)} \cdot s = \sum_{\mu=1}^{N} \omega_\mu^{(0)} s_\mu = 0 \}$, the Hamiltonian K_0 satisfying (47) is given by

$$K_0 = \sum_{\ell - m \in \Delta} \tilde{C}_{\ell m} z^\ell \bar{z}^m \ , \qquad z^\ell \bar{z}^m = \prod_{\mu=1}^{N} z_\mu^{\ell_\mu} \bar{z}_\mu^{m_\mu} \ , \qquad (49)$$

where $z_\mu = x_\mu + i y_\mu$, $\bar{z}_\mu = x_\mu - i y_\mu$. The normal form in this case is <u>not</u> integrable in general (see [3]).

5b. The linearized KdV equation.

It is easy to see that a linear dispersive wave equation given by

$$\dot{v} = v_{3x} + \sum_{n\geq 1} \varepsilon^n a_1^{(n)} v_{(2n+3)x} , \tag{50}$$

where $a_1^{(n)}$'s are constants, is the integrable normal form on Γ_0 (i.e. (50) has the same set of integrals $\{I_\nu^{(0)}[u]\}_{\nu=1}^{\infty}$ in (30) of (11)). Let us see the normal forms on the constant surfaces given by the finite sets of integrals $\{I_\nu^{(0)}[u]=\text{const.}\}_{\nu\in\Gamma_1}$ for $\Gamma_1=\{1\}$ and $\Gamma_1=\{1,2\}$. The normal form on $\Gamma_1=\{1\}$ is given by the vector field (12) satisfying

$$(X_0^{(1)} \cdot \nabla I_1^{(0)})[v] = \int_{-\infty}^{\infty} X_0^{(1)}(v) v \, dx = 0 , \tag{51}$$

which implies that $a_3^{(1)}=2a_2^{(1)}$ and $a_1^{(1)}$, $a_2^{(1)}$ are arbitrary, i.e.

$$X_0^{(1)}(v) = a_1^{(1)} v_{5x} + a_2^{(1)}(vv_{3x}+2v_x v_{2x}) . \tag{52}$$

Similarly, the normal form on $\Gamma_1=\{1,2\}$ is given by the conditions (51) and

$$(X_0^{(1)} \cdot \nabla I_2^{(0)})[v] = \int_{-\infty}^{\infty} X_0^{(1)}(v) v_{2x} \, dx = 0 , \tag{53}$$

which lead to $a_2^{(1)}=a_3^{(1)}=0$, and $a_1^{(1)}$ is arbitrary. Consequently, the normal form on the constant surface determined by the two constants $I_\nu^{(0)}[v]$, $\nu=1,2$, is nothing but the linear equation (50) up to order ε, i.e.

$$X_0^{(1)}(v) = a_1^{(1)} v_{5x} = \partial_0^{(0)} \nabla H_0^{(1)}[v] + \partial_0^{(1)} \nabla H_0^{(0)}[v] , \tag{54}$$

where the Hamiltonian structure is given by

$$\mathcal{J}_0^{(0)} = \partial_x \,, \qquad \mathcal{J}_0^{(1)} = b_1^{(1)}\partial_x^3 \,, \qquad\qquad (55)$$
$$H_0^{(0)}[v] = I_2^{(0)}[v] \,, \quad H_0^{(1)}[v] = c_2^{(1)} I_3^{(0)}[v] \,,$$

with $I_\nu^{(0)}[v]$ in (30), and $b_1^{(1)}+c_2^{(1)}=a_1^{(1)}$ (i.e. (51) and (53) give $b_2^{(1)}=b_3^{(1)}=0$ in (14)). Thus, the normal form given by (54) is actually the normal form on Γ_0, i.e. $X_0^{(1)}\cdot\nabla I_\nu^{(0)}=0$ for any $\nu\in\Gamma_0$.

5c. The KdV equation.

Since the results for the KdV equation are similar to those of the previous example 4b, we just state the results. The normal form on the constant surface given by $I_\nu^{(0)}[v]$=const. for ν=1,2 (defined in (40)) is

$$X_0^{(1)}(v) = a_1^{(1)}(v_{5x} + 10vv_{3x} + 20v_xv_{2x} + 30v^2v_x)$$
$$= \mathcal{J}_0^{(0)}\nabla H_0^{(1)}[v] + \mathcal{J}_0^{(1)}(v)\nabla H_0^{(0)}[v] \,, \qquad (56)$$

where the Hamiltonian structure is given by

$$\mathcal{J}_0^{(0)} = \partial_x \,, \qquad \mathcal{J}_0^{(1)}(v) = b_1^{(1)}(\partial_x^3+2(\partial_x v+v\partial_x)) \,, \qquad\qquad (57)$$
$$H_0^{(0)}[v] = I_2^{(0)}[v] \,, \qquad H_0^{(1)}[v] = c_2^{(1)} I_3^{(0)}[v] \,,$$

with $I_\nu^{(0)}[v]$ in (40) and $b_1^{(1)}+c_2^{(1)}=a_1^{(1)}$ (arbitrary). Note that the normal form given by (56) is also an <u>integrable</u> system (known as the Lax hierarchy of the KdV equation, and its set of integrals is the same as that of the KdV equation, i.e. $\Gamma_1=\Gamma_0$, [2]). Also, it can be shown easily that at the next order, the normal form is an integrable equation (in the Lax hierarchy), <u>if</u> there exist $I_\nu^{(2)}$ for ν=1,2,3. (i.e. at this order, we need one more integral $I_3^{(2)}$ to have the integrable equation in the Lax type as the normal form).

Thus, the second order equation may <u>not</u> be integrable, since $I_3^{(2)}$ does not exist in general (see the comment at the end of the previous section).

6. Transformations

As we have seen in the previous section, it is not practical to find the integrals $I_\nu[u;\varepsilon]$ of the perturbed equation for <u>all</u> $\nu \in \Gamma_0$. However, we have noticed in the cases 4b and 4c that the perturbed equation and the corresponding normal form are characterized by the <u>few</u> integrals (<u>not</u> all of them). So that if there is a transformation between the perturbed equation and the normal form, it may be constructed from those integrals only, and if the normal form is integrable, so is the perturbed equation.

We consider such transformation in the following power series,

$$u = \phi(v;\varepsilon) = v + \varepsilon\phi^{(1)}(v) + \cdots , \tag{58}$$

where u and v are the solutions of the perturbed equation (1) and its normal form (44), respectively. Then the equations for $\phi^{(\ell)}$'s are given by

$$\sum_{m=0}^{\ell} \{X_0^{(\ell-m)}\cdot\nabla\phi^{(m)} - \lim_{\delta\to 0}\frac{1}{m!}\frac{d^m}{d\delta^m}X^{(\ell-m)}(v+\delta\phi^{(1)}+\cdots)\} = 0$$

$$\text{for} \quad 1 \le \ell \le n , \tag{59}$$

where we have assumed $\phi^{(0)}(v)=v$. For $\ell=1$, we have the following equation for $\phi^{(1)}$,

$$[X^{(0)},\phi^{(1)}] := X^{(0)}\cdot\nabla\phi^{(1)} - \phi^{(1)}\cdot\nabla X^{(0)}$$

$$= X^{(1)} - X_0^{(1)} , \tag{60}$$

where [·,·] is the Lie bracket. Under the transformation ϕ, the integrals $I_\nu[u;\varepsilon]$ of the perturbed equation (1) are transformed into those of the normal form, i.e.

$$I_\nu[u;\varepsilon] = (\phi^* I_\nu)[v;\varepsilon] = I_\nu^{(0)}[v] + \sum_{\ell=1}^{n} \varepsilon^\ell J_\nu^{(\ell)}[v] + O(\varepsilon^{n+1}) ,$$

$$\text{for} \quad \nu \in \Gamma_n , \tag{61}$$

where $J_\nu^{(\ell)}[v]$ are the integrals of the normal form i.e. $X_0^{(0)} \cdot \nabla J_\nu^{(\ell)} = 0$, and ϕ^* is the pull-back map. Especially, for the Hamiltonians H and H_0, we have

$$H[u,\varepsilon] = (\phi^* H)[v;\varepsilon] = H_0[v;\varepsilon] . \tag{62}$$

For each order of ε, (61) can be written in the form

$$\lim_{\delta\to 0} \sum_{\ell=0}^{m} \frac{1}{\ell!} \frac{d^\ell}{d\delta^\ell} I_\nu^{(m-\ell)}[v+\varepsilon\phi^{(1)}(v)+\cdots] = J_\nu^{(m)}[v] ,$$

$$\text{for} \quad 1 \le m \le n , \ \nu \in \Gamma_n . \tag{63}$$

Thus, the equation for $\phi^{(1)}$ in terms of the integrals (instead of the vector fields as (60)) is

$$\phi^{(1)} \cdot \nabla I_\nu^{(0)} + I_\nu^{(1)} = J_\nu^{(1)} . \tag{64}$$

Note that (63) can be derived directly from (59). In order to find $\phi^{(1)}$, we rather use (64) than (60) which is more difficult to solve (because the dimension of differential polynomials (DP) is much greater than that of functional DP, and for (64) we deal only few numbers of ν's (e.g. ν=1,2 for the examples 2b, 2c)).

In connection with Hamiltonian formalism, let us discuss how the Hamiltonian structure changes under the transformation. We note that by the transformation ϕ in (58), the gradient of a functional, $\nabla_u K[u]$, becomes, for any vector field Y,

$$Y\cdot\nabla_u K[u] = \lim_{\delta\to 0}\frac{d}{d\delta}K[u+\delta Y]$$

$$= Y\cdot\nabla_v K_0[v] - \varepsilon(Y\cdot\nabla_v\phi^{(1)}(v))\cdot\nabla_v K_0[v] + O(\varepsilon^2)\ , \qquad (65)$$

where $K_0[v]=(\phi^*K)[v]=K[u]$. From (1) and (44) in the Hamiltonian forms, and using (65), we obtain the relation between the Hamiltonian operators $\mathcal{J}$ in (1) and $\mathcal{J}_0$ in (44) at order ε,

$$(\mathcal{J}^{(1)}(v)-\mathcal{J}_0^{(1)}(v))\nabla K_0[v]$$

$$= (\mathcal{J}_0^{(0)}\nabla K_0[v])\cdot\nabla\phi^{(1)}(v) + \mathcal{J}^{(0)}(\quad\cdot\nabla\phi^{(1)}(v))\cdot\nabla K_0[v]\ , \quad (66)$$

where $(\quad\cdot\nabla\phi^{(1)})\cdot\nabla K_0$ is defined by for any vector field Y,

$$(Y\cdot\nabla\phi(v))\cdot\nabla K_0[v] = \lim_{\delta\to 0}\frac{d}{d\delta}K_0[v+\delta Y\cdot\nabla\phi(v)]\ . \qquad (67)$$

In the sense (66) (i.e. $\phi^{(1)}$ connects the Hamiltonian structures for the systems (1) and (44)), the transformation ϕ may be considered as "a canonical transformation". It is interesting to note that in finite dimensional case, the canonical transformation $\phi^{(1)}$ which may be given by $\phi^{(1)}=\mathcal{J}^{(0)}\nabla S^{(1)}$ with $\mathcal{J}^{(0)}$ in (8) and a function $S^{(1)}$ (so-called the generator of the ransformation) does not change the Hamiltonian structure (i.e. $\mathcal{J}^{(1)}=\mathcal{J}_0^{(1)}$). On the other hand, in infinite dimensional case, $\phi^{(1)}$ generally changes the Hamiltonian structure, and the generator for $\phi^{(1)}$ may not exist (except for the auto-canonical transformations, (see [9,10])).

We now construct $\phi^{(1)}$ for the examples given in the previous sections. Here we consider only the first order problem, but the higher order problems can be discussed in the same way.

6a. N-uncoupled harmonic oscillators.

Here we recover the classical result [3] in which the perturbed equation,

$$\dot{u} = \mathcal{J}^{(0)}\nabla(H^{(0)}[u] + \varepsilon H^{(1)}[u] + \cdots) , \tag{68}$$

with $\mathcal{J}^{(0)}$ in (8) and $H^{(\ell)}$ in (10), can be canonical transformed into the normal form

$$\dot{v} = \mathcal{J}_0^{(0)}\nabla(H_0^{(0)}[v] + \varepsilon H_0^{(1)}[v] + \cdots) , \tag{69}$$

where $\mathcal{J}_0^{(0)}=\mathcal{J}^{(0)}$ and $\{H_0^{(0)},H_0^{(\ell)}\}^{(0)}=\nabla H_0^{(0)}\cdot\mathcal{J}^{(0)}\nabla H_0^{(\ell)}=0$.

In the equation (62) at order ε, i.e.

$$\phi^{(1)}\cdot\nabla H^{(0)} + H^{(1)} = H_0^{(1)} , \tag{70}$$

we look for the solution $\phi^{(1)}$ in the form given by the generator $S^{(1)}$,

$$\phi^{(1)} = \mathcal{J}^{(0)}\nabla S^{(1)} . \tag{71}$$

Then, (70) can be written by

$$L_{X^{(0)}}S^{(1)} = \sum_{\mu=1}^{N}\omega_\mu \frac{\partial}{\partial\theta_\mu} S^{(1)} = H^{(1)} - H_0^{(1)} . \tag{72}$$

The solution $S^{(1)}$ can be found by choosing $H_0^{(1)}$ as the part of the kernel of $L_{X^{(0)}}$ in $H^{(1)}$, i.e.

$$H_0^{(1)} = (\ker L_{X^{(0)}}) \cap H^{(1)} , \tag{73}$$

which is nothing but the <u>resonant</u> term in $H^{(1)}$.

The higher order problems which is given by

$$L_{X^{(0)}}S^{(\ell)} = P^{(\ell)} - H_0^{(\ell)} , \tag{74}$$

with $P^{(\ell)}=P^{(\ell)}[H_0^{(0)},\cdot\cdot,H_0^{(\ell-1)},S^{(1)},\cdot\cdot,S^{(\ell-1)}]$ (Deg $P^{(\ell)}$=Deg $H^{(\ell)}$ $=\ell+2$), can be solved in the same way. Namely, choose $H_0^{(\ell)}$ as

$$H_0^{(\ell)} = (\ker L_{X^{(0)}}) \cap P^{(\ell)} , \tag{75}$$

(Note that the decomposition of $P^{(\ell)}$ into two parts, ker $L_{x^{(0)}}$ and im $L_{x^{(0)}}$ (the image of $L_{x^{(0)}}$) is unique). This is the result of the Binkhoff normal form theorem (see the section 3 and ref.[3]).

6b. The linearized KdV equation.

For the perturbed equation.

$$\dot{u} = \mathcal{J}^{(0)}\nabla H^{(0)}[u] + \varepsilon(\mathcal{J}^{(0)}\nabla H^{(1)}[u] + \mathcal{J}^{(1)}\nabla H^{(0)}[u]) + O(\varepsilon^2) \quad , \tag{76}$$

with the Hamiltonian structure given by (14), we have shown in the previous sections that, up to order ε, (76) has at least three integrals given by (35), and the corresponding normal form is

$$\dot{v} = \mathcal{J}_0^{(0)}\nabla H^{(0)}[v] + \varepsilon(\mathcal{J}_0^{(0)}\nabla H_0^{(1)}[v] + \mathcal{J}_0^{(1)}\nabla H_0^{(0)}[v]) + O(\varepsilon^2) \quad , \tag{77}$$

with (55). The transformation $\phi^{(1)}$ between (76) and (77) can be constructed by the equations (64) for ν=1 and 2, i.e.

$$\left.\begin{aligned}
(\phi^{(1)}\cdot\nabla I_1^{(0)})[v] &= \int_{-\infty}^{\infty} \phi^{(1)}(v)v\,dx \\
&= (\gamma_1^{(1)} - c_1^{(1)})I_2^{(0)}[v] - b_{11}^{(1)}\int_{-\infty}^{\infty} v^3\,dx \;, \\
(\phi^{(1)}\cdot\nabla I_2^{(0)})[v] &= \int_{-\infty}^{\infty} \phi^{(1)}(v)v_{2x}\,dx \\
&= (\gamma_2^{(1)} - c_2^{(1)})I_3^{(0)}[v] - b_{12}^{(1)}\int_{-\infty}^{\infty} vv_x^2\,dx \;,
\end{aligned}\right\} \tag{78}$$

where we have chosen $J_\nu^{(1)} = \gamma_\nu^{(1)} I_{\nu+1}^{(0)}$ with constants $\gamma_\nu^{(1)}$ (note Deg $J_\nu^{(1)}$=Deg $I_{\nu+1}^{(0)}$), and the integrals of (77), $I_\nu^{(0)}$, are given by (30). Here, we have used the fact that the normal form has the integrals $I_\nu^{(0)}$ for ν=1,2,3. In order to solve (78), we make an anstz for $\phi^{(1)}$ having Deg$(\phi^{(1)})$=4 [7,11],

$$\phi^{(1)}(v) = \alpha_1^{(1)}v^2 + \alpha_2^{(1)}v_x\int_{-\infty}^{x} v\,dx + \alpha_3^{(1)}v_{2x} \;, \tag{79}$$

where $\{\alpha_\ell^{(1)}\}_{\ell=1}^3$ are constants determined from (78). Substituting (79) into (78), we obtain the constants from the coefficient of the independent FDPs, i.e. $\int v^3dx$, $\int v_x^2dx$, $\int vv_x^2dx$ and $\int v_{2x}^2dx$,

$$\left.\begin{aligned} \alpha_1^{(1)} &= \tfrac{1}{3}\,(-b_{11}^{(1)}+b_{12}^{(1)}) = \tfrac{1}{3}\,(b_2^{(1)}-b_3^{(1)}) \,, \\ \alpha_2^{(1)} &= \tfrac{2}{3}\,(2b_{11}^{(1)}+b_{12}^{(1)}) = -\tfrac{2}{3}\,(2b_2^{(1)}+b_3^{(1)}) \,, \\ \alpha_3^{(1)} &= \tfrac{1}{2}\,(\gamma_1^{(1)}-c_1^{(1)}) = \tfrac{1}{2}\,(\gamma_2^{(1)}-c_2^{(1)}) \,. \end{aligned}\right\} \qquad (80)$$

Also, it is easily checked from (66) that under this tranformation the Hamiltonian operator $\mathcal{J}^{(1)}=b_1^{(1)}\partial_x^3+b_2^{(1)}(\partial_x u+u\partial_x)$ is transformed into $\mathcal{J}_0^{(1)}=b_1^{(1)}\partial_x^3$.

6c. The KdV equation.

In the similar discussions as the previous example, we obtain the result where the perturved equation (17) is transformed into the integrable normal form (56) by a transformation having the same form as (76) [7].

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