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On the Topological Structure of Tensor Algebras and the Closure of the Cone of Positive Elements

by

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0. Introduction

One motivation for the study of tensor algebras comes from quantum field theory because every Garding-Wightman field (/6/) describes a Wightman functional (i.e. a positive, Poincaré invariant, continuous linear functional on the tensor algebra over the Schwartz space $\mathcal{S}(\mathbb{R}^4)$, /22/), and vice versa every Wightman functional gives a Garding-Wightman field by /3/,/23/.

This paper is organized as follows. The definition of tensor algebras and some algebraic properties of them are given in Section 1. In Section 2 we introduce locally convex (l.c.) topologies on tensor algebras $E_{\otimes}$ over a l.c. space $E[t]$, discuss the order relations between these topologies and their connection with the topological structure of $E[t]$ (Theorem 2.1), and list some properties of these topologies for the case that $E[t]$ is a Frechet space (Theorem 2.3).

The third point of this paper is aimed at the structure of the cone of positive elements $E_{\otimes}^{+}$, (3.1, 3.2), and at its

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topological closure, (3.3). Finally in Sect.4 the results are illustrated by some examples.
For the definitions and concepts from the theory of topological vector spaces and ordered vector spaces used in the following we refer to /19/.

1. Definition and basic properties of tensor algebras

1.1
Let E be a vector space over $\mathbb{C}$ (the complex plane) with an involution *, i.e. antilinear mapping $f \longmapsto f^*$ with $f^{**}=f$ for all $f \in E$. Then let us put

$E_{\otimes}:=E_o \oplus E_1 \oplus E_2 \oplus \ldots$, $E_o:=\mathbb{C}$, $E_n:= E \otimes \ldots \otimes E$ (the n-fold algebraic tensor product).

Thus the elements $f \in E_{\otimes}$ are terminating sequences

$$f=(f_o,f_1,\ldots,f_N,0,0,\ldots),\ f_i \in E_i,\ i=0,1,2,\ldots .$$

Let us define componentwise the following algebraic operations on $E_{\otimes}$:

$$(f+g)_n= f_n + g_n ,$$

$$(fg)_n = f_o g_n + f_1 \otimes g_{n-1} +\ldots+ f_{n-1} \otimes g_1 + f_n g_o ,$$

$$(f^*)_n=(f_n)^*= \sum_{\substack{i_1\ldots i_n \\ \text{finite}}} \overline{\alpha}_{i_1\ldots i_n} e^{(i_n)*} \otimes e^{(i_{n-1})*} \otimes \ldots \otimes e^{(i_1)*}$$

for $f_n= \sum_{i_1\ldots i_n} \alpha_{i_1\ldots i_n} e^{(i_1)} \otimes \ldots \otimes e^{(i_n)}$, $e^{(i_j)} \in E$, $j=1,\ldots,n$,
$\alpha_{i_1\ldots i_n} \in \mathbb{C}$, $\overline{\alpha}$ denotes the conjugate complex value of α ,

with $f,g \in E_{\otimes}$, $n=0,1,2,\ldots$.
Thus $E_{\otimes}$ becomes a *-algebra with unity $\mathbb{1}=(1,0,0,\ldots)$.
For $f=(0,\ldots,0,f_L,\ldots,f_N,0,0,\ldots) \in E_{\otimes}$, $f_L \neq 0$, $f_N \neq 0$ let us put

$$\mathrm{Grad}(f) = \begin{cases} N & \text{if } f \neq \mathbb{0}=(0,0,\ldots) \\ -\infty & \text{if } f= \mathbb{0} \end{cases},\quad \mathrm{grad}(f) = \begin{cases} L & \text{if } f \neq \mathbb{0} \\ \infty & \text{if } f = \mathbb{0} \end{cases}.$$

Then one sees readily

$$\begin{aligned} &\mathrm{Grad}(fg)=\mathrm{Grad}(f)+\mathrm{Grad}(g),\quad \mathrm{grad}(fg)=\mathrm{grad}(f)+\mathrm{grad}(g), \\ &\mathrm{Grad}(f+g) \leq \max\{\mathrm{Grad}(f),\mathrm{Grad}(g)\}, \\ &\mathrm{grad}(f+g) \geq \min\{\mathrm{grad}(f),\mathrm{grad}(g)\}, \end{aligned} \tag{1}$$

$\mathrm{Grad}(f^*)=\mathrm{Grad}(f)$, $\mathrm{grad}(f^*)=\mathrm{grad}(f)$,

for $f,g\in E_\otimes$. If $f\neq g$ then the "="-sign occurs in (1).

Let be

$$E_\otimes^+ = \left\{\sum_{i=1}^{M} a^{(i)*}\, a^{(i)};\ a^{(i)}\in E_\otimes,\ M\in\mathbb{N}\right\}$$

the cone of positive elements in $E_\otimes$. $h(E_\otimes):=\{f\in E_\otimes;\ f=f^*\}$ is the hermitean part of $E_\otimes$. $h(E_\otimes)$ is a vector space over $\mathbb{R}$, (the real numbers). Then one gets the decomposition

$$E_\otimes = h(E_\otimes) + i\, h(E_\otimes),\ i \text{ denotes the imaginary unit},$$

by $f=f^{(1)}+i\, f^{(2)}$ with $f^{(1)}=\frac{1}{2}(f+f^*)\in h(E_\otimes)$, $f^{(2)}=\frac{1}{2}(f^*-f)\in h(E_\otimes)$.

Some properties concerning the algebraic structure of $E_\otimes$ are listed in the following

Statement 1.1:

i) $E_\otimes$ is a commutative *-algebra iff $\dim(E)=1$, ($\dim(E)$ denotes the dimension of E).

ii) It is

$$Z(E_\otimes)=\{f\in E_\otimes; fg=gf \text{ for all } g\in E_\otimes\}=\begin{cases} E_\otimes & \text{if } \dim(E)=1 \\ \mathbb{C} & \text{otherwise}\end{cases}$$

for the centre of $E_\otimes$.

iii) $E_\otimes$ has no divisors of zero.

iv) The only invertible elements of $E_\otimes$ are the elements from $\mathbb{C}\setminus\{0\}$.

v) 0, 1 are the only idempotent elements in $E_\otimes$.

vi) $E_\otimes$ has no minimal ideals.

vii) $E_\otimes$ is semisimple.

These properties were proved for $E=\mathcal{S}(\mathbb{R}^4)$ by Borchers and Wyss, /4/,/25/. The proof of Statement 1.1 is in analogy to that of $E=\mathcal{S}(\mathbb{R}^4)$.

2. Topologies on $E_\otimes$

2.1

Let $E[t]$ be a l.c. vector space, i.e. there is a system of seminorms $\mathcal{P}(t)=\{p_\alpha;\ \alpha\in A\}$, A is a directed set of indexes, describing the topology t.

Following Schatten, Grothendick, Pietsch, (/20/,/8/,/17/), there are the following three important topologies on E_n, $n=2,3,\ldots$:

i) The injective topology ε_n given by the system of seminorms

$$f_n \longrightarrow \check{p}_{\alpha_1\ldots\alpha_n}(f_n)=\sup\Big\{\Big|\sum_{i_1\ldots i_n} T^{(1)}(g^{(i_1)})\ldots T^{(n)}(g^{(i_n)})\Big| ;$$
$$|T^{(1)}(.)|\le p_{\alpha_1}(.),\ldots |T^{(n)}(.)|\le p_{\alpha_n}(.)\Big\}$$
$$f_n=\sum_{\substack{i_1\ldots i_n\\ \text{finite}}} g^{(i_1)}\otimes\ldots\otimes g^{(i_n)}\in E_n,\ \alpha_i\in A.$$

ii) The projective topology π_n given by

$$f_n \longrightarrow \hat{p}_{\alpha_1\ldots\alpha_n}(f_n)= \inf\Big\{\sum_{i_1\ldots i_n} p_{\alpha_1}(h^{(i_1)})\ldots p_{\alpha_n}(h^{(i_n)});$$
$$f_n=\sum_{\substack{i_1\ldots i_n\\ \text{finite}}} h^{(i_1)}\otimes\ldots\otimes h^{(i_n)}\Big\}$$

π_n is also the strongest l.c. topology on E_n such that its topological dual is topological isomorphic to the jointly continuous multilinear forms $B(E_n)$ on E_n;

$(E_n[\pi_n])' \underset{\text{top.}}{\cong} B(E_n)$.

iii) The inductive topology i_n is defined as the strongest l.c. topology on E_n, such that $(E_n[i_n])' \underset{\text{top}}{\cong} B_s(E_n)$, where $B_s(E_n)$ denotes the separately continuous multilinear forms on E_n.

Let $\tau \prec \tau'$ denote $\mathcal{P}(\tau)\subset\mathcal{P}(\tau')$, that means the l.c. topology τ' is stronger (finer) than the l.c. topology τ respectively τ is weaker (coarser) than τ'.

Then $\varepsilon_n \prec \pi_n \prec i_n$, n=2,3,..., follows immediately.

Now let us define l.c. topologies on $E_\otimes$ connected with $E_n[\varepsilon_n]$. We denote by $\varepsilon_\otimes$ the topology of the direct sum of the spaces $E_n[\varepsilon_n]$ and by $\varepsilon_{\mathcal{P}}$ the restriction of the topology of the direct product $\bigotimes_{n=0}^{\infty} E_n[\varepsilon_n]$ to its subspace $E_\otimes$.

Then $\varepsilon_\otimes$ (resp. $\varepsilon_{\mathcal{P}}$) is the strongest (resp. weakest) l.c. topology on $E_\otimes$ with $\varepsilon_\otimes|_E=t$, $\varepsilon_\otimes|_{E_n}=\varepsilon_n$, (resp. $\varepsilon_{\mathcal{P}}|_E=t$, $\varepsilon_{\mathcal{P}}|_{E_n}=\varepsilon_n$) n=2,3,..., and $\tau|_G$ denotes the restriction of the topology τ to a subspace G.

A further important topology is ε_∞ defined as the strongest l.c. topology on $E_\otimes$ such that the multiplication $f,g \longrightarrow fg$ is jointly continuous as mapping

$E_{\otimes}[\varepsilon_\infty] \times E_{\otimes}[\varepsilon_\infty] \longrightarrow E_{\otimes}[\varepsilon_\infty]$.

The topology ε_∞ was introduced by Lassner /13/.

Let $\mathbb{N}^{\mathbb{N}}$ denote the set of all sequences $(\gamma_n)_{n=o}^{\infty}$ of natural numbers including 0, $A^m = A \times \ldots \times A$ (m times), $m=1,2,\ldots$, $A^o=\{1\}$, $\bigotimes_{n=o}^{\infty} A^n$ the set of all sequences $(\nu^n)_{n=o}^{\infty}$, $\nu^n \in A^n$, and $A_\infty := \{(\nu^n)_{n=o}^{\infty} = (\nu,\nu,\ldots,\nu)_{n=o}^{\infty} \in \bigotimes_{n=o}^{\infty} A^n;\ \nu \in A\}$.

Then the above introduced topologies can be given by the following systems of seminorms:

$$\mathcal{P}(\varepsilon_\otimes) = \{f \to \check{p}_{(\gamma_n)(\nu^n)}(f) := \sum_{n=o}^{\infty} \gamma_n \check{p}_{(\nu^n)}(f_n);\ (\gamma_n) \in \mathbb{N}^{\mathbb{N}},\ (\nu^n) \in \bigotimes_{n=o}^{\infty} A^n\},$$

$$\mathcal{P}(\varepsilon_\infty) = \{f \to \check{p}_{(\gamma_n)(\nu^n)}(f) := \sum_{n=0}^{\infty} \gamma_n \check{p}_{(\nu^n)}(f_n);\ (\gamma_n) \in \mathbb{N}^{\mathbb{N}},\ (\nu^n) \in A_\infty\},$$

$$\mathcal{P}(\varepsilon_p) = \{f \to \check{p}_{(\nu^n)}(f_n);\ n=0,1,\ldots,\ (\nu^n) \in A^n\},$$

$f=(f_0,f_1,\ldots,f_N,0,0,\ldots) \in E_\otimes$, $p_{(\nu^o)}(f_o) = |f_o|$.

We get readily $\varepsilon_p \prec \varepsilon_\infty \prec \varepsilon_\otimes$.

Analogously we define the topologies $\pi_p, \pi_\infty, \pi_\otimes, i_p, i_\infty, i_\otimes$.

2.2

It is obvious that the following order relations between the topologies defined in 2.1 are valid:

$$\begin{array}{ccccc} \varepsilon_p & \prec & \varepsilon_\infty & \prec & \varepsilon_\otimes \\ \curlywedge & & \curlywedge & & \curlywedge \\ \pi_p & \prec & \pi_\infty & \prec & \pi_\otimes \\ \curlywedge & & \curlywedge & & \curlywedge \\ i_p & \prec & i_\infty & \prec & i_\otimes \end{array}$$

A connection between the coincidence of some of these topologies and the topological structure of $E[t]$ is given by the

Theorem 2.1:

a) $E[t]$ is normable iff one of the following equivalent conditions $\varepsilon_\infty = \varepsilon_\otimes, \pi_\infty = \pi_\otimes, i_\infty = i_\otimes$ is satisfied.

b) If $E[t]$ is nuclear then $\varepsilon_p = \pi_p$, $\varepsilon_\infty = \pi_\infty$, $\varepsilon_\otimes = \pi_\otimes$. Conversely, if there is a system of Hilbertian seminorms describing the topology t then $\varepsilon_p = \pi_p$ or $\varepsilon_\infty = \pi_\infty$ or $\varepsilon_\otimes = \pi_\otimes$ implies the nuclearity of $E[t]$.

c) Every separately continuous multilinear form on E_n, $n=2,3,\ldots$, is jointly continuous iff $i_p = \pi_p$ or $i_\infty = \pi_\infty$ or $i_\otimes = \pi_\otimes$.

The proof of this theorem is contained in /12/.

Let us make the following

Remarks to Theorem 2.1:

i) Pisier (/18/) constructed an example of an infinite dimensional Banach space B with the property $\varepsilon_2 = \pi_2$. Because B is not nuclear that example indicates the need of a further assumption for $\mathcal{P}(t)$ to prove the second statement of Theorem 2.1 b).

ii) The assertions of Theorem 2.1 can be illustrated by the following figure.

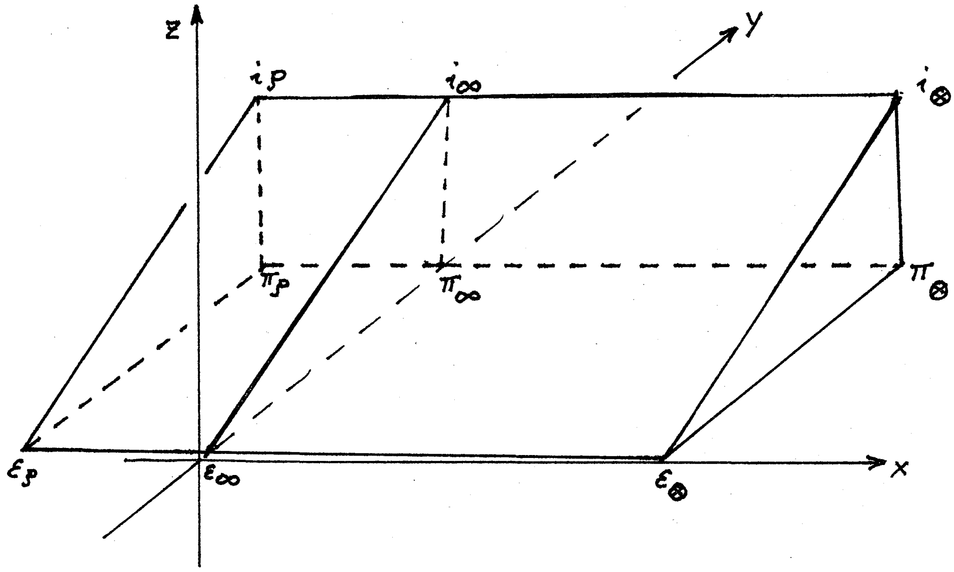

Fig.1

Every point of the wedge ε_p, π_p, i_p, $\varepsilon_\otimes$, $\pi_\otimes$, $i_\otimes$ illustrates a l.c. topology on $E_\otimes$ and the semiordering "$<$" between these topologies is given by the cone $\{(x,y,z);\ x \geq 0,\ y \geq 0,\ z \geq 0\}$.

If the assertion a) (b) resp. c)) is valid then we have to carry out the orthogonal projection into the yz-plane (xz-plane resp. xy-plane) in Fig.1.

That means that this wedge of topologies collapses to the smaller wedge with the corners ε_p, π_p, i_p, $\varepsilon_\infty = \varepsilon_\otimes$, $\pi_\infty = \pi_\otimes$, $i_\infty = i_\otimes$ (to the rectangle $\varepsilon_p = \pi_p$, $\varepsilon_\otimes = \pi_\otimes$, $i_\otimes$, i_p in the xz-plane resp. to the rectangle ε_p, $\varepsilon_\otimes$, $\pi_\otimes = i_\otimes$, $\pi_p = i_p$ in the x,y-plane) iff the assumptions of Theorem 2.1 a) (b) resp. c)) are valid.

An easy consequence of Theorem 2.1 is the

Corollary 2.2:

i) If $E[t]$ is finite dimensional then
$\varepsilon_p = \pi_p = i_p \lneq \varepsilon_\infty = \pi_\infty = i_\infty = \varepsilon_\otimes = \pi_\otimes = i_\otimes$.

ii) If $E[t]$ is a Frechet space or an LB-space (i.e. strict inductive limit of Banach spaces) then $\pi_p = i_p$, $\pi_\infty = i_\infty$, $\pi_\otimes = i_\otimes$.

iii) If $E[t]$ is a nuclear Frechet space then $\varepsilon_p = \pi_p = i_p$, $\varepsilon_\infty = \pi_\infty = i_\infty$, $\varepsilon_\otimes = \pi_\otimes = i_\otimes$.

Proof; i) If $E[t]$ is finite dimensional then the assumptions of Theorem 2.1 a), b), c) are satisfied and thus assertion i) follows.
ii) follows by /19; III.5.1/ and the definition of i_n, $n=2,3,\ldots$.
iii) is a consequence of Theorem 2.1 b) and Corollary 2.2 ii).

2.3
In the following let $\overline{M}^\tau$ denote the closure of a set M with respect to the l.c. topology τ .
Let $E_{\hat\otimes}$ ($\hat{E}_n$, $n=2,3,\ldots$, resp. $\hat{E}$) denote the completion of $E_\otimes[\pi_\otimes]$ ($E_n[\pi_n]$ resp. $E[t]$). Then

$$E_{\hat\otimes} = \mathbb{C} \oplus \hat{E} \oplus \hat{E}_2 \oplus \hat{E}_3 \oplus \ldots$$

follows by /19; II.6.2/. Further a set $M \subset E_{\hat\otimes}$ is called graded if $\{Q^{(m)}f;\ f \in M\} \subset M$ for all $m=0,1,2,\ldots$, where

$$Q^{(m)}(f_o,f_1,\ldots,f_m,\ldots,f_N,0,0,\ldots) := (f_o,f_1,\ldots,f_m,0,0,\ldots) .$$

If $E[t]$ is a Frechet space then the following topological properties of $E_{\hat\otimes}$ are valid.

Theorem 2.3:

a) Let $E[t]$ be a Frechet space and τ a l.c. topology on $E_{\hat\otimes}$ with $\pi_p < \tau < \pi_\otimes$. Then
i) $\overline{M}^{\pi_p} = \overline{M}^\tau = \overline{M}^{\pi_\otimes}$ holds for every graded set $M \subset E_{\hat\otimes}$;
ii) $E_{\hat\otimes}$ is barrelled iff $\tau = \pi_\otimes$.

b) Let $E[t]$ be a Frechet space containig a continuous norm and η a l.c. topology on $E_{\hat\otimes}$ with $\pi_\infty < \eta < \pi_\otimes$. Then
i) if there is a base of neighborhoods describing η and containing graded sets only then $E_{\hat\otimes}[\eta]$ is complete.

ii) $E_{\hat{\otimes}}[\eta]$ and $E_{\hat{\otimes}}[\pi_{\otimes}]$ have the same bounded sets.
iii) $E_{\hat{\otimes}}[\eta]$ is bornological iff $\eta = \pi_{\otimes}$.

The proof of this theorem is contained in /12/. Let us make some

Remarks:

i) If $E[t]$ is a Frechet space which has no continuous norm then $E_{\hat{\otimes}}[\pi_{\infty}]$ is not complete, /12/.

ii) In /12/ there is also an example of a l.c. topology η^*, $\pi_{\infty} < \eta^* < \pi_{\otimes}$ having nongraded neighborhoods in every base with the property that $E_{\hat{\otimes}}[\eta^*]$ is not complete.

i),ii) show that one connot spare the additional assumption of Theorem 2.3 b).

3. On the cone of positive elements

3.1

We show some basic facts on the cone of positive elements $E_{\otimes}^{+}$ in this section. Let us remark that all considerations are also valid if we replace $E_{\otimes}$ and $E_{\otimes}^{+}$ by $E_{\hat{\otimes}}$ and

$$E_{\hat{\otimes}}^{+}=\{\sum_{i=1}^{M} a^{(i)}*a^{(i)};\ a^{(i)}\in E_{\hat{\otimes}},\ M\in \mathbb{N}\}.$$

A linear functional T on $E_{\otimes}$ is called positive if $T(g)\geqslant 0$ for all $g\in E_{\otimes}^{+}$. Further a linear functional S on a complex vector space F with an involution * is called hermitean if

$$S(f^*) = \overline{S(f)},$$

and $\overline{S(f)}$ denotes the conjugate complex value of $S(f)$.
Then every positive linear functional is hermitean and satisfies the Cauchy-Schwarz inequality

$$|T(f^*g)|^2 \leq T(f^*f)\ T(g^*g), \quad (/16/).$$

Further there is an isomorphism between the set of linear hermitean functionals $L^*(E_{\otimes},\mathbb{C})$ on $E_{\otimes}$ and the set of real linear functionals $L(h(E_{\otimes}),\mathbb{R})$ on the real vector space $h(E_{\otimes})$ given by $\varkappa: L^*(E_{\otimes},\mathbb{C}) \longrightarrow L(h(E_{\otimes}),\mathbb{R})$ with

$$\varkappa T=T,\quad T\in L^*(E_{\otimes},\mathbb{C}),$$

$$(\varkappa^{-1}L)(f)=L(f^{(1)})+iL(f^{(2)}),\quad L\in L(h(E_{\otimes}),\mathbb{R}),$$

$$f^{(1)}= \tfrac{1}{2}(f+f^*),\quad f^{(2)}= \tfrac{i}{2}(f^*-f).$$

Because of the duality of the direct sum and the direct product of l.c. spaces (/19; IV.4/) every linear functional T on $E_\otimes$ can be written as $T=(T_0,T_1,T_2,\ldots)$, and T_j is a linear functional on E_j, $j=0,1,2,\ldots$.

The following lemma is important for the proof of the theorem of this section.

Lemma 3.1:

Let be $0\neq k\in E_\otimes^+$. Then

i) grad(k) and Grad(k) are even numbers;

ii) if grad(k)=2n, Grad(k)=2N then $(T_n\otimes T_n)(k_{2n})\geq 0$, $(T_N\otimes T_N)(k_{2N})\geq 0$ hold for every hermitean linear functional T_n on E_n and T_N on E_N;

iii) there are hermitean linear functionals T_n^o on E_n and T_N^o on E_N with $(T_n^o\otimes T_n^o)(k_{2n})>0$, $(T_N^o\otimes T_N^o)(k_{2N})>0$.

Proof: We are giving the proof for the highest nonvanishing component, i.e. we regard Grad(k). The corresponding proofs for grad(k) are analogously.

i) Let be $k=\sum_{i=1}^{M} a^{(i)}{*}a^{(i)}\in E_\otimes^+$, $a^{(i)}\in E_\otimes$, $N:=\max\{\mathrm{Grad}(a^{(1)}),\ldots$ $\ldots,\mathrm{Grad}(a^{(M)})\}$, $a_N^{(1)},\ldots,a_N^{(M')}\neq 0$, $a_N^{(M'+1)}=a_N^{(M'+2)}=\ldots=a_N^{(M)}=0$, $M'\in\mathbb{N}$, $1\leq M'\leq M$, and $\{a_N^{(1)},\ldots,a_N^{(M')}\}$ linear independent.

Now let us assume $k_{2N}=\sum_{i=1}^{M'} a_N^{(i)}{*}a_N^{(i)}=0$.

The linear independence of $\{a_N^{(1)},\ldots,a_N^{(M')}\}$ implies the linear independence of $\{a_N^{(1)*},\ldots,a_N^{(M')*}\}$ and thus $k_{2N}=0$ yields $a_N^{(1)}=\ldots=a_N^{(M')}=a_N^{(1)*}=\ldots=a_N^{(M')*}=0$. Thus we have

$k_{2N-1}=\sum_{i=1}^{M}(a_N^{(i)*}\otimes a_{N-1}^{(i)}+a_{N-1}^{(i)*}\otimes a_N^{(i)})=0$ which proves that Grad(k) is even.

ii) It is

$$(T_N\otimes T_N)(k_{2N})=(T_N\otimes T_N)\Big(\sum_{i=1}^{M} a_N^{(i)*}\otimes a_N^{(i)}\Big)=\sum_{i=1}^{M} T(a_N^{(i)*})T(a_N^{(i)})$$
$$=\sum_{i=1}^{M}|T_N(a_N^{(i)})|^2\geq 0.$$

iii) Let be $a_N^{(1)}\neq 0$. Then $a_N^{(1)*}+a_N^{(1)}\neq 0$ or $a_N^{(1)*}-a_N^{(1)}\neq 0$, and thus there is a real linear functional L_N^o on $h(E_N)=\{f_N\in E_N;\ f_N^*=f_N\}$ with $L_N^o(a_N^{(1)*}+a_N^{(1)})\neq 0$ or $L_N^o(i(a_N^{(1)*}-a_N^{(1)}))\neq 0$. Then

$$(T_N^o \otimes T_N^o)(k_{2N}) \geq |T_N^o(a_N^{(1)})|^2 = |\tfrac{1}{2}L_N^o(a_N^{(1)*}+a_N^{(1)})+\tfrac{i}{2}L_N^o(i(a_N^{(1)*}-a_N^{(1)}))|^2$$
$$> 0 \quad \text{holds for } T_N^o = \chi^{-1}L_N^o .$$

Some basic facts on $E_\otimes^+$ are stated in

<u>Theorem 3.2:</u>

a) $E_\otimes^+$ is a proper cone, i.e. $k,-k \in E_\otimes^+$ imply $k=0$.

b) $E_\otimes^+$ is generating for $h(E_\otimes)$, i.e. $h(E_\otimes)=\{k^{(1)}-k^{(2)};\ k^{(1)}, k^{(2)} \in E_\otimes^+\}$.

c) $E_\otimes^+$ has no topological interior points with respect to every l.c. topology τ with $\varepsilon_p < \tau < i_\otimes$.

d) $E_\otimes^+$ is not a lattice cone in $h(E_\otimes)$.

Proof: a) is an immediate consequence of Lemma 3.1 ii), iii).
b) Because of $(\mathbb{1}+f)^*(\mathbb{1}+f)-(\mathbb{1}-f)^*(\mathbb{1}-f)=2(f+f^*)$, $f\in E_\otimes$, and $h(E_\otimes)=\{f+f^*;\ f\in E_\otimes$ assertion b) is valid.
c) To every $k\in E_\otimes^+$ there is a τ-neighborhood U of zero and an $u\in U$ with $\mathrm{Grad}(u)=2s+1 > \mathrm{Grad}(k)$, $s\in\mathbb{N}$. Then $\mathrm{Grad}(k+u)=2s+1$ and thus $k+u \notin E_\otimes^+$ because of Lemma 3.1 i).
d) Let be $[x,y]:=\{f\in h(E_\otimes);x\leq f\leq y\};=\{f\in h(E_\otimes);\ f-x\in E_\otimes^+, y-f\in E_\otimes^+\}$, $x,y\in h(E_\otimes)$, the orderintervall generated by the cone $E_\otimes^+$.
Let be $r,s\in\mathbb{N}$, $r<s$, $0\neq g_r\in E_r$, $0\neq g_s\in E_s$, $a=(0,\dots,0,g_r,0,\dots,0,-g_s,0,0,\dots)$, $b=(0,\dots,0,g_r,0,\dots,0,g_s,0..)$ and $u=(0,\dots,0,2g_r^*\otimes g_r,0,\dots)$, $v=(0,\dots,0,2g_s^*\otimes g_s,0,\dots)\in E_\otimes^+$.
Then $a^*a+b^*b=(0,\dots,0,2g_r^*\otimes g_r,0,\dots,0,2g_s^*\otimes g_s,0,0,\dots)$ and

$$a^*a \in [\phi, a^*a+b^*b] = [\phi, u+v] \tag{1}$$

follow. But on the other side $f=(f_o,\dots,f_N,0,\dots)\in[\phi,u]$ resp. $h=(h_o,\dots,h_{N'},0,\dots)\in[\phi,v]$ implies $f_i=0$ for $i\neq 2r$ resp. $h_j=0$ for $j\neq 2s$. This yields $a^*a\notin[\phi,u]+[\phi,v]$, and thus because of (1) the Riesz decomposition property is not valid. Then d) follows by /19;V.1.1/.

The assertions of Theorem 3.2 a),b),c) were proved for $E=\mathcal{S}(\mathbb{R}^4)$ at first by Borchers and Wyss (/4/,/25/).

3.2

Now let us regard a net $k^{(\beta)}=(k_o^{(\beta)},\dots,k_{2N}^{(\beta)},0,0,\dots)\in E_{\hat\otimes}^+$, $\beta\in B$, B is a directed set of indexes, and $k_{2N}^{(\beta)} \longrightarrow 0$ with respect to π_{2N}. Then $k_{2N-1}^{(\beta)} \xrightarrow[\pi_{2N-1}]{} 0$ follows because of Lemma 3.1 i).

This indicates that the components k_i of an element

$k=(k_o,k_1,\ldots,k_{2N},0,\ldots)\in E_{\otimes}^{+}$ are not independent of each other. The aim of this section is to give a quantitative estimation of this dependency.

Let us write $\tilde{f}_n$ for $(0,\ldots,0,f_n,0,\ldots)$, $f_n\in E_n$, $n=0,1,\ldots$. Then we say a mapping $\pounds\colon E_x\longrightarrow\mathbb{C}$ has the property (A) if the following three conditions are fulfilled:

(A_i) $|\pounds(f+g)|\leq|\pounds(f)|+|\pounds(g)|$ for all $f,g\in E_{\otimes}$; $\pounds(\phi)=0$;

(A_{ii}) $|\pounds(\sum_{i=1}^{M}\tilde{a}_n^{(i)}*\tilde{a}_n^{(i)})|\geq 0$ for all $\tilde{a}_n^{(i)}\in E_{\otimes}$, $M\in\mathbb{N}$;

(A_{iii}) $|\pounds(\pm\sum_{i=1}^{M}\tilde{a}_n^{(i)}*\tilde{a}_m^{(i)})|^2\leq\pounds(\sum_{i=1}^{M}\tilde{a}_n^{(i)}*\tilde{a}_n^{(i)})\pounds(\sum_{i=1}^{M}\tilde{a}_m^{(i)}*\tilde{a}_m^{(i)})$

for all $\tilde{a}_n^{(i)}$, $\tilde{a}_m^{(i)}\in E_{\otimes}$, $n,m\in\mathbb{N}$.

Further let us put

$$L_n^{\pounds}(k):=(\pounds(\sum_{i=1}^{M}\tilde{a}_n^{(i)}*\tilde{a}_n^{(i)}))^{1/2},\quad [\![k]\!]^{\pounds}:=\sum_{n=o}^{\infty}2^{4^n}|\pounds(\tilde{k}_{2n})|$$

for $k=(k_o,\ldots,k_{2N},0,0,\ldots)=\sum_{i=1}^{M}a^{(i)}*a^{(i)}$, $a^{(i)}\in E_{\otimes}$.

If there is no possibility of confusion then let us write $[\![k]\!]$, L_n instead of $[\![k]\!]^{\pounds}$, $L_n^{\pounds}(k)$.

Some relations between $\pounds$, L_n and $[\![\,.\,]\!]$ are proved in the technical

<u>Lemma 3.3:</u>

Let $\pounds$ have the property (A), $k\in E_{\otimes}^{+}$, $n\in\mathbb{N}$. Then

i) $|\pounds(\tilde{k}_n)|\leq\sum_{j=o}^{n}L_{n-j}L_j$;

ii) $L_n^2-2\sum_{j=1}^{n}L_{n+j}L_{n-j}\leq|\pounds(\tilde{k}_{2n})|$;

iii) $\sum_{n=o}^{\infty}(L_n)^2\leq[\![k]\!]$.

Proof: i) is a consequence of

$$|\pounds(\sum_{i=1}^{M}\sum_{r+s=n}\tilde{a}_r^{(i)}*\tilde{a}_s^{(i)})|\overset{(A_i)}{\leq}\sum_{r+s=n}|\pounds(\sum_{i=1}^{M}\tilde{a}_r^{(i)}*\tilde{a}_s^{(i)})|$$

$$\overset{(A_{ii})}{\leq}\sum_{r+s=n}(\pounds(\sum_{i=1}^{M}\tilde{a}_r^{(i)}*\tilde{a}_r^{(i)})\pounds(\sum_{i=1}^{M}\tilde{a}_s^{(i)}*\tilde{a}_s^{(i)}))^{1/2}=\sum_{r+s=n}L_r L_s .$$

ii) follows from

$$|\pounds(\tilde{k}_{2n})|=|\pounds(\sum_{r+s=2n}\sum_{i=1}^{M}\tilde{a}_r^{(i)}*\tilde{a}_s^{(i)})|\overset{(A_i)}{\geq}\pounds(\sum_{i=1}^{M}\tilde{a}_n^{(i)}*\tilde{a}_n^{(i)})-$$

$$-\sum_{\substack{r+s=2n\\ r\neq s}}|\pounds(-\sum_{i=1}^{M}\tilde{a}_r^{(i)}*\tilde{a}_r^{(i)})|\overset{(A_{iii})}{\geq}\pounds(\sum_{i=1}^{M}\tilde{a}_n^{(i)}*\tilde{a}_n^{(i)})-$$

$$-\sum_{\substack{r+s=2n\\ r\neq s}}(\pounds(\sum_{i=1}^{M}\tilde{a}_r^{(i)}*\tilde{a}_r^{(i)})\pounds(\sum_{i=1}^{M}\tilde{a}_s^{(i)}*\tilde{a}_s^{(i)}))^{1/2}=(L_n)^2-2\sum_{j=1}^{n}L_{n+j}L_{n-j}.$$

iii) Because of

$$\|k\|^{\pounds}=\sum_{n=o}^{\infty} 2^{4^n}|\pounds(\tilde{k}_{2n})| \overset{(A_{ii})}{\geq} \sum_{n=0}^{\infty} 2^{4^n}((L_n)^2-\sum_{j=1}^{n} L_{n+j}\, L_{n-j})$$

we have to show the matrix inequality $\mathbb{A}\geq\mathbb{I}$ with $\mathbb{I}=(\delta_{ij})_{i,j=o}^{\infty}$, $\mathbb{A}=(a_{ij})_{i,j=o}^{\infty}$,

$$\delta_{ij}=\begin{cases}1 & \text{for } i=j\\ 0 & \text{for } i\neq j\end{cases}, \quad a_{ij}=\begin{cases}2^{4^j} & \text{for } i=j\\ 0 & \text{for } i+j \text{ is odd}\\ -2^{4^s} & \text{for } i+j=2s,\ s=1,2,..\end{cases},$$

i.e. $\sum_{i,j=o}^{\infty} a_{ij}\bar{l}_i l_j \geq \sum_{i,j=o}^{\infty}\delta_{ij}\bar{l}_i l_j$ for every terminating sequence $(l_o,l_1,\ldots,l_N,0,0,\ldots)$, $l_i\in\mathbb{C}$.

By repeated application of

$$\sum_{i,j=o}^{n} a_{ij}\bar{l}_i l_j=\sum_{i,j=o}^{n-1} a_{ij}\bar{l}_i l_j + a_{nn}|l_n|^2 + 2RE(\sum_{j=o}^{n-1} a_{nj}\bar{l}_n l_j)$$

$$\geq \sum_{i,j=o}^{n-1} a_{ij}\bar{l}_i l_j + a_{nn}|l_n|^2-c_n|l_n|^2-c_n^{-1}|\sum_{j=o}^{n-1} a_{nj}l_j|^2$$

$$\geq \sum_{i,j=o}^{n-1} a_{ij}\bar{l}_i l_j-x_n(\sum_{j=o}^{n-1}|l_j|^2)+(a_{nn}-c_n)|l_n|^2,$$

with $c_n>0$, $x_n=c_n^{-1}\sum_{j=o}^{n-1}|a_{nj}|^2$, RE(.) denotes the real part of (.), we get

$$\sum_{i,j=o}^{N} a_{ij}\bar{l}_i l_j \geq (a_{NN}-c_N)|l_N|^2+(a_{N-1,N-1}-x_N-c_{N-1})|l_{N-1}|^2+ \qquad (2)$$
$$+(a_{N-2,N-2}-x_N-x_{N-1}-c_{N-2})|l_{N-2}|^2+\ldots+(a_{22}-x_N-x_{N-1}-\ldots-x_3-c_2)$$
$$|l_2|^2+(a_{11}-x_N-x_{N-1}-\ldots-x_2)(|l_1|^2+|l_2|^2).$$

Let be $\left[\frac{n}{2}\right]=s$ for $n=2s$ or $n=2s+1$, $s\in\mathbb{N}$, and $c_n=\left[\frac{n}{2}\right]2^{4^{n-1}+n+1}$.
Then all coefficients of $|l_o|^2,\ldots,|l_N|^2$ in (2) are greater than 1. Thus $\mathbb{A}\geq\mathbb{I}$ is satisfied. A similar proof is in /15/.

Let us give two

<u>Examples of mappings with property (A):</u>

i) Let $T=(T_o,T_1,\ldots)$ be a positive linear functional on $E_{\otimes}$. Then T fulfills (A). (A_i) is a consequence of the linearity of T, (A_{ii}) of the positivity and (A_{iii}) of the Cauchy-Schwarz inequality.

ii) Let us regard the seminorm $f\longrightarrow\check{p}(f)=\sum_{n=o}^{\infty}\check{p}_n(f_n)$ with

$$\check{p}_n(f_n)=\sup\{|\sum T^{(1)}(g^{(i_1)})\ldots T^{(n)}(g^{(i_n)})|;\ |T^{(1)}(.)|\leq p(.),\ldots$$
$$\ldots,|T^{(n)}(.)|\leq p(.)\},$$

$f_n=\sum_{\substack{i_1 i_2\ldots i_n\\ \text{finite}}} g^{(i_1)}\otimes\ldots\otimes g^{(i_n)}$ and a fixed seminorm $p\in\mathcal{P}(t)$ with

$p(f_1^*)=p(f_1)$ for all $f_1 \in E$. Then $f \longrightarrow \check{p}(f)$ has the property (A) because (A_i), (A_{ii}) are fulfilled by definition and (A_{iii}) follows by

$$(\check{p}(\sum_{i=1}^{M} \tilde{a}_n^{(i)} * \tilde{a}_m^{(i)}))^2 \leq (\check{p}_{n+m}(\sum_{i=1}^{M} \tilde{a}_n^{(i)} * \tilde{a}_n^{(i)}))^2 =$$

$$= \sup \{ |(T^{(1)} \otimes \ldots \otimes T^{(n+m)})(\sum_{i=1}^{M} a_n^{(i)} {}^* \otimes a_m^{(i)})|^2 ; T^{(s)}(.) \leq p(.), \; s=1,2,\ldots,n+m \}$$

$$\leq \sup \{ \sum_{i=1}^{M} |(T^{(1)} \otimes \ldots \otimes T^{(n)})(a_n^{(i)*})|^2 ; \; |T^{(r)}(.)| \leq p(.), \; r=1,2,\ldots,n \}$$

$$\sup \{ \sum_{i=1}^{M} |(T^{(1)} \otimes \ldots \otimes T^{(m)})(a_m^{(i)})|^2 ; |T^{(u)}(.)| \leq p(.), \; u=1,\ldots,m \}$$

$$= \check{p}_{2n}(\sum_{i=1}^{M} a_n^{(i)} * a_n^{(i)}) \; \check{p}_{2m}(\sum_{i=1}^{M} a_m^{(i)} * a_m^{(i)}).$$

Let be $\{n\}=s$ for $n=2s$ or $n=2s-1$, $s=1,2,\ldots$ and $\mathbb{B}=(\beta_\mu^\nu(c))_{\mu,\nu=1}^{\infty}$ an infinite dimensional matrix of elements $\beta_\mu^\nu > 0$ depending on a constant $c>0$ and given by

$\beta_\mu^\nu = 0$ for $\mu=1,2,\ldots,\{\nu\}-1$,

$\beta_{\{\nu\}}^\nu = \min\{(c/(\nu+1))^2, 1\}$, $\beta_{n+1}^\nu = (\beta_n^\nu/(4n))^2$, $n=\{\nu\}+1, \{\nu\}+2, \ldots$.

<u>Theorem 3.4:</u>

Let be $k=\sum_{i=1}^{M} a^{(i)} * a^{(i)} \in E_{\otimes}^+$, £ a mapping with property (A) and $L_n^£(k) \leq 1$, $n=0,1,\ldots$. Then:

a) If there is a $c>0$ and an odd index ν with $|£(\tilde{k}_\nu)|=c>0$ then there is an other index $2l > \nu$ with $|£(\tilde{k}_{2l})| > \frac{1}{2} \beta_l^\nu$.

b) If there is an even index $\nu=2s$ and a constant $\varrho>0$ with $|£(\tilde{k}\ )|=c'>0$, $(L_s^£(k))^2 \leq \varrho \sum_{j=1}^{s} L_{s+j}^£(k) \, L_{s-j}^£(k)$

then there is an other index $2l>\nu$ with $|£(\tilde{k}_{2l})| > \frac{1}{2} \beta_l^\nu(\frac{2c'}{\varrho+2})$

($\beta^\cdot_\cdot(\frac{2c'}{\varrho+2})$ means that we have to put $\frac{2c'}{\varrho+2}$ for c in the definition of $\beta^\cdot_\cdot$.)

c) If there is an index ν with $L_\nu \geq c'' > 0$ then there exists an other index $l>\nu$ with $|£(\tilde{k}_{2l})| \geq \frac{1}{2} \beta_l^\nu((\nu+1)c'')$.

Proof:

a) Because of $L_n \leq 1$, $n=0,1,2,\ldots$,

$$c=|£(\tilde{k}_\nu)|=|£(\sum_{s+t=\nu} \sum_{j=1}^{M} \tilde{a}_s^{(j)} * \tilde{a}_t^{(j)})| \overset{(A_{iii})}{\leq} \sum_{m=0}^{\nu} L_{\nu-m} L_m \leq (\nu+1) L_{m_1}$$

follows for an index $m_1 > \nu/2$. Thus

$$L_{m_1} \geqslant \frac{c}{\vartheta+1} = (\beta^{\vartheta}_{\{\vartheta\}})^{1/2} \geqslant (\beta^{\vartheta}_{m_1})^{1/2} \tag{3}$$

follows. Now let us regard $|\pounds(\tilde{k}_{2m_1})|$. Then there are two possibilities:

$$|\pounds(\tilde{k}_{2m_1})| > \frac{1}{2}\beta^{\vartheta}_{m_1}, \tag{I$_1$}$$

$$|\pounds(\tilde{k}_{2m_1})| \leqslant \frac{1}{2}\beta^{\vartheta}_{m_1}. \tag{II$_1$}$$

The assertion is proved for (I_1). The following inequalities follow if (II_1) is fulfilled.

$$\frac{1}{2}\beta^{\vartheta}_{m_1} - \sum_{j=1}^{m_1} L_{m_1-j}L_{m_1+j} \overset{(3)}{\leqslant} \frac{1}{2}(L_{m_1})^2 - \sum_{j=1}^{m_1} L_{m_1-j}L_{m_1+j} \overset{(*)}{\leqslant}$$

$$\leqslant \frac{1}{2}|\pounds(\tilde{k}_{2m_1})| \overset{(II_1)}{\leqslant} \frac{1}{4}\beta^{\vartheta}_{m_1}, \tag{4}$$

which implies

$$\frac{1}{4}\beta^{\vartheta}_{m_1} \leqslant \sum_{j=1}^{m_1} L_{m_1-\vartheta}L_{m_1+\vartheta} \leqslant m_1 L_{m_1+m_2} \tag{5}$$

for an index m_2 with $1 \leqslant m_2 \leqslant m_1$, and

$$L_{m_1+m_2} \geqslant (4m_1)^{-1}\beta^{\vartheta}_{m_1} \geqslant (\beta^{\vartheta}_{m_1+m_2})^{1/2}. \tag{6}$$

Next let us regard $k_{2(m_1+m_2)}$. There are again two possibilities for $|\pounds(\tilde{k}_{2(m_1+m_2)})|$:

$$|\pounds(\tilde{k}_{2(m_1+m_2)})| > \frac{1}{2}\beta^{\vartheta}_{m_1+m_2}, \tag{I$_2$}$$

$$|\pounds(\tilde{k}_{2(m_1+m_2)})| \leqslant \frac{1}{2}\beta^{\vartheta}_{m_1+m_2}. \tag{II$_2$}$$

(I_2) proves the assertion. In case of (II_2) we get an index m_3, $1 \leqslant m_3 \leqslant m_2$, with $L_{m_1+m_2+m_3} \geqslant (\beta^{\vartheta}_{m_1+m_2+m_3})^{1/2}$ by analogous consideration. Thus the above defined algorithm goes on. However, because of $L_n=0$ for $n > \frac{1}{2}\mathrm{Grad}(k)$ the possibility (I_1) must occur after l steps. This proves a).

((*) follows by Lemma 3.3 ii).)

b) It is

$$c' = |\pounds(\tilde{k}_{2s})| \overset{(**)}{\leqslant} (L_s)^2 + 2\sum_{j=1}^{s} L_{s-j}L_{s+j} \leqslant (\varrho+2)\sum_{j=1}^{s} L_{s-j}L_{s+j}$$

$$\leqslant (\varrho+2)sL_{s+j_1}$$

for an index $1 \leqslant j_1 \leqslant s$. Then

$L_{m_1} \geq c'/(\varsigma+2)s = (\beta_s^{2s}(c(2s+1)/(\varsigma+2)s))^{1/2} \geq$

$\geq (\beta_s^{2s}(2c/(\varsigma+2)))^{1/2} \geq (\beta_{m_1}^{2s}(2c/(\varsigma+2)))^{1/2}$

follows for $m_1=s+j_1$. This is the corresponding inequality to (3) and the further proof is in analogy to that of a). ((**) is a consequence of Lemma 3.3 i).)

c) We take $L_\nu \geq c'' \geq (\beta_\nu^\nu((\nu+1)c''))^{1/2}$ for (3). The further proof is analogously to that of a).

3.3

In this section let us regard the topological closure of $E_\otimes^+$ resp. $E_{\hat\otimes}^+$.

One motivation for the study of this closure is the following statement by Wyss and Yngvason.

Statement 3.5:

Let $E= \mathcal{S}(\mathbb{R}^4)$. Then every linear functional T on $E_{\hat\otimes}$ with $T(f) \geq 0$ for all $f \in \overline{E_{\hat\otimes}^+}^{\varepsilon_\otimes}$ is

i) $\varepsilon_\otimes$-continuous, (/25/),

ii) $\mathcal{N}$-continuous, where $\mathcal{N}$ is a l.c. topology defined on $E_{\hat\otimes}$ with $\varepsilon_\infty \precneqq \mathcal{N} \precneqq \varepsilon_\otimes$, (/26/).

Beside $\overline{E_\otimes^+}^\tau$ let us regard the sets

$E_x^{+,f\tau} = \{g \in E_\otimes;$ there is a sequence $(g^{(n)})_{n=1}^\infty$, $g^{(n)} \in E_\otimes^+$ with $g^{(n)} \xrightarrow[\tau]{} g\}$,

$E_x^{+,s\tau} = \{\sum_{i=1}^\infty a^{(i)} * a^{(i)};\ a^{(i)} \in E_\otimes,\ \Sigma$ is τ-convergent$\}$

for a l.c. topology $\varepsilon_p \prec \tau \prec i_\otimes$ on $E_\otimes$. Then it is

$$\begin{array}{ccccc} \overline{E_\otimes^+}^\tau & \supset & E_\otimes^{+,f\tau} & \supset & E_\otimes^{+,s\tau} \\ \cap & & \cap & & \cap \\ \overline{E_\otimes^+}^{\tau'} & \supset & E_\otimes^{+,f\tau'} & \supset & E_\otimes^{+,s\tau'} \end{array}$$

for $\tau' \prec \tau$. The aim of this section is to show that these sets coincide for a large class of topological vector spaces $E[t]$ and l.c. topologies τ', τ.

Let us use the notations $E^n := \{(f_0, f_1, \ldots, f_N, 0, \ldots) \in E_\otimes;\ f_m=0$ if $m>n\}$,

$\delta_{2n}(e) := (\beta_n^1(e))^{-1}$, $n \in \mathbb{N}$. Then one can prove the

Lemma 3.6:

Let τ be a l.c. topology on $E_\otimes$, s.t. there is a system of seminorms $\mathfrak{P}(\tau)$ describing τ with the properties:

i) every $p \in \mathfrak{P}(\tau)$ satisfies (A_{iii});

ii) to every $p \in \mathfrak{P}(\tau)$ there are seminorms p_n on E_n, $n=0,1,\dots,$ such that $p(f)=\sum_{n=0}^{\infty} p_n(f_n)$ for all $f=(f_0,f_1,\dots,f_N,0,\dots)\in E_\otimes$;

iii) to every $c>0$, $p\in\mathfrak{P}(\tau)$ there is a τ-continuous seminorm $f \longrightarrow \tilde{p}(f)=\sum_{n=o}^{\infty} \tilde{p}_n(f_n)$, $\tilde{p}_n$ are seminorms on E_n, such that $\tilde{p}(f) \geq p(f)$, $\| f\|^p \leq \tilde{p}(f)$, $\delta_{2n}(c)p_{2n}(f_{2n}) \leq \tilde{p}_{2n}(f_{2n})$ for all $n\in\mathbb{N}$, $f\in E_\otimes$.

Then $\overline{E_\otimes^+}^{\tau} \cap E^{2n} = \overline{E_x^+ \cap E^{2n}}^{\tau}$, $n=0,1,2,\dots,$ follows.

Proof: Let be $\overline{E_\otimes^+}^{\tau} \cap E^{2n_o} \supsetneqq \overline{E_\otimes^+ \cap E^{2n_o}}^{\tau}$ for an $n_o \in\mathbb{N}$, i.e. there is an element

$g=(g_o,\dots,g_{2n_o},0,0,\dots)\in(\overline{E_\otimes^+}^{\tau}\cap E^{2n_o})\setminus(\overline{E_\otimes^+ \cap E^{2n}}^{\tau})$.

Thus there are a seminorm $p\in\mathfrak{P}(\tau)$, a τ-continuous seminorm $\tilde{p}$ given by assumption iii) and a constant C depending on p,g with

$$1/2 \geq C > 0\,, \tag{7}$$

$$p(g-\sum_{i=1}^{M} f^{(i)}*f^{(i)}) > C \quad \text{for all } f^{(i)}\in E_\otimes^+\cap E^{n_o},\ M\in\mathbb{N},$$

$\| g\|^p=1/2$ and

$$p(g-h) \leq \tilde{p}(g-h) \leq C/2 \leq 1/4 \tag{8}$$

for some $h=\sum_{i=1}^{M'} b^{(i)}*b^{(i)} \in E_\otimes^+$, $b^{(i)}=(b_o^{(i)},\dots,b_{n_i}^{(i)},0,\dots)\in E_\otimes^+\cap E^{n_i}$, $M',n_i\in\mathbb{N}$. Then there is an index l_o, $n_o < l_o \leq 2n_o$, with

$$C/2 \overset{(8)}{\leq} C-p(g-h) \overset{(7)}{<} p(g-\sum_{i=1}^{M'}(Q^{(n_o)}b^{(i)})*(Q^{(n_o)}b^{(i)}))-p(g-Q^{(2n_o)}h)$$

$$\leq p(\sum_{i=1}^{M'}(Q^{(n_o)}b^{(i)})*(Q^{(n_o)}b^{(i)})-Q^{(2n_o)}h)=p(\sum_{i=1}^{M'}\sum_{r=n_o+1}^{2n_o}\sum_{\mu+\nu=r}$$

$$(b_\mu^{(i)}*b_\nu^{(i)}+b_\nu^{(i)}*b_\mu^{(i)}) \overset{(*)}{\leq} \sum_{r=n_o+1}^{2n_o}\sum_{\substack{\mu+\nu=r\\ \mu\geq n_o+1}} 2\, L_\mu^p(h)\, L_\nu^p(h) \overset{(**)}{\leq}$$

$$\leq \sum_{r=n_o+1}^{2n} (r-n_o)\, L_r^p(h) \leq 2n_o(l_o-n_o)\, L_{l_o}^p(h). \tag{9}$$

(9) and Theorem 3.4 iii) imply the existence of an index l_1, $l_1 > l_o$, such that

$$p(\tilde{h}_{2l_1}) \geq \frac{1}{2}\beta_{l_1}^{l_o}(e(l_o+1)) \geq \frac{1}{2}\beta_{l_1}^{l_o}(C/4l_o) \geq \frac{1}{2}\beta_{l_1}^{1}(C) \tag{10}$$

for $e=C/(2(2n_o)(l_o-n_o)) \geq C/(4l_o^2)$. Then

$$\tilde{p}(g-h) \geq \tilde{p}_{2l_1}((g-h)_{2l_1}) \overset{(***)}{=} \tilde{p}(\tilde{h}_{2l_1}) \overset{(iii)}{\geq} \delta_{2l_1}(C)p_{2l_1}(h_{2l_1}) \geq$$

$$\overset{(10)}{\geq} \frac{1}{2}\delta_{2l_1}(C)\,\beta_{l_1}^{1}(C) = \frac{1}{2}$$

is a contradiction to (8).

((*) follows by (A_{iii}) and the definition of L_n^p.

(**) is a consequence of Lemma 3.3 iii) and

$$\sum_{n=o}^{\infty}(L_n^p(h))^2 \leq [\![h]\!]^p \leq [\![g]\!]^p + [\![g-h]\!]^p \overset{(iii)(8)}{\leq} 1/2 + \tilde{p}(g-h) \leq$$

$$\overset{(8)}{\leq} 1/2 + C/2 \overset{(7)}{<} 1 .$$

(***) is valid because of $\mathrm{Grad}(g) \leq 2n_o < 2l_1$.)

This completes the proof of Lemma 3.6.

Remarks:

i) The assumptions of Lemma 3.6 are satisfied for ε_∞ and $\varepsilon_\otimes$. But there are also l.c. topologies τ, $\tau \not\leq \varepsilon_\infty$, satisfying these assumptions.

ii) In /9/ there are examples of sets $M \subset (\mathcal{S}(\mathbb{R}^4))_\otimes$ with $\overline{M \cap E^n}^{\varepsilon_\otimes} \neq \overline{M}^{\varepsilon_\otimes} \cap E^n$ for some $n \in \mathbb{N}$. That shows that the structure of $E_\otimes^+$ is important for the proof of Lemma 3.6.

Let us give two corollaries of Lemma 3.6.

<u>Corollary 3.7:</u>

Let τ_1 be a l.c. topology on $E_\otimes$ satisfying the assumptions of Lemma 3.6, and let be τ_2 a further l.c. topology on $E_\otimes$ with $\tau_2 \succ \tau_1$, $\tau_1 \restriction_{E^{2n}} = \tau_2 \restriction_{E^{2n}}$, n=0,1,... .

Then $\overline{E_\otimes^+}^{\tau_1} = \overline{E_\otimes^+}^{\tau_2}$ follows.

Proof:

$$\overline{E_\otimes^+}^{\tau_2} \subset \overline{E_\otimes^+}^{\tau_1} = \bigcup_{n=o}^{\infty} \overline{E_\otimes^+}^{\tau_1} \cap E^{2n} = \bigcup_{n=o}^{\infty} \overline{E_\otimes^+ \cap E^{2n}}^{\tau_1} = \bigcup_{n=o}^{\infty} \overline{E_\otimes^+ \cap E^{2n}}^{\tau_2} \subset \overline{E_\otimes^+}^{\tau_2}$$

implies the assertion.

Corollary 3.8:

Let τ satisfy the assumptions of Lemma 3.6, and let further be $E[t]$, $t=\tau\restriction_E$, an LF-space (i.e. strict inductive limes of Frechet spaces).
Then $\overline{E_{\otimes}^{+}}^{\tau} = E_{\otimes}^{+,f\tau}$ follows.

Proof; Let be $E[t] = \underset{n\to\infty}{\underrightarrow{\lim}}\ {}^{(n)}E[t^{(n)}]$, ${}^{(n)}E[t^{(n)}]$ Frechet spaces,

$E \supset \ldots \supset {}^{(n+1)}E \supset {}^{(n)}E \supset \ldots$, $t^{(n+1)}\restriction_{{}^{(n)}E} = t^{(n)}$, $n=1,2,\ldots$.

If $k \in \overline{E_{\otimes}^{+}}$ then there is a net $(k^{(\alpha)})_{\alpha\in A}$, A is a directed set of indexes, $k^{(\alpha)} \in E_{\otimes}^{+}$ and $k^{(\alpha)} \xrightarrow[\tau]{} k$.
However, $k \in \overline{E_{\otimes}^{+}}$ implies further that there are indexes n',N with $k \in ({}^{(n')}E)_{\otimes} \cap E^{N}$. Then there is a cofinal subset $A' \subset A$ with $k^{(\alpha)} \in ({}^{(n')}E)_{\otimes} \cap E^{N}$, $\alpha \in A'$, by Lemma 3.6. Because $\tau\restriction_{({}^{(n')}E)_{\otimes}\cap E^{N}}$ is metrizable there is a sequence $(g^{(n)})_{n=1}^{\infty}$, $g^{(n)} \in E_{\otimes}^{+} \cap E^{N}$ with $g^{(n)} \xrightarrow[n\to\infty]{} k$ with respect to τ.
This proves the Corollary.

There is the following lemma proved by Borchers.

Lemma 3.9 (/5/):

If $E[t]$ is a nuclear LF-space then $E_{\otimes}^{+,f\varepsilon_{\otimes}} = E_{\otimes}^{+,s\varepsilon_{\otimes}}$ follows.

Combining Lemma 3.6, Corollaries 3.7, 3.8 and Lemma 3.9 one gets the

Theorem 3.10:

Let $E[t]$ be a nuclear LF-space and τ a l.c.topology on $E_{\otimes}$ satisfying the assumption of Lemma 3.6 and $\tau\restriction_{E^{2N}} = \varepsilon_{\otimes}\restriction_{E^{2N}}$.
Then $E_{\otimes}^{+,s\tau} = \overline{E_{\otimes}^{+}}^{\varepsilon_{\otimes}}$.

Remarks:

i) Theorem 3.10 holds especially for $\tau = \varepsilon_{\infty}$, i.e. $E_{\otimes}^{+,s\varepsilon_{\infty}} = \overline{E_{\otimes}^{+}}^{\varepsilon_{\otimes}}$.
All assertions of Theorem 3.10 remain valid if one replaces $E_{\otimes}$ and $E_{\otimes}^{+}$ by $E_{\hat{\otimes}}$ and $E_{\hat{\otimes}}^{+}$.

ii) This theorem was firstly proved for $E = \mathcal{S}(\mathbb{R}^4)$ by Borchers and the author (/5/,/9/,/10/). Later there are also proofs in /1/,/21/.

iii) One can prove an analogous theorem for cones of "positive

type", i.e. cones satisfying Theorem 3.4 or a similar version of it. Further one can prove an analogous theorem for the union of some cones of positive type.
This will be treated in a subsequent paper.

One can easily extend the proofs of Theorem 3.2 to $E_{\otimes}^{+,s\ \varepsilon_\infty}$.
Thus the following Corollary 3.11 is an important consequence of Theorem 3.10.

Corollary 3.11:

All assertions of Theorem 3.2 are valid for $\overline{E_{\otimes}^{+}}^{\varepsilon_\otimes}$ and $\overline{E_{\hat{\otimes}}^{+}}^{\varepsilon_\otimes}$.

4. Examples

Let us discuss our results for some examples.

4.1

Let be $E=\mathbb{C}$. Then $E_{\otimes}$ is *-isomorphic with the algebra of polynomials $\mathcal{P}$ in one real variable t. This *-isomorphism is given by

$$f=(f_o,\dots,f_N,0,0,\dots) \longleftrightarrow \hat{f}(t)=\sum_{n=0}^{\infty} f_n t^n,\ f_n\in\mathbb{C},\ t\in\mathbb{R}.$$

Let the algebraic operations in $\mathcal{P}$ given by $\hat{f}(t)+\hat{g}(t), \hat{f}(t)\hat{g}(t)$, $\hat{f}^*(t)=\overline{\hat{f}(t)}$.

Readily one sees

$$\widehat{f+g} = \hat{f} + \hat{g},\quad \widehat{fg} = \hat{f}\hat{g},\quad \widehat{f^*}=\hat{f}^*,\ \hat{f},\hat{g}\in\mathcal{P}.$$

We have further

$$\widehat{\mathbb{C}_{\otimes}^{+}} = \{\hat{f}(t)\in\mathcal{P}\ ;\ \hat{f}(t)\geq 0 \text{ for all } t\in\mathbb{R}\}. \tag{1}$$

Proof:

$\hat{C}_{\otimes}^{+} \subset \{\dots\}$ follows immediately.

Otherwise one has

$0\leq \hat{f}(t)=(t-a_1)^{2\alpha_1}\dots(t-a_n)^{2\alpha_n}(t^2+b_1^2)^{\beta_1}\dots(t^2+b_r^2)^{\beta_r}$, $\alpha_i,\beta_i\in\mathbb{N}$,

$a_1,\dots,a_n\in\mathbb{R}$, $a_1<a_2<\dots<a_n$, $b_1,\dots,b_r\in\mathbb{R}$,

because $\hat{f}(t)$ is real for all t the conjugate complex value of every root must be a root too, and because $\hat{f}(t)\geq 0$ the exponents of the factors $(t-a_j)$, $j=1,2,\dots,n$, have to be even. Thus

$$\hat{f}(t)=\sum_{\substack{k=o\\ i_1<\dots<i_{s-k}\\ i_s\in\{1,\dots,s\}}}^{s} ((t-a_1)^{\alpha_1}\dots(t-a_n)^{\alpha_n}(t^2+b_{s+1}^2)^{\varepsilon_{s+1}}\dots(t^2+b_r^2)^{\varepsilon_r} b_{i_1\dots i_{s-k}} t^k)^2 \in \widehat{\mathbb{C}_{\otimes}^{+}}$$

for $\beta_j=2\gamma_j+1$, $j=1,\dots,s$, $s\leq r$, $\beta_l=2\varepsilon_l$, $l=s+1,\dots,r$.
This proves (1). (Further properties of $\mathbb{C}_\otimes$ are proved in /14/.)
Because of Corollary 2.2 one has $\varepsilon_p=\pi_p=i_p \neq \varepsilon_\infty=\pi_\infty=i_\infty=\varepsilon_\otimes=\pi_\otimes=$ $= i_\otimes$. The topologies ε_p and $\varepsilon_\otimes$ can be described by the following systems of seminorms:

$$\varepsilon_p:\ \{f\longrightarrow |f_n|;\ n=0,1,\dots\}$$

$$\varepsilon_\otimes:\ \{f\longrightarrow p_{(\gamma_n)}(f)=\sum_{n=0}^{\infty}\gamma_n|f_n|;\ (\gamma_n)_{n=0}^{\infty}\in\mathbb{N}^{\mathbb{N}}\},$$

$f=(f_0,f_1,\dots,f_N,0,0,\dots)\in\mathbb{C}_\otimes$. Further one has the

<u>Statement 4.1:</u>

i) $\overline{\mathbb{C}_\otimes^+}^{\varepsilon_\otimes}=\mathbb{C}_\otimes^+$, ii) $\mathbb{C}_\otimes^+ \subsetneqq \overline{\mathbb{C}_\otimes^+}^{\varepsilon_p}$.

Proof: i) The image $\hat{\varepsilon}_\otimes$ on $\mathcal{P}$ of the topology $\varepsilon_\otimes$ is stronger than the topology of the pointwise convergence on $\mathcal{P}$ because for $(\gamma_n(s))_{n=0}^{\infty}\in\mathbb{N}^{\mathbb{N}}$, $s=1,2,\dots$, with $\gamma_n(s)=s^n$ we have

$$|\hat f(t_0)|\leq|f_0|+|f_1||t_0|+\dots+|f_N||t|^N\leq p_{(\gamma_n(s))}(f)\quad\text{for } s>|t_0|.$$

Thus $\hat f\in\widehat{\overline{\mathbb{C}_\otimes^+}^{\varepsilon_\otimes}}$ implies $\hat f(t)\geq 0$ for all $t\in\mathbb{R}$, and $\hat f\in\widehat{\mathbb{C}_\otimes^+}$ follows by (1). This proves i).
ii) Let us regard the sequence $(\xi_n)_{n=0}^{\infty}$, $\xi_n\in\mathbb{R}$, defined by $\xi_0=-\xi_1=1$, $\sum_{i+j=m}\xi_i\xi_j=0$, $m=2,3,\dots$. Then $(\xi_n)_{n=0}^{\infty}$ is not terminating, i.e. to every $n_0\in\mathbb{N}$ there is an $n\in\mathbb{N}$, $n>n_0$ with $\xi_n\neq 0$.
Let us regard $f^{(n)}=(\xi_0,\xi_1,\dots,\xi_n,0,0,\dots)\in\mathbb{C}_\otimes$. Then we have

$$f^{(n)}*f^{(n)}=(1,-1,0,\dots 0,\sum_{\substack{i+j=n+1\\ i\geq 1}}\xi_i\xi_j,\sum_{\substack{i+j=n+2\\ i,j\geq 2}}\xi_i\xi_j,\dots,\xi_n^2,0,\dots)\in\mathbb{C}_\otimes^+,$$

$$f^{(n)}*f^{(n)}\xrightarrow[n\to\infty]{}(1,-1,0,0,\dots)\text{ with respect to }\varepsilon_p.$$

Because of $(1,-1,0,0,\dots)\notin\mathbb{C}_\otimes^+$ ii) is proved.

Remarks:

i) Because of $E_\otimes^+\subset E_\otimes^{+,s\varepsilon_\otimes}\subset E_\otimes^{+,f\varepsilon_\otimes}\subset\overline{E_\otimes^+}^{\varepsilon_\otimes}$ for any l.c. space E Statement 4.1 i) and $\varepsilon_\infty=\varepsilon_\otimes$ gives the assertion of Theorem 3.10 with $\tau=\varepsilon_\infty$.

ii) Statement 4.1 ii) is valid for arbitrary tensoralgebras.

4.2

Let us regard $E=\mathcal{S}(\mathbb{R}^d)$, the Schwartz space of test function over $\mathbb{R}^d$, $d\in\mathbb{N}$.

$\mathcal{S}(\mathbb{R}^d)$ is a non normable, nuclear Frechet space having continuous norms, /7/, /22/. Thus Theorem 2.1 implies
$\varepsilon_\infty=\pi_\infty=i_\infty \lneq \varepsilon_\otimes=\pi_\otimes=i_\otimes$.
The corresponding assertions of Theorem 2.3 and Theorem 3.10 are proved for E= $\mathcal{S}(\mathbb{R}^d)$ in /9/,/11/,/4/ and /5/,/10/.

4.3
Let $\mathcal{D}(\mathbb{R}^d)$ denote the Schwartz space of the complex valued smooth functions on $\mathbb{R}^d$, $d\in\mathbb{N}$, with compact support. Further let us regard strongest l.c. topology t on $\mathcal{D}(\mathbb{R}^d)$ which induces the topology given by
$\{f \longrightarrow p_n(f)=\max\{|D_1^{\alpha_1}\ldots D_d^{\alpha_d}\ f(x_1,\ldots,x_d)|;\ \alpha_1,\ldots,\alpha_d\leq n\};n=0,1,..\}$,
$D_i=\frac{\partial}{\partial x_i}$, i=1,2,...,d, on every subspace
$\mathcal{D}_a=\{f\in\mathcal{D}(\mathbb{R}^d);\ \mathrm{supp}(f)\subset\{|x_1|^2+\ldots+|x_d|^2\}\leq a\},\ a>0$.
Then $\mathcal{D}(\mathbb{R}^d)$ is a nonmetrizable, nuclear LF-space, /7/,/22/.
Theorem 2.1 implies

$$\begin{array}{ccc} i_\infty & \lneq & i_\otimes \\ \rotatebox{90}{$\gneq$} & & \rotatebox{90}{$\gneq$} \\ \varepsilon_\infty=\pi_\infty & \lneq & \varepsilon_\otimes=\pi_\otimes \end{array}$$

Let $(\mathcal{D}_\otimes[\tau])'$ denote the set of the τ-continuous linear functionals on $(\mathcal{D}(\mathbb{R}^d))_\otimes$. Then $(\mathcal{D}_\otimes[i_\otimes])'\supsetneqq(\mathcal{D}_\otimes[\pi_\otimes])'$ was proved by Alcantara, /2/.

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