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Dynamical System Arising in Nonstationary Motion of a Free Boundary of a Perfect Fluid

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ABSTRACT

We consider a free boundary problem for nonstationary motion of a perfect fluid. We propose a simple model which describes an irrotational flow around a celestial body. Our main concern is to see the PDEs which describe the nonstationary motion of the free boundary from a viewpoint of the dynamical system. Therefore we study the asymptotic behavior of free boundary for a given initial data rather than existence of the solution itself or the regularity. Of course it is very hard to acomplish it rigorously. So numerical simulation is a powerful tool to see the system as a dynamical system. We derive an evolution equation which is equivalent to the free boundary problem. This rewriting enables us to give a simple numerical algorithm which simulate the evolution of the free boundary.

§1. Introduction.

We consider a free boundary problem which is a model for a flow around a celestial body. We consider a flow in a plane which contains the equator of the celestial body. In this plane we consider a irrotational flow of perfect fluid around the equator. For a fixed time t we denote the equator and the free boundary by Γ and $\gamma(t)$, respectively (see Fig. I). Then the doubly connected domain enclosed by Γ and $\gamma(t)$ is the flow region, which is denoted by $\Omega(t)$. Our problem is now formulated as follows:

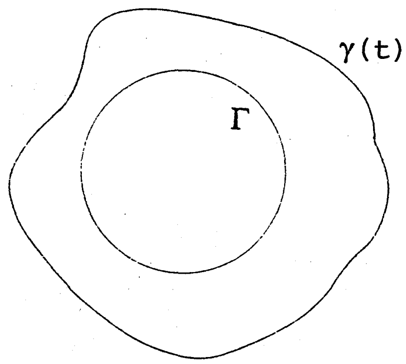


Fig. I

<u>Find a time dependent closed Jordan curve</u> $\gamma(t)$ <u>and functions</u> $V = V(t,r,\theta)$, $P = P(t,r,\theta)$ <u>satisfying the conditions (1.1-8) below.</u>

(1.1) $\quad \Delta V = 0 \qquad (0 < t,\ (r,\theta) \quad \Omega(t)),$

(1.2) $\quad V = 0 \qquad \text{on } \Gamma,$

(1.3) $\quad \frac{\partial}{\partial\theta} V(t,\gamma(t,\theta),\theta) = \gamma(t,\theta)\frac{\partial\gamma}{\partial t}(t,\theta) \qquad (0 < t,\ 0 \leqq \theta < 2\pi),$

(1.4) $\quad \frac{1}{r}\frac{\partial^2 V}{\partial t\partial\theta} + \frac{\partial}{\partial r}\left(\frac{1}{2}|\nabla V|^2 + P - \frac{g}{r} \right) = 0 \qquad (0 < t,\ (r,\theta) \quad \Omega(t))$

(1.5) $\quad -\frac{\partial^2 V}{\partial t\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{2}|\nabla V|^2 + P - \frac{g}{r} \right) = 0 \qquad (0 < t,\ (r,\theta) \quad \Omega(t))$

$$(1.6) \qquad P = \sigma K_{\gamma(t)} \qquad \text{on} \quad \gamma(t),$$

$$(1.7) \qquad V(0,t,\theta) = V_0(r,\theta), \quad \gamma(0,\theta) = \gamma_0(\theta),$$

$$(1.8) \qquad |\Omega(t)| = \omega_0.$$

where

V : stream function   P: pressure

$K_{\gamma(t)}$: curvature of $\gamma(t)$, σ: surface tension coefficient

$|\Omega(t)|$: area of $\Omega(t)$.

We transform this system of PDEs to an abstract evolution equation, which enables us to discretize our problem in a simple difference scheme. The resulting equation is of the following form:

$$(1.9) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial \theta} A(u, \dfrac{\partial u}{\partial t}) & (0 < t), \\ u(0) = u_0, \quad \dfrac{\partial u}{\partial t}(0) = u_1. \end{cases}$$

Here A is a nonlinear operator of u and $\frac{\partial u}{\partial t}$, and is a second order "pseudo differential" operator to be defined in the following. The unknown u is a function of (t,θ), where $0 < t$, $0 \leqq \theta < 2\pi$, with a period 2π with respect to θ. The evolution equation (1.9) is derived in §2 where precise definition of A will be given. We close this section by some explanations and comments on (1.1-8).

That V is a stream function implies that the velocity vector of the flow is given by $(-\frac{\partial V}{\partial y}, \frac{\partial V}{\partial x})$. Therefore (1.1) implies that the fluid is incompressible. The equation (1.3) means that the fluid particles on the free boundary move on the free boundary throughout the motion. The equations (1.4,5) are the Euler equation written by V. The outside of the curve $\gamma(t)$ is assumed to be a vacuum. Hence the equation (1.6) implies that the pressure difference across the free boundary is proportional to the curvature of the free boundary. The equation (1.6) is called the Laplace equation (see, e.g., Landau and Lifschitz [2]).

§2. Transformation to an evolution equation.

In this section we define the operator A and derive (1.9) from (1.1-8). We first fix functions $\gamma(t,\theta)$ and $\frac{\partial\gamma}{\partial t}(t,\theta)$, and we consider closed Jordan curves with parameter t defined as follows:

$$\gamma(t) = \{ (r,\theta) \; ; \; r = \gamma(t,\theta) \; , \quad 0 \leqq \theta < 2\pi \}.$$

Here and in what follows we use the polar coordinates (r,θ). We restrict ourselves to solutions which are close to a special solution which we call a trivial solution. It is defined to be a stationary solution with radial symmetry. More concretely, we define $r_0 > 1$ by $\pi r_0^2 - \pi = \omega_0$ and we consider a circle $\gamma^{\#}$ of radius r_0 with the origin as its center. On the other hand, we define $V^{\#}$ with a real parameter a by

$$V^{\#} = \frac{a}{\log r_0} \log r \qquad\qquad (1 < r < r_0).$$

We also define $P^{\#} = \frac{g}{r} - \frac{1}{2}\left(\frac{a}{r\log r_0} \right)^2 + \frac{\sigma - g}{r_0} + \frac{1}{2}\left(\frac{a}{r_0 \log r_0} \right)^2$. Then $\{\gamma^{\#}, V^{\#}, P^{\#}\}$ is a solution to (1.1-8), which we call a trivial solution. The parameter a expresses a magnitude of circulation, whence the flow speed becomes larger if a becomes larger.

Now we regard $\gamma(t,\theta)$ and $\frac{\partial\gamma}{\partial t}(t,\theta)$ as given functions which are close to r_0 and 0, respectively. Then we can define a function V by solving the Dirichlet Problem (1.1-3). More precisely we consider an integrated form of (1.3), i.e.,

$$(1.3)^* \quad V(t,\gamma(t,\theta),\theta) = \int_0^{\theta} \gamma(t,\phi)\frac{\partial\gamma}{\partial t}(t,\phi)d\phi + f(t) \qquad (0 \leqq \theta < 2\pi),$$

where f does not depend on θ. For a fixed time t the equations

(1.1,2) and (1.3)* form a dirichlet problem. Hence it is uniquely solvable with respect to V, if $f(t)$ is known and if γ and $\frac{\partial\gamma}{\partial t}$ are sufficiently smooth and γ is sufficiently close to r_0. The function f is determined by the requirement that the circulation is constant (Kelvin's theorem). This is expressed by the equation

$$(2.1) \qquad \int_0^{2\pi}\frac{\partial V}{\partial r}(t,1,\theta)d\theta = \int_0^{2\pi}\frac{\partial V_0}{\partial r}(1,\theta)d\theta.$$

(Actually this condition is a necessary condition for the existence of the solution, since (1.5) must hold on $r = 1$.) Now we give a rigorous definition of V and f: first, define V_1 to be a unique solution of $\Delta V_1 = 0$ in $\Omega(t)$, $V_1 = 0$ on $r = 1$, $V_1 = \int_0^\theta \gamma(t,\phi)\frac{\partial\gamma}{\partial t}(t,\phi)d\phi$ on $\gamma(t)$. Replacing the boundary condition on $\gamma(t)$ by $V_2 = 1$, we define V_2 in the same way. Then the functionn f is determined by the equality

$$f(t)\int_0^{2\pi}\frac{\partial V_2}{\partial r}(t,1,\theta)d\theta + \int_0^{2\pi}\frac{\partial V_1}{\partial r}(t,1,\theta)d\theta = \int_0^{2\pi}\frac{\partial V_0}{\partial r}(1,\theta)d\theta$$

We put $V = V_1 + f(t)V_2$, which completes the definition. Note that V is determined by γ, $\frac{\partial\gamma}{\partial t}$ and the initial condition.

The next step is to solve the Euler equation (1.4-5). We begin with a heuristic argument. Putting

$$R = \frac{\partial V}{\partial t}, \quad Q = \frac{1}{2}|\nabla V|^2 + P - \frac{g}{r},$$

the equations (1.4-5) implies that the functions R and Q must satisfy

$$(2.2)\quad \begin{cases} \dfrac{1}{r}\dfrac{\partial R}{\partial \theta} + \dfrac{\partial Q}{\partial r} = 0 & \text{in } \Omega(t), \\ -\dfrac{\partial R}{\partial r} + \dfrac{1}{r}\dfrac{\partial Q}{\partial \theta} = 0 & \text{in } \Omega(t). \end{cases}$$

This is a Cauchy-Riemann equation written in the polar coordinates. Hence Q and R must be harmonic functions. Furthermore $R = 0$ on $r = 1$, because of the condition (1.2). Therefore $\frac{\partial Q}{\partial r} = 0$ on $r = 1$. In view of these facts and (1.6), we define a function Q to be a solution of

$$(2.3)\quad \begin{cases} \Delta Q = 0 & \text{in } \Omega(t), \\ \dfrac{\partial Q}{\partial r} = 0 & \text{on } r = 1, \\ Q = \dfrac{1}{2}|\nabla V|^2 + \sigma K_{\gamma(t)} - \dfrac{g}{\gamma(t,\theta)} & \text{on } \gamma(t), \end{cases}$$

where V is the function defined in the previous step. Then we define R by the equation below:

$$(2.4)\qquad R(t,r,\theta) = \int_1^r \frac{1}{\rho}\frac{\partial Q}{\partial \theta}(t,\rho,\theta)\,d\rho.$$

It is easy to check that R and Q defined in this way satisfies the equation (2.2). By the definition we see that R must coincide with $\frac{\partial V}{\partial t}$, if the solution exists.

We are now in a position to derive an equation which $\gamma(t,\theta)$ has to satisfy. This is done by differentiating (1.3) in t:we obtain

$$\frac{\partial^2}{\partial t^2}\gamma(t,\theta)^2 = 2\frac{\partial}{\partial \theta}\left\{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial r}\frac{\partial \gamma}{\partial t}\right\} \qquad (0 < t,\ 0 \leqq \theta < 2\pi).$$

Consequently the function γ satisfies

$$\frac{\partial^2}{\partial t^2}\gamma(t,\theta)^2 = 2\frac{\partial}{\partial\theta}\left(R|_{\gamma(t)} + \frac{\partial V}{\partial r}\Big|_{\gamma(t)}\cdot\frac{\partial\gamma}{\partial t}\right).$$

Introducing a new function u by $\gamma^2 = r_0^2 + u$, we have

(2.5) $$\frac{\partial^2 u}{\partial t^2} = 2\frac{\partial}{\partial\theta}\left(R|_{\gamma(t)} + \frac{\partial V}{\partial r}\Big|_{\gamma(t)}\cdot\frac{\partial\gamma}{\partial t}\right),$$

which is of the form (1.9). Finally the condition (1.8) is equivalent to

(2.6) $$\int_0^{2\pi} u(t,\theta)d\theta = 0 \qquad (0 < t).$$

Therefore we have arrived an evolution equation (2.5) to be solved in some space of functions with zero mean ((2.6)).

To show that (2.5) is well-posed in the sense of the evolution equation is not succeeded until now. But we think that the method in Yosihara [6,7] will prove the well-posedness, although (2.5) is completely different from that in [6,7]. We derived (2.5) mainly to use it for the numerical simulation. In the form of (2.5), numerical analysis is much eqsier than the numerical analysis applied directly to (1.1-8). Our scheme is presented in §5.

§3. Linearized equation.

In this section we study the linearized equation for (2.5). The eqation (2.5) is highly nonlinear and its analysis is difficult. But, as we will show later, its linearized equation is easy to analyze. Regarding the right hand side of (2.5) as a function of u and $\frac{\partial u}{\partial t}$, we take the Fréchet derivative at $u = 0$. Then we obtain the following equation:

$$(3.1) \qquad \frac{\partial^2 u}{\partial t^2} - 2b\frac{\partial^2 u}{\partial t \partial \theta} + b^2 \frac{\partial^2 u}{\partial \theta^2} + H^{-1}\left(b^2 + \frac{\sigma - g}{r_0^3} \right)\frac{\partial u}{\partial \theta} + \frac{\sigma}{r_0^3} H^{-1} \frac{\partial^3 u}{\partial \theta^3} = 0$$

$$(3.2) \qquad u(0,\theta) = u_0(\theta), \qquad \frac{\partial u}{\partial t}(0,\theta) = u_1(\theta),$$

where $b = \dfrac{a}{r_0^2 \log r_0}$, H is an operator defined by

$$(3.3) \qquad H : \sum_{n\neq 0} v_n e^{in\theta} \to \sum_{n\neq 0} (-i)\frac{r_0^n + r_0^{-n}}{r_0^n - r_0^{-n}} v_n e^{in\theta} .$$

Observe that the coefficient $R_n \doteq \dfrac{r_0^n + r_0^{-n}}{r_0^n - r_0^{-n}}$ tends to 1 as $n \to +\infty$, and to -1 as $n \to -\infty$. Consequently H is equal to a Hilbert transform modulo a smoothing operator. Now we see that the equation (3.1) is of second order in time and of third order in the space variable θ. The derivation of (3.1) is found in [3].

We consider a special solution of (3.1-2) which is of the form $\hat{u} = \exp(\lambda t + in\theta)$. This function $\hat{u}$ satisfies (3.1) if and only if the parameter λ satisfies

$$(3.4)\qquad \lambda^2 - 2inb\lambda - n^2b^2 - \left[\left(b^2 + \frac{\sigma-g}{r_0^3}\right)in + \frac{\sigma}{r_0^3}(in)^3\right]/(iR_n) = 0.$$

Putting $b_n = \frac{1}{r_0}\left[\frac{\sigma(n^2-1)/r_0^2 + g/r_0^2}{(1+nR_n)/r_0}\right]^{1/2}$, we define $\lambda_+(n)$ and $\lambda_-(n)$ by

$$(3.5)\qquad \lambda_\pm(n) = i\left[bn \pm \sqrt{\frac{n}{R_n}\left[(1+nR_n)b_n^2 - b^2\right]}\right].$$

Then for each n $\mathbb{Z}$ $\{0\}$, we have two solutions of (3.1) represented by $\hat{u} = \exp(\lambda_\pm(n)t + in\theta)$. To study $\hat{u}$ we put

$$\beta(n) = \sqrt{\frac{n}{R_n}\left[(1+nR_n)b_n^2 - b^2\right]},$$

and $\quad b^* = \min_{n\geq 1} b_n, \qquad b^{**} = \min_{n\geq 1} \sqrt{1+nR_n}\, b_n.$

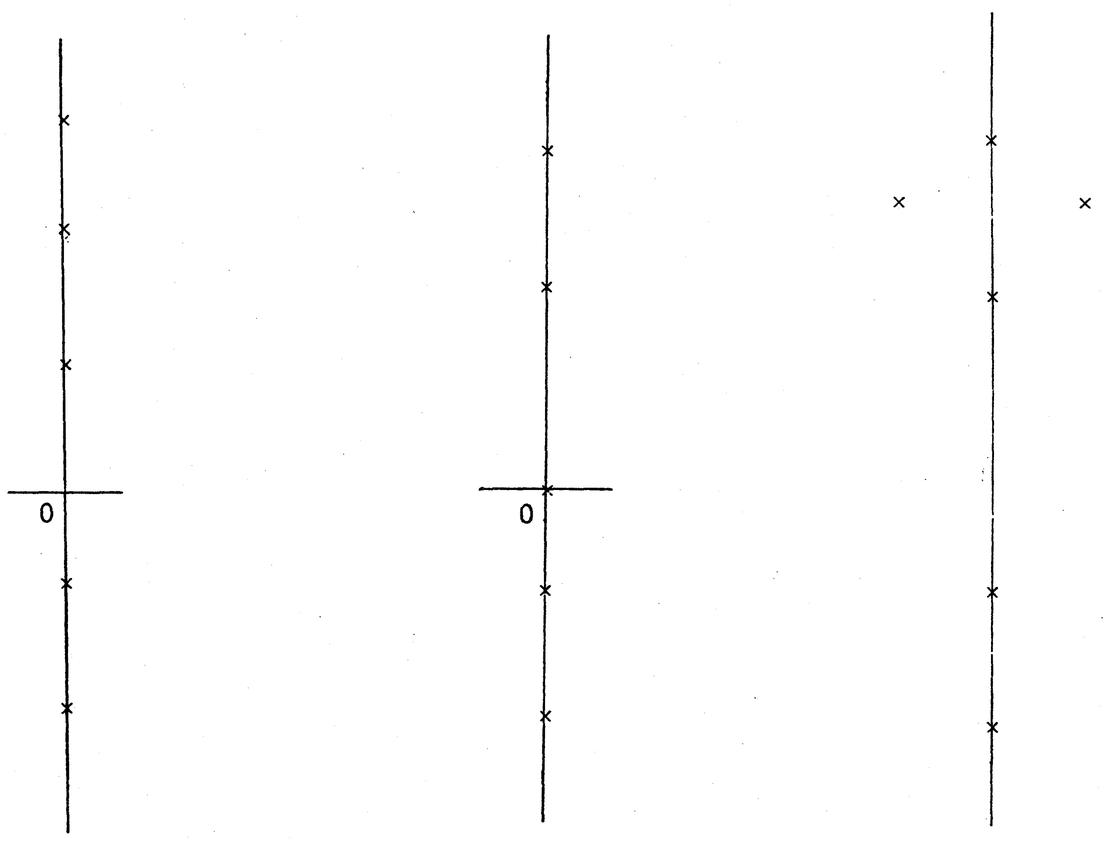
Then we see

i) that $\beta(n)$ is real and nonzero if $0 < b < b^*$,

ii) that $\beta(n)$'s are still real but some of them vanish if $b^* \leq b < b^{**}$,

iii) and that $\beta(n)$ is a pure imaginary number for some $n \neq 0$ if $b^{**} < b$. Therefore $\lambda_\pm(n)$ is a pure imaginary number for all $n \neq 0$ if $0 < b < b^{**}$, and the real part of $\lambda_-(n)$ is positive for some n if $b^{**} < b$ (see Fig. II). If we take an integer n such that $\mathrm{Re}\,\lambda_-(n) > 0$, then $\hat{u} = \exp(\lambda_-(n)t + in\theta)$ is an exponentially growing solution if $b^{**} < b$. For $0 < b < b^*$ the solution $\hat{u}$ is a bounded oscillating solution. Taking the real part of $\hat{u}$, we see that the function

$$(3.6)\quad w(t,\theta) = \cos(n\theta + nbt)\cos(\beta(n)t) + \frac{nb}{\beta(n)}\sin(n\theta + nbt)\sin(\beta(n)t)$$



$0 < a < a^*$ $\quad$ $a = a^*$ $\quad$ $a^{**} < a$

FIG. II

satiafies the equation (3.1) and

$$(3.7) \qquad w(0,\theta) = \cos(n\theta), \quad \frac{\partial w}{\partial t}(0,\theta) = 0.$$

Therefore we can conclude that w is periodic if $nb/\beta(n) \quad Q$ and that w is aperiodic if $nb/\beta(n) \quad Q$. Since $\beta(n)$ depends on b continuously, both cases take place. We hope that there is periodic solutions and quasi-periodic solutions to the nonlinear equation (2.5) for some initial value $u_0 = \varepsilon\cos(n\theta) + O(\varepsilon^2)$, $u_1 = O(\varepsilon^2)$ with a small parameter ε, which is asymptotically equal to $\varepsilon w(t,\theta)$ as $\varepsilon \to 0$. In Fig. III we show a numerical experiment to this nonlinear problem.

§4. Stability of the trivial solution.

In this section we state results and conjecture concerning the stability of the trivial solution. In [5] it is proved that branches of nontrivial solutions bifurcates from the trivial solution with a as a bifurcation paramater. It is also shown that there are countable number of bifurcation points $\{a_n\}$ which are explicitly calculated. The shape of the free boundary of the bifurcating solution is computed numerically (see Fujita et al. [1]). We put $a^* = \min\{ a_n ; n = 1,2,\cdots\}$. Then we can see that $a^* = b^*/(r_0^2 \log r_0)$ (see [5] and (3.5)). Now one may conjecture that the trivial solution is stable if the velocity of the flow is so small that $0 < a < a^*$ and that the trivial solution loses the stability at $a = a^*$. Concerning this conjecture it is worthy of notice that in the linearized equation the trivial solution is unstable if $a > a^{**} \equiv b^{**}/(r_0^2 \log r_0)$. But we cannot say anything for $0 < a < a^{**}$ since all the eigenvalues lie on the imaginary axis (see Fig. II). On the other hand, the existence of progressive wave solutions is proved in [4]. They exists even when $0 \leqq a < a^*$. Since they are a kind of periodic solution, we find that the trivial solution might be stable for $0 < a < a^*$ but it is not asymptotically stable. This is supported in another aspect. Indeed, the simulation (Fig. III) shows that there is an orbit which does not approaches to the trivial solution.

On the other hand, we conjecture that the trivial solution is unstable if the parameter a is greater than a^{**}. In Fig. IV we show a numerical computation which shows that there is no time-global solution if $a > a^{**}$.

Finally we consider the stability of the progressive wave solutions. We computed numerically the solution with an initial value which is close to a progressive wave. We have found that it "progresses" for a long time and we have found a little change of the shape. So we think that the progressive wave is stable.

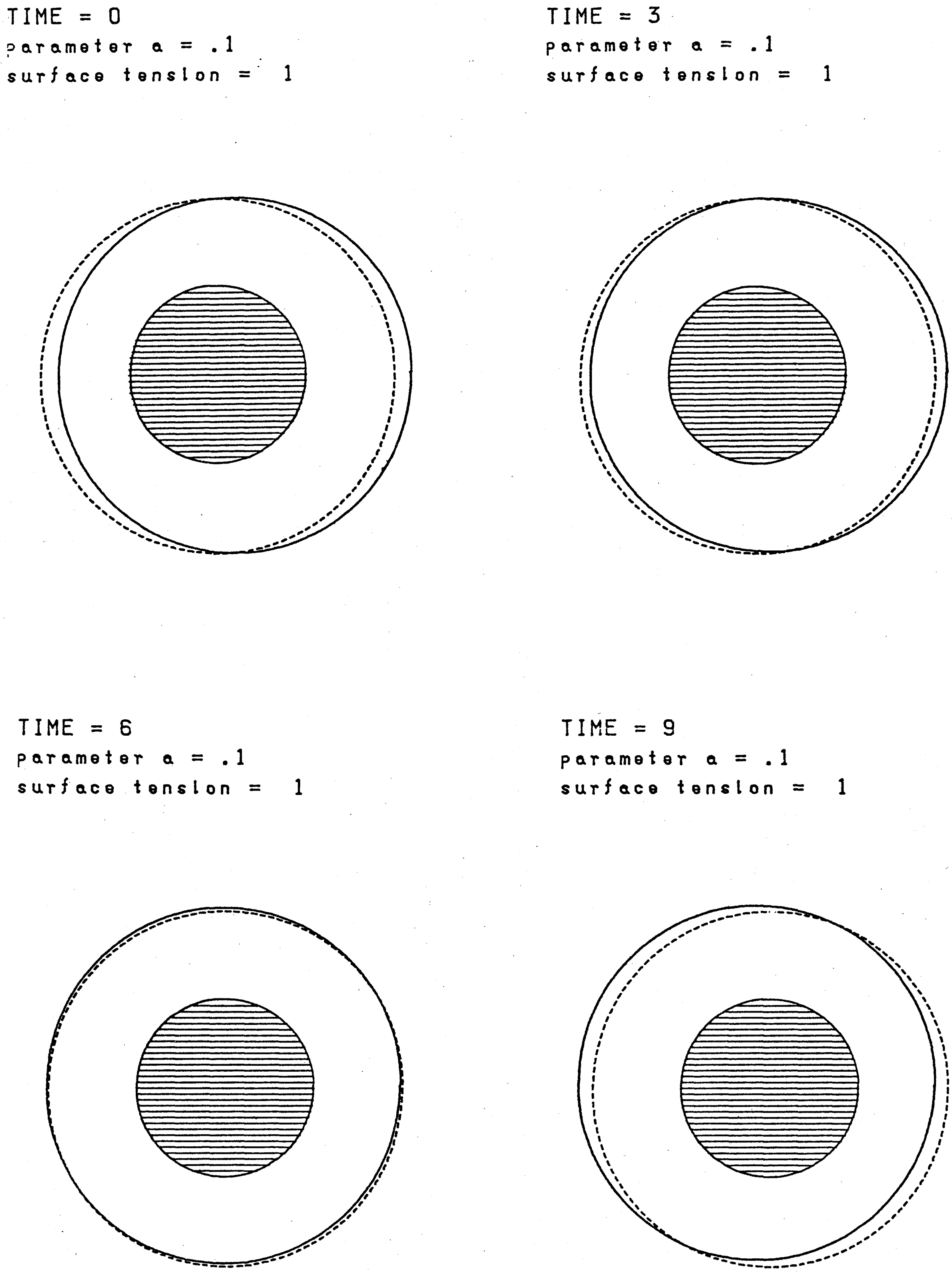

TIME = 0
parameter a = .1
surface tension = 1
TIME = 3
parameter a = .1
surface tension = 1
TIME = 6
parameter a = .1
surface tension = 1
TIME = 9
parameter a = .1
surface tension = 1

Fig. III

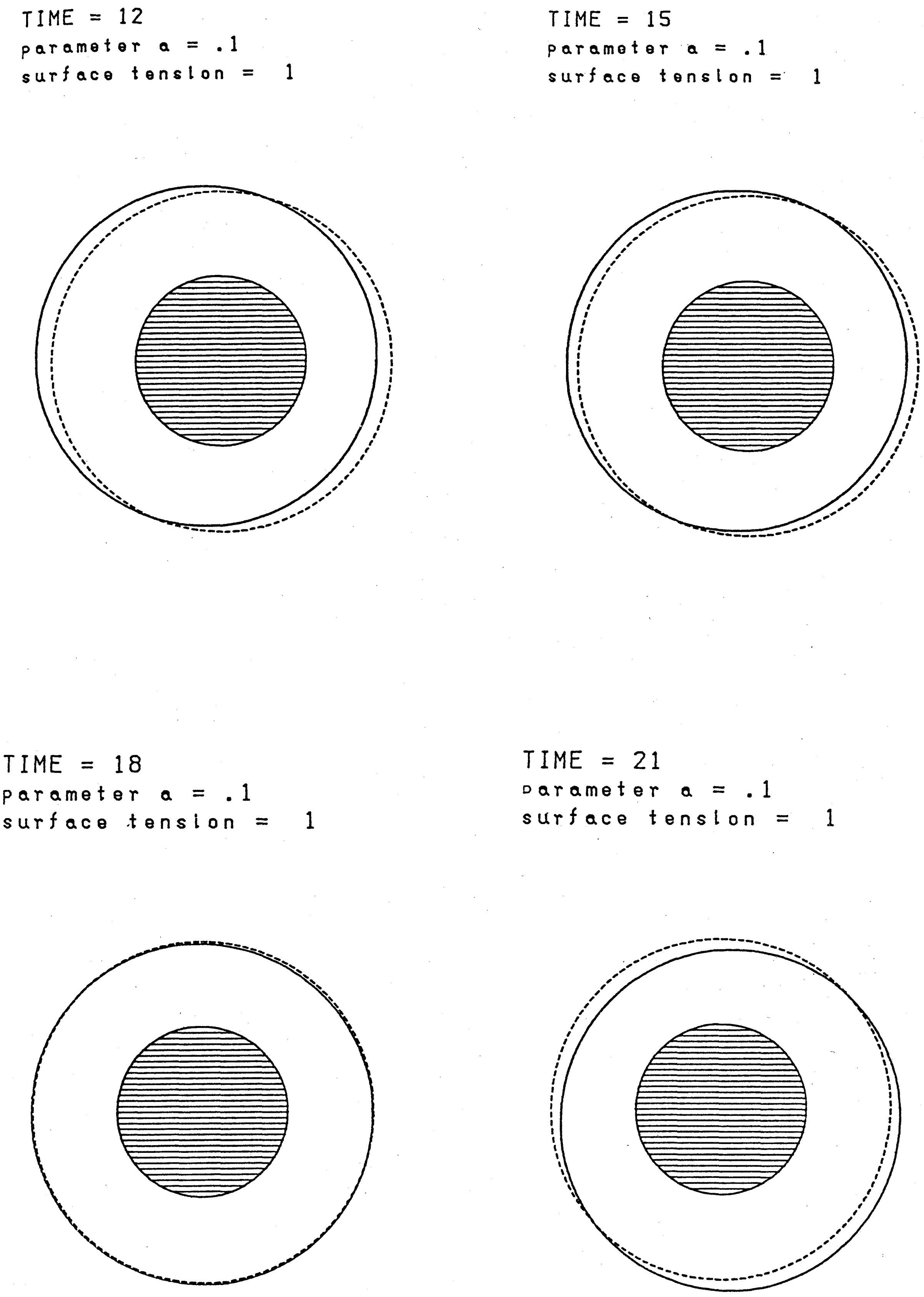

TIME = 12
parameter a = .1
surface tension = 1
TIME = 15
parameter a = .1
surface tension = 1
TIME = 18
parameter a = .1
surface tension = 1
TIME = 21
parameter a = .1
surface tension = 1

Fig. III

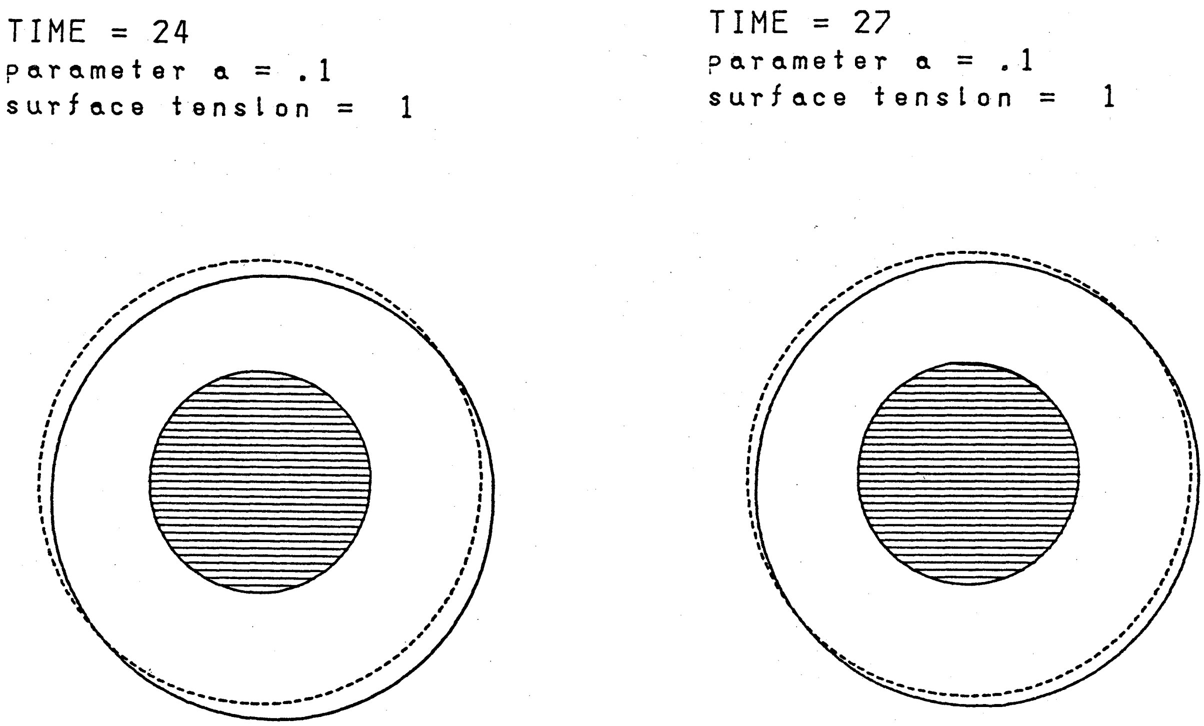

TIME = 24
parameter a = .1
surface tension = 1
TIME = 27
parameter a = .1
surface tension = 1

Fig. III

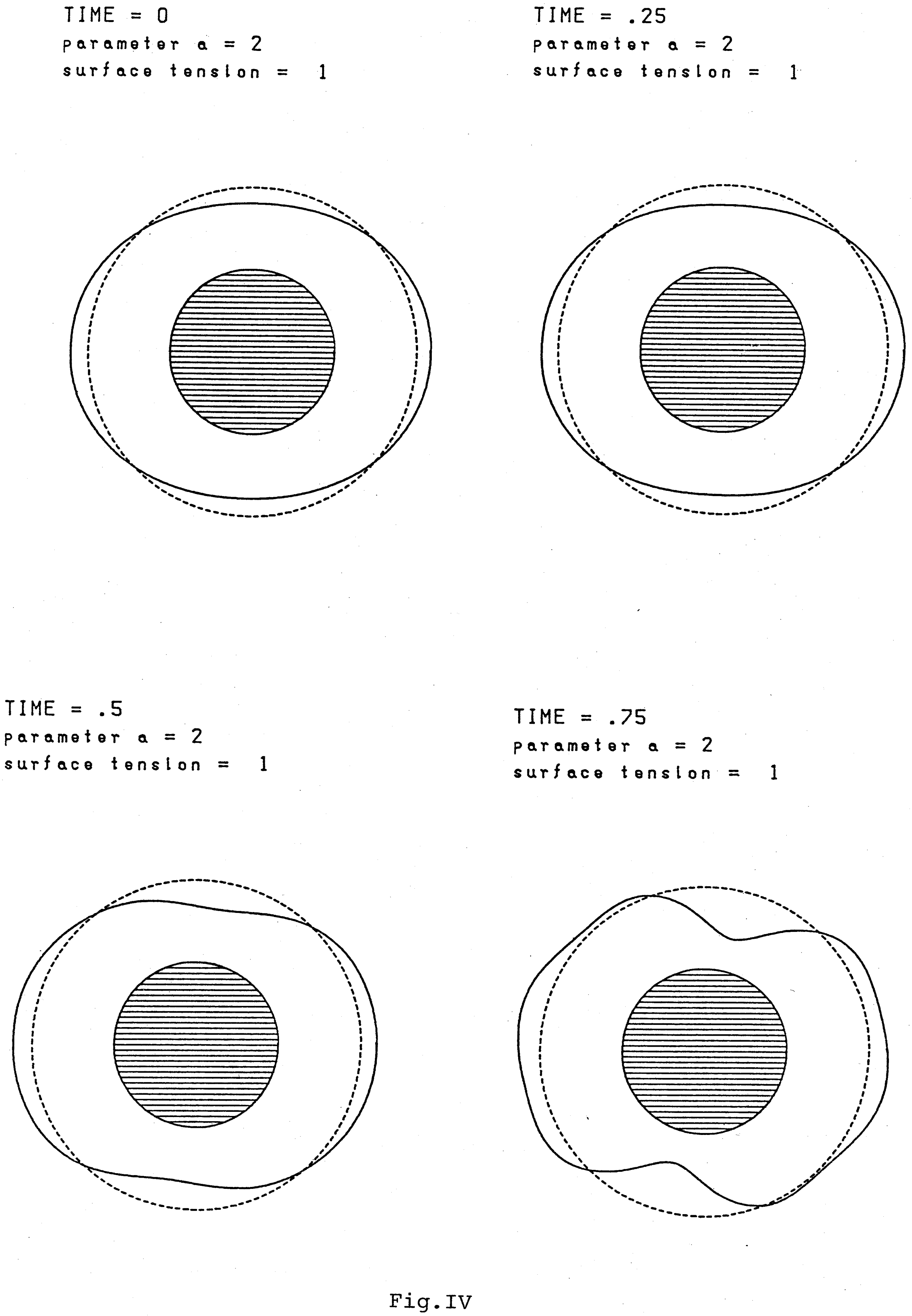

TIME = 0
parameter a = 2
surface tension = 1
TIME = .25
parameter a = 2
surface tension = 1
TIME = .5
parameter a = 2
surface tension = 1
TIME = .75
parameter a = 2
surface tension = 1

Fig.IV

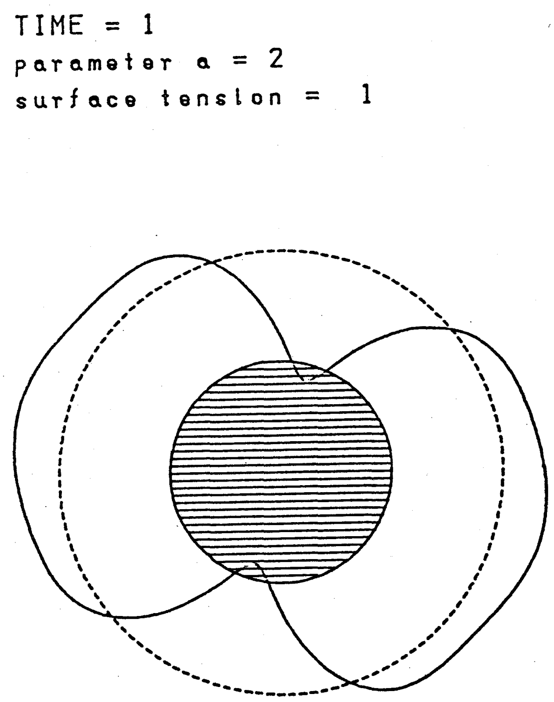


Fig. IV

§5. Discretization.

In this section we give a scheme by which we computed the nonlinear evolution equation (2.5). We identify S^1 with the interval $[0,2\pi)$ and we divide it into N subintervals of equal length. We denote the mid-point of the subintervals by θ_j ($j = 1,2,\cdots, N$), whence $\theta_j = \frac{2j-1}{N}\pi$. We first define the initial data $\{\gamma(0,k)\}_{k=1}^N$ and $\{\dot{\gamma}(0,k)\}_{k=1}^N$ by

$$(5.1) \qquad \gamma(0,k) = \gamma_0(\theta_k),\ \dot{\gamma}(0,k) = \frac{\partial}{\partial\theta} V_0(\gamma_0(\theta),\theta)|_{\theta=\theta_k} \quad (1 \leqq k \leqq N).$$

We put $\gamma(1,k) = \gamma(0,k) + \tau\dot{\gamma}(0,k)$ ($k = 1,2,\cdots, N$), where τ is the time mesh. When we have $\{\gamma(m,k)\}_{k=1}^N$ for $m = 1,2,\cdots,n$, we put $\dot{\gamma}(n,k) = (\gamma(n,k) - \gamma(n-1,k))/\tau$ and define $\gamma(n+1,k)$ by the following scheme.

<u>1-st step</u>. We define functions Φ_j:

$$(5.2) \qquad \Phi_j(r,\theta) = \frac{-1}{4\pi} \log \frac{r^2 + 4r_0^2 - 4r_0 r\cos(\theta-\theta_j)}{4r_0^2 r^2 + 1 - 4r_0 r\cos(\theta-\theta_j)} \qquad (1 \leqq j \leqq N).$$

Observe that Φ_j vanishes on Γ and is harmonic in $\mathbb{R}^2\backslash\{(2r_0,\theta_j)\}$, whence the singular points lie on the circle of radius $2r_0$. Therefore the free boundary near the trivial solution does not touch the circle. We then define functions V_1 and V_2 by

$$V_1(n,r,\theta) = \sum_{j=1}^N \alpha_j\Phi_j(r,\theta), \quad V_2(n,r,\theta) = \sum_{j=1}^N \beta_j\Phi_j(r,\theta),$$

where the coefficients $\{\alpha_j\}$ and $\{\beta_j\}$ are determined by

$$V_1(n,\gamma(n,k),\theta_k) = \sum_{s=1}^{k-1} \frac{2\pi}{N}\gamma(n,s)\dot{\gamma}(n,s) + \frac{\pi}{N}\gamma(n,k)\dot{\gamma}(n,k),$$

$$V_2(n,\gamma(n,k),\theta_k) = 1$$

(see the definition of V_1 and V_2 in §2). We then define $f(n)$ by

$$f(n)\int_0^{2\pi} \frac{\partial V_2}{\partial t}(n,1,\theta)d\theta + \int_0^{2\pi} \frac{\partial V_1}{\partial t}(n,1,\theta)d\theta = \int_0^{2\pi} \frac{\partial V}{\partial t}(n-1,1,\theta)d\theta.$$

Actually this is equivalent to

$$\left\{ \sum_{j=1}^{N} \beta_j \right\} f(n) + \sum_{j=1}^{N} \alpha_j = \int_0^{2\pi} \frac{\partial V}{\partial t}(n-1,1,\theta)d\theta.$$

Now we put $V(n,r,\theta) = V_1(n,r,\theta) + f(n)V_2(n,r,\theta)$ and

(5.3) $$t_k = |\nabla V(n,r,\theta)|^2 \Big|_{(r,\theta)=(\gamma(n,\theta_k),\theta_k)} \qquad (0 \leqq k \leqq N).$$

<u>2-nd step</u>. For $1 \leqq j \leqq N$ we define harmonis functions Ψ_j by

$$\Psi_j(r,\theta) = \frac{-1}{4\pi} \log (r^2 + 4r_0^2 - 4r_0 r\cos(\theta-\theta_j))$$
$$- \frac{-1}{4\pi}\log (4r_0^2 r^2 + 1 - 4r_0 r\cos(\theta-\theta_j)) + \frac{1}{2\pi} \log r.$$

Observe that Ψ_j is harmonic in $\mathbb{R}^2\backslash\{(2r_0,\theta_j)\}$ and satisfies $\frac{\partial \Psi_j}{\partial r} = 0$ on Γ. We define Q by $Q = \sum_{j=1}^{N} \lambda_j \Psi_j(r,\theta)$, where $\{\lambda_j\}$ is determined by

$$Q(n,\gamma(n,k),\theta_k) = \frac{1}{2}t_k + \sigma K(n,k) - \frac{g}{\gamma(n,k)} \qquad (1 \leqq k \leqq N).$$

Here t_k is defined in (5.3) and $K(n,k)$ is a discretized curvature defined by

(5.4) $$K(n,k) = \frac{\gamma(n,k)^2 + 2\gamma'(n,k)^2 - \gamma(n,k)\gamma''(n,k)}{\left\{ \gamma(n,k)^2 + \gamma'(n,k)^2 \right\}^{3/2}},$$

where we have used the following abbreviation:

$$\gamma'(n,k) = \frac{\gamma(n,k+1) - \gamma(n,k-1)}{2h} \qquad (h = 2\pi/N),$$

$$\gamma''(n,k) = \frac{\gamma(n,k+1) - 2\gamma(n,k) + \gamma(n,k-1)}{h^2} .$$

3-rd step. We define $R(n,r,\theta)$ by

$$R(n,r,\theta) = \int_1^r \frac{1}{\rho} \frac{\partial Q}{\partial \theta}(n,\rho,\theta)d\rho.$$

This integral is explicitly calculated and we obtain

$$R(n,r,\theta) = \sum_{j=1}^{N} \frac{-1}{2\pi} \left\{ \tan^{-1} \frac{2r_0(r-1)\sin(\theta-\theta_j)}{4r_0^2 + r - 2r_0(r+1)\cos(\theta-\theta_j)} + \tan^{-1} \frac{2r_0(r-1)\sin(\theta-\theta_j)}{4r_0^2r^2 + 1 - 2r_0(r+1)\cos(\theta-\theta_j)} \right\}.$$

We also put $Z(n,k) = \frac{\partial V}{\partial r}(n,\gamma(n,k),\theta_k)\dot{\gamma}(n,k)$.

4-th step. We determine $\{\gamma(n+1,k)\}_{k=1}^{N}$ by the following scheme:

$$\left\{ \gamma(n+1,k)^2 - 2\gamma(n,k)^2 + \gamma(n-1,k)^2 \right\}/(2\tau^2)$$

$$= - \gamma(n,k)\frac{\partial Q}{\partial r}(n,\gamma(n,k),\theta_k) + \frac{1}{\gamma(n,k)} \frac{\partial Q}{\partial \theta}(n,\gamma(n,k),\theta_k)\cdot\gamma'(n,k)$$

$$+ \frac{Z(n,k+1) - Z(n,k-1)}{2h} .$$

This discretization is obtained from (2.5) in the following way. We first carry out the differentiation with respect to θ and then we replace the derivative of R by those of Q by means of (2.2).

To study the stability we have linearized this scheme and found that it is stable if $\tau^2/h^3 < r_0^3/(2\sigma)$ and if we choose h sufficiently small for a fixed mode n. For general initial values we cannot conclude the stability. This is a problem to be solved in the future.

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