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Note on the class of unit in $K_0(C(S^1)\times_r\Gamma_g)$

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1. Prologue:

In [2] A. Connes showed, among other things, that the unit of the reduced crossed product $C(S^1)\times_r\Gamma$ is a torsion element of the K_0-group for every torsionfree cocompact discrete subgroup Γ of $PSL_2(\mathbb{R})$ acting on S^1, which we view as the boundary of the upper half-plane, given by the linear fractional transformations. He showed this in the framework of his noncommutative differential geometry. More precisely, he showed that for a suitable element x of the geometric group $K^0(S^1,\Gamma)$ which corresponds to the class of unit through the index map $\mu : K^0(S^1,\Gamma) \to K_0(C(S^1)\times_r\Gamma)$, the Chern character $ch(x)$ is equal to zero. Consequently, x is a torsion element, hence so is the class of unit.

The purpose of this note is to give an elementary proof of this result in a purely C*-algebraic manner.

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2. The group Γ_g :

Let Γ_g be the fundamental group of a closed Riemannian surface of genus $g > 1$. We can regard Γ_g as a torsion-free cocompact discrete subgroup of $PSL_2(\mathbb{R})$.

It is well-known that Γ_g is generated by 2g generators α_1, $\beta_1,\cdots,\alpha_g,\beta_g$ with a single relation:

$$[\alpha_1,\beta_1]\cdots[\alpha_g,\beta_g] = 1.$$

Put $\gamma = [\beta_1,\alpha_1]$ ($= [\alpha_2,\beta_2]\cdots[\alpha_g,\beta_g]$). Let G (resp. S) be the group generated by α_1,β_1 (resp. $\alpha_2,\beta_2,\cdots,\alpha_g,\beta_g$). Then G and S are free groups with 2 and 2g-2 generators, respectively. From the above relation, it follows that

$$\Gamma_g = G*_H S,$$

where H is the infinite cyclic subgroup generated by γ .

Let A, A_G, A_S and A_H denote the reduced crossed products $C(S^1)\rtimes_r\Gamma_g$, $C(S^1)\rtimes_r G$, $C(S^1)\rtimes_r S$ and $C(S^1)\rtimes_r H$, respectively. Let κ^1 (resp. κ^2) be a natural inclusion of A_H into A_G (resp. A_S), and let ε^1 (resp. ε^2) be a natural inclusion of A_G (resp. A_S) into A. Then

$$\varepsilon^1\circ\kappa^1 = \varepsilon^2\circ\kappa^2 .$$

In the latter sections, we compute the induced maps κ^1_* and κ^2_* in K-theory.

3. Rieffel projection:

Let h be an orientation preserving homeomorphism of S^1, which is not the identity.

<u>Definition</u> 1. A <u>foundation</u> of h is a quadruple $T = (t_0,t_1,t_2,t_3)$ of mutually distinct points of S^1 such that

1) $h(t_0) = t_2$, $h(t_1) = t_3$;

2) $t_0,\cdots,t_3$ are contained in the same connected component I of $S^1 \setminus \mathrm{Fix}(h)$ and are sitting in the negative direction in I, which we view as an oriented open submanifold of S^1.

We give examples. Let h be induced from a matrix of the form

$$\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

with $p > 0$. Then h has no foundations. On the other hand, if h is induced from a hyperbolic element, then h always has a foundation. Thus, an orientation preserving homeomorphism h may or may not have a foundation. However, at least one of h and h^{-1} has a foundation.

Let $T = (t_0,t_1,t_2,t_3)$ be a foundation of h . The homeomorphism h gives rise to an action of $\mathbb{Z}$ on $C(S^1)$. Consequently, the crossed product $C(S^1) \rtimes \mathbb{Z}$ is defined.

For $i = 0,1,2$, let $[t_i,t_{i+1}]$ denote the closure of the connected component of $S^1 \setminus \{t_i,t_{i+1}\}$ which does not contain t_k, where $k \neq i$, $k \neq i+1$.

On $[t_0,t_1]$ let f be a continuous function with values in $[0,1]$ and with $f(t_0) = 0$ and $f(t_1) = 1$. On $[t_2,t_3]$ define f by

$$f(t) = 1 - f(h^{-1}(t)),$$

while on $[t_1,t_2]$ let f have value 1. On the complement of

$[t_0,t_1] \cup [t_1,t_2] \cup [t_2,t_3]$, let f have value 0. Define g by

$$g(t) = (f(t)(1-f(t)))^{1/2}.$$

Put $e = (gU_h)^* + f + gU_h$, where U_h is the canonical unitary of $C(S^1) \rtimes \mathbb{Z}$ corresponding to h. Then, from the construction, it follows that e is a projection of $C(S^1) \rtimes \mathbb{Z}$. We call e a <u>Rieffel projection associated to</u> h.

Notice that as h is orientation preserving, the action of $\mathbb{Z}$ on $K_1(C(S^1))$ is trivial.

Let δ_1 be the connecting homomorphism from $K_0(C(S^1) \rtimes \mathbb{Z})$ into $K_1(C(S^1))$ associated to a six-term exact sequence obtained in [6]. Let z be the canonical unitary of $C(S^1)$. Then by a direct computation (cf., [Appendix, 6]) we get:

Lemma 2. <u>We have</u> $\delta_1([e]) = [z]$ <u>in</u> $K_1(C(S^1))$.

By Lemma 2 together with the six-term exact sequence mentioned above, it follows that $K_0(C(S^1) \rtimes \mathbb{Z})$ is a free abelian group generated by [1] and [e]. We denote the class of e by $\beta(h,T)$.

Let T, T' be foundations of h. Then:

Lemma 3. <u>For some</u> $m \in \mathbb{Z}$, <u>we have</u>

$$\beta(h,T) - \beta(h,T') = m[1].$$

The proof is easy and omitted here.

4. Computations:

In this section we compute the maps κ_*^1 and κ_*^2.

First, notice that Γ_g has no elliptic elements, because it acts freely on the upper half plane. By the construction of the embedding of Γ_g into $PSL_2(\mathbb{R})$, we can easily see that the α_i's and β_j's have their own foundations.

We compute $\kappa_*^1 : K_0(A_H) \longrightarrow K_0(A_G)$. Choose a foundation of γ to get a Rieffel projection e corresponding to γ. Since γ has a fixed point, there exists a normalized trace τ on A_H. Using the induced map $\tau_* : K_0(A_H) \longrightarrow \mathbb{R}$ together with Lemma 3, we can show that the class $[e]$ of e does not depend on the choice of a foundation.

Take foundations and construct Rieffel projections p_1 and q_1 corresponding to α_1 and β_1, respectively. Consider the C*-dynamical system $(C(S^1), \mathbb{Z}, \rho)$ determined by α_1. Let B be the reduced crossed product of the system $(C(S^1), \mathbb{Z}, \rho)$. By the same argument as above, we can show that the class $[p_1]$ of p_1 in $K_0(B)$ is independent of the choice of a foundation of α_1, because α_1 has a fixed point. Consequently, the class $[p_1]$ in $K_0(A_G)$ is also independent of the choice of a foundation. Similarly, the class of q_1 in $K_0(A_G)$ is uniquely determined.

As we have seen in the previous section, $K_0(A_H)$ is a free abelian group generated by $[1]$ and $[e]$. By using a six-term exact sequence for A_G (cf., [7]), we can see that $K_0(A_G)$ is a free abelian group with generators $[1]$, $[p_1]$ and $[q_1]$.

Obviously, $\kappa_*^1([1]) = [1]$. Put

$$\kappa_*^1([e]) = k[p_1] + m[q_1] + n[1].$$

Consider the six-term exact sequence mentioned above:

$$\begin{array}{ccccc} K_0(C(S^1)) & \longrightarrow & K_0(C(S^1)\rtimes_r G') & \longrightarrow & K_0(A_G) \\ {\scriptstyle\delta_0}\big\downarrow & & & & \big\downarrow{\scriptstyle\delta_1} \\ K_1(A_G) & \longleftarrow & K_1(C(S^1)\rtimes_r G') & \longleftarrow & K_1(C(S^1)), \end{array}$$

where G' is the infinite cyclic subgroup of G generated by α_1. By a direct computation using the argument of [Appendix, 6], we have that

$$\delta_1(\kappa_*^1([e])) = 0.$$

This means that $m = 0$, because $\delta_1([q_1]) = [z]$.

If we exchange the roles of α_1 and β_1, we can see that k is also equal to zero. Thus we obtain:

$$\kappa_*^1([e]) = k[1]$$

for some $k \in \mathbb{Z}$.

Let $\tilde{\alpha}_1$, $\tilde{\beta}_1$ and $\tilde{\gamma}$ be the lifts of α_1, β_1 and γ, respectively, so that $\tilde{\alpha}_1$, $\tilde{\beta}_1$ and $\tilde{\gamma}$ have fixed points. As $\gamma = [\beta_1, \alpha_1]$, we see that

$$\tilde{\gamma} = [\tilde{\beta}_1, \tilde{\alpha}_1]\tau_n ,$$

where τ_n is the translation of $\mathbb{R}$ by n, namely, $\tau_n(x) = x + n$ with $n \in \mathbb{Z}$.

Proposition 4. <u>We have</u> $k = n$.

<u>Proof.</u> Let $j \in \mathbb{N}$ be fixed. Consider the j-fold covering $\pi_j : S^1 \longrightarrow S^1$. The diffeomorphisms $\tilde{\gamma}$, $\tilde{\alpha}_1$ and $\tilde{\beta}_1$ descend to diffeomorphisms $\bar{\gamma}$, $\bar{\alpha}_1$ and $\bar{\beta}_1$ of the total space of π_j, respectively. Since $\tilde{\gamma} = [\tilde{\beta}_1, \tilde{\alpha}_1]\tau_n$, we have that

$$\bar{\gamma} = [\bar{\beta}_1, \bar{\alpha}_1]\,\tau(n/j),$$

where $\tau(n/j)$ is the rotation by the angle $(2\pi n)/j$.

Let D be the crossed product $C(S^1) \rtimes \mathbb{Z}/(j)$, where the action is given by the rotation $\tau(1/j)$ with angle $(2\pi)/j$.

It is easy to see that $K_0(D)$ is isomorphic to $\mathbb{Z}$, in particular the class $[1_D]$ of the unit 1_D is torsion-free.

Notice that the diffeomorphisms $\bar{\alpha}_1$ and $\bar{\beta}_1$ commute with $\tau(1/j)$. Let σ be the action of the free group F_2 with two generators, defined by $\bar{\alpha}_1$ and $\bar{\beta}_1$, and let E be the associated reduced crossed product. Using the six-term exact sequence for crossed products by free groups [7], we can see that the class of the unit 1_E of E is not a torsion element of $K_0(E)$.

Let $\Pi = \pi_j^* : C(S^1) \longrightarrow C(S^1)$ be the map induced from π_j. Put $\Pi(\alpha_1) = \bar{\alpha}_1$ and $\Pi(\beta_1) = \bar{\beta}_1$. Then Π extends to a homomorphism from the full crossed product $C(S^1) \times G$ into E. Since G is K-amenable [3], Π induces a homomorphism

$$\Pi_* : K_*(A_G) \longrightarrow K_*(E),$$

and $\Pi_*(\kappa_*^1([e])) = k[1_E]$.

Taking a foundation of $\bar{\gamma}$ which covers that of γ, we construct a Rieffel projection e' corresponding to $\bar{\gamma}$. Then, by the construction, we get:

$$\Pi_*(\kappa_*^1([e])) = j[e'].$$

On the other hand, from the equality $\bar{\gamma} = [\bar{\beta}_1, \bar{\alpha}_1]\tau(n/j)$, it follows that in $K_0(E)$, we have

$$[e'] = n[e''] + i[1_E]$$

for some $i \in \mathbb{Z}$, where e'' is a Rieffel projection corresponding to $\tau(1/j)$. Therefore

$$k[1_E] = jn[e''] + ji[1_E].$$

It is not difficult to see that $j[e''] = [1_D]$ in $K_0(D)$. Consequently, $j[e''] = [1_E]$ in $K_0(E)$. Hence $(k-n)[1_E] = ji[1_E]$. Since $[1_E]$ is not a torsion element of $K_0(E)$, we see that $k-n = ji$. This means that $k - n$ is divisible by j. As j is arbitrary, $k - n = 0$. Q.E.D.

Since $\gamma = [\alpha_2, \beta_2]\cdots[\alpha_g, \beta_g]$, there exists $m \in \mathbb{Z}$ such that

$$\tilde{\gamma} = [\tilde{\alpha}_2, \tilde{\beta}_2]\cdots[\tilde{\alpha}_g, \tilde{\beta}_g]\tau_m.$$

Then as in the proof of Proposition 4, we see that

$$\kappa^2_*([e]) = m[1].$$

We have now an equality

$$[\tilde{\alpha}_1, \tilde{\beta}_1][\tilde{\alpha}_2, \tilde{\beta}_2]\cdots[\tilde{\alpha}_g, \tilde{\beta}_g] = \tau_{(k-m)}.$$

Then, by [8], we have $k - m = 2g - 2$.

Recall that $\varepsilon^1 \cdot \kappa^1 = \varepsilon^2 \cdot \kappa^2$. Therefore,

$$\begin{aligned} k[1_A] &= (\varepsilon^1 \cdot \kappa^1)_*([e]) \\ &= (\varepsilon^2 \cdot \kappa^2)_*([e]) \\ &= m[1_A]. \end{aligned}$$

From this it follows

$$(k-m)[1_A] = (2g-2)[1_A] = 0.$$

Thus we have:

Theorem 5 ([Corollary 6.7, 1]). For the fundamental group $\Gamma_g \subset PSL_2(\mathbb{R})$ of a closed Riemannian surface of genus $g \geq 2$, in the reduced crossed product $A = C(S^1) \rtimes_r \Gamma_g$ the unit 1_A is a torsion element of $K_0(A)$.

5. Epilogue:

The computations carried in the previous sections are exactly what we omitted in [Epilogue, 5].

We briefly review the main result of [5]. Let G be an amalgamated product of countable groups G_1 and G_2 along G_0, namely,

$$G = G_1 *_{G_0} G_2 .$$

Let (A, α, G) be a C*-dynamical system. In [5] we showed that if one of (G_1, G_0) and (G_2, G_0) has property Λ, then there exists a six-term exact sequence:

$$\begin{array}{ccccc} K_0(A\rtimes_r G_0) & \longrightarrow & K_0(A\rtimes_r G_1) \oplus K_0(A\rtimes_r G_2) & \longrightarrow & K_0(A\rtimes_r G) \\ \uparrow & & & & \downarrow \\ K_1(A\rtimes_r G) & \longleftarrow & K_1(A\rtimes_r G_1) \oplus K_1(A\rtimes_r G_2) & \longleftarrow & K_1(A\rtimes_r G_0). \end{array}$$

Let $\Gamma_g = G *_H S$ be as in the preceding sections. So far, it is not known whether (G,H) or (S,H) has property Λ.

We have the following complex of abelian groups:

$$K_*(A_H) \xrightarrow{\kappa_*^1 - \kappa_*^2} K_*(A_G) \oplus K_*(A_S) \xrightarrow{\varepsilon_*^1 + \varepsilon_*^2} K_*(A_{\Gamma_g}).$$

As we have seen in Section 4, the maps $\kappa_*^1 - \kappa_*^2 : K_0(A_H) \to K_0(A_G) \oplus K_0(A_S)$ is given by the following matrix:

$${}^t\begin{pmatrix} 1 & 0 & 0 & -1 & 0\cdots0 \\ m & 0 & 0 & 2g-2-m & 0\cdots0 \end{pmatrix} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \oplus \mathbb{Z}^{2g-1}.$$

It is rather easier to describe $\kappa_*^1 - \kappa_*^2 : K_1(A_H) \to K_1(A_G) \oplus K_1(A_S)$. This map is given by

$${}^t\begin{pmatrix} 1 & 0 & 0 & -1 & 0\cdots0 \\ 0 & 0 & 0 & 0 & 0\cdots0 \end{pmatrix} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \oplus \mathbb{Z}^{2g-1}.$$

Now assume the existence of a six-term exact sequence given in [Thm. A1, 5], for the C*-dynamical system $(C(S^1), \Gamma_g, \alpha)$. Then

$$K_0(C(S^1) \rtimes_r \Gamma_g) \simeq \mathbb{Z}^{2g+1} \oplus \mathbb{Z}/(2g-2), \text{ and}$$

$$K_1(C(S^1) \rtimes_r \Gamma_g) \simeq \mathbb{Z}^{2g+1}.$$

On the other hand, as $C(S^1) \rtimes_r \Gamma_g$ is stably isomorphic to the C*-algebra of an Anosov foliation on the unit tangent bundle of M_g, we can directly compute $K_*(C(S^1) \rtimes_r \Gamma_g)$ by using the Thom isomorphism (cf., [1]). The result coincides with the one above.

To conclude this section, we give the following:

Problem. Show that (G,H) or (S,H) has property Λ.

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