1. Introduction

We consider a finite graph $G$ which may have multiple edges but has no loops. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of $G$, respectively. We write $d_G(x)$ for the degree of a vertex $x$ in $G$. Let $a$, $b$ and $r$ be integers such that $0 \leq a \leq b$ and $r > 0$. A spanning subgraph $F$ of $G$ is called an $[a,b]$-factor of $G$ if $a \leq d_F(x) \leq b$ for all $x \in V(G)$, and we call an $[r,r]$-factor an $r$-factor. An $r$-regular graph is a graph in which each vertex has degree $r$.

Tutte [8] ([3], p. 77) proved that for any odd integer $r$ and any integer $k$ ($0 \leq k \leq r$), every $r$-regular graph has a $[k-1,k]$-factor. It was proved in [5], [9] that every regular graph has a $[1,2]$-factor each of whose components is regular. Enomoto and Saito [4] gave the following conjecture: Every $r$-regular graph has a $[k-1,k]$-factor each of whose components is regular for any $k$, $0 < k < r$. Note that this conjecture is true when $r$ is even by Petersen's 2-factorable theorem (see Lemma 1). So the essential part of this conjecture is the case that $r$ is odd. We obtain the following two theorems.

Theorem 1. Let $r$ and $k$ be positive integers. If $k \leq 2(2r + 1)/3$, then every $(2r+1)$-regular graph has a $[k-1,k]$-factor each
of whose components is regular.

**Theorem 2.** Let \( k \) and \( r \) be positive integers. If \( 2r+3 - \sqrt{2r+1} < k \leq 2r \), then there exists a simple \((2r+1)\)-regular graph that has no \([k-1,k]\)-factor each component of which is regular.

It seems that there exists a \((2r+1)\)-regular graph that has no \([k-1,k]\)-factor with regular components if \( 2(2r+1)/3 < k \leq 2r \). Some results related to our results can be found in a survey article [1].

2. Proofs of Theorems

Let \( G \) be a graph, and \( g \) and \( f \) be integer-valued functions defined on \( V(G) \) such that \( g(x) \leq f(x) \) for all \( x \in V(G) \). A spanning subgraph \( F \) of \( G \) is called a \((g,f)\)-factor if \( g(x) \leq d_F(x) \leq f(x) \) for all \( x \in V(G) \). A \((g,f)\)-factor satisfying \( g(x) = f(x) \) for all \( x \in V(G) \) is briefly called an \( f\)-factor. For a vertex subset \( X \) of \( G \), we write \( G \setminus X \) for the graph obtained from \( G \) by deleting the vertices in \( X \) together with their incident edges. Similarly, for an edge subset \( Y \) of \( G \), \( G \setminus Y \) denotes the graph obtained from \( G \) by deleting all the edges in \( Y \). For two disjoint subsets \( S \) and \( T \) of \( V(G) \), we denote by \( e_G(S,T) \) the number of edges of \( G \) joining \( S \) and \( T \).

**Lemma 1.** (Petersen [7], [2] p.166) Every \( 2r \)-regular graph has a \( 2k \)-factor for every integer \( k \), \( 0 < k < r \).

**Lemma 2** [6] Let \( G \) be an \( n \)-edge-connected graph \( (n \geq 1) \), \( \theta \) be a real number such that \( 0 \leq \theta \leq 1 \), and \( f \) be an integer-valued function defined on \( V(G) \). Suppose (1) and (2) hold. Moreover, if one of (3a) and (3b) holds, then \( G \) has an \( f \)-factor.

1. \[ \sum_{x \in V(G)} f(x) = 0 \pmod{2}. \]
(2) \( \varepsilon = \sum_{x \in V(G)} |f(x) - \theta d_G(x)| < 2. \)

(3a) \( \{f(x) \mid x \in V(G)\} \) consists of even numbers, and \( m(1-\theta) \geq 1 \), where \( m \in \{n, n+1\} \) and \( m \equiv 1 \pmod{2} \).

(3b) \( \{d_G(x), f(x) \mid x \in V(G)\} \) consists of odd numbers, and \( m \theta \geq 1 \), where \( m \in \{n, n+1\} \) and \( m \equiv 1 \pmod{2} \).

**Lemma 3.** Let \( G \) be an \( n \)-edge-connected graph \( (n \geq 1) \), \( \theta \) be a real number such that \( 0 < \theta < 1 \), and \( g \) and \( f \) be integer-valued functions defined on \( V(G) \) such that \( g(x) \leq f(x) \) for all \( x \in V(G) \). Suppose (1) and (2) hold. Moreover, if one of (3a) and (3b) holds, then \( G \) has a \( (g,f) \)-factor.

(1) \( G \) has at least one vertex \( v \) such that \( g(v) < f(v) \).

(2) \( \varepsilon = \sum_{x \in V(G)} (\max\{0, g(x) - \theta d_G(x)\} + \max\{0, \theta d_G(x) - f(x)\}) < 1. \)

(3a) \( \{f(x) \mid f(x) = g(x), x \in V(G)\} \) consists of even numbers, and \( m(1-\theta) \geq 1 \), where \( m \in \{n, n+1\} \) and \( m \equiv 1 \pmod{2} \).

(3b) \( \{f(x), d_G(x) \mid f(x) = g(x), x \in V(G)\} \) consists of odd numbers, and \( m \theta \geq 1 \), where \( m \in \{n, n+1\} \) and \( m \equiv 1 \pmod{2} \).

**Lemma 4.** Let \( G \) be a 2-edge-connected \( (2r+1) \)-regular graph, and \( h \) be a positive integer. If \( 2h \leq 2(2r+1)/3 \), then \( G \) has a \( 2h \)-factor. If \( (2r+1)/3 \leq 2h + 1 \leq (2r+1) \), then \( G \) has a \( (2h+1) \)-factor.

In particular, for every integer \( k \), \( 0 < k < 2r+1 \), \( G \) has a \([k-1,k]\)-factor each component of which is regular.

**Proof** Define a function \( f \) on \( V(G) \) by \( f(x) = 2h \) for all \( x \in V(G) \), and set \( \theta = 2h/(2r+1) \). We show that \( G, f \) and \( \theta \) satisfy conditions (1), (2) and (3a) of Lemma 2. Since \( G \) is of even order, (1) holds, and (2) is trivial as \( \varepsilon = 0 \). Furthermore, (3a) follows from \( m = 3 \) and \( 2h \leq 2(2r+1)/3 \). Hence \( G \) has an \( f \)-factor,
that is, $G$ has a 2h-factor. Similarly, we can prove that $G$
has a $\{2h+1\}$-factor if $2h+1 \geq (2r+1)/3$ by using (3b) instead of
(3a). Since one of $\{k-1, k\}$ is odd and the other is even, the
last statement is an easy consequence of the two results proved
above.

Lemma 5. Let $G$ be a 2-edge-connected $[2r, 2r+1]$-graph
having exactly one vertex $w$ of degree $2r$. Then
(1) if $0 < 2k \leq 2(2r+1)/3$, then $G$ has a $2k$-factor; and
(2) if $(2r+1)/3 \leq 2k+1 \leq 2r+1$, then $G$ has a $[2k, 2k+1]$-factor $F$ such
that $d_F(w) = 2k$ and $d_F(x) = 2k+1$ for all $x \in V(G) \setminus \{w\}$.

Proof. We prove only (2). It is clear that we may assume
$2r > 2k$. Define two functions $g$ and $f$ on $V(G)$ by
\[
g(x) = \begin{cases} 2k & \text{if } x = w \\ 2k+1 & \text{otherwise,} \end{cases}
\quad \text{and } f(x) = 2k+1 \text{ for all } x \in V(G).
\]

Put $\theta = (2k+1)/(2r+1)$. We show that $G$, $g$, $f$ and $\theta$ satisfy condi-
tions (1), (2) and (3b) of Lemma 3. Since $g(w) > f(w)$, (1) holds.
It is immediate that $g(w) < \theta d_G(w) < f(w)$. Thus (2) holds. Since
$\{d_G(x), f(x) \mid f(x) = g(x), x \in V(G)\} = \{2r+1, 2k+1\}$ and $m = 3$, (3b)
follows. Therefore, $G$ has a $(g, f)$-factor $F$, which is a $[2k, 2k +1]$-factor. Since $G$ is of odd order, we have $d_F(w) = 2k$. Conse-
quently, $F$ is a desired factor.

The next lemma plays an important role in the proof of
Theorem 1, and its proof is not so short.

Lemma 6. Let $G$ be a connected $(2r+1)$-regular graph with
at least two bridges, and $k$ be a positive integer. If $(2r+1)\
/3 \leq k \leq 2(2r+1)/3$, then $G$ has a $[k-1, k]$-factor each component of
which is regular.
Proof of Theorem 1  We prove the theorem by induction on \(2r+1\). Let \(G\) be a \((2r+1)\)-regular graph and \(k\) be an integer such that \(2 \leq k \leq 2(2r+1)/3\). Note that every regular graph has a \([0,1]\)-factor with regular components since it has a 0-factor. By Lemma 4, we may assume \(G\) is not 2-edge-connected. Suppose \(G\) has one bridge \(vw\). Then each component \(C\) of \(G-vw\) is a 2-edge-connected \([2r,2r+1]\)-graph possessing one vertex of degree \(2r\). Thus \(C\) has a \(k\)-factor or a \((k-1)\)-factor by Lemma 5. Therefore \(G\) has a \(k\)-factor or a \((k-1)\)-factor, and the theorem holds. Consequently, we may assume \(G\) has at least two bridges.

By Lemma 6, a 3-regular graph with at least two bridges has a \([1,2]\)-factor with regular components. Hence every 3-regular graph has a \([1,2]\)-factor with regular components, and so the theorem is true if \(2r+1=3\). Similarly, we can show that every 5-regular graph has a \([2,3]\)-factor \(F_1\) with regular components. Since 3-regular components of \(F_1\) has a \([1,2]\)-factor with regular components, \(F_1\) has a \([1,2]\)-factor with regular components, which is of course a desired \([1,2]\)-factor of \(G\). Hence the theorem follows for \(2r+1=5\). In general, if a \((2r+1)\)-regular graph \(G\) has an \([h-1,h]\)-factor \(F_2\) with regular components and if each component of \(F_2\) has a \([k-1,k]\)-factor with regular components, then \(G\) has a \([k-1,k]\)-factor with regular components. By this argument, we can verify that if \(2r+1\leq13\), then the theorem holds. Suppose \(2r+1\geq15\). If \((2r+4)/3 \leq k \leq 2(2r+1)/3\), then a \((2r+1)\)-regular graph \(G\) has a \([k-1,k]\)-factor with regular components by Lemma 6. Hence we may assume \(k<(2r+4)/3\). Let \(h\) be the greatest integer not exceeding \(2(2r+1)/3\). Then \(G\) has an
[h-1,h]-factor F with regular components. Since $2(h-1)/3 \geq 2(4r -1)/9$ and $2r+1 \geq 5$, we have $2(h-1)/3 \geq (2r+4)/3 > k$. Hence each component of F has a $[k-1,k]$-factor with regular components by Lemma 1 or by the inductive hypothesis. Therefore G has a $[k-1,k]$-factor with regular components, and we conclude that the proof of Theorem 1 is complete.

**Proof of Theorem 2.** Let k and r be positive integers such that $2r+2 - \sqrt{2r+1} < 2k \leq 2r$. Let $k'$ be an odd integer that is one of $\{2k-1, 2k+1\}$ and not equal to $2r+1$. Let $K_{2r+2}$ denote the complete graph with vertex set $\{a_1, \ldots, a_{2r+2}\}$. We obtain the graph $R$ from $K_{2r+3}$ by deleting edges $a_1a_2, a_1a_3, \ldots, a_1a_{2r-2k+5}, a_{2r-2k+6}a_{2r-2k+7}, \ldots, a_{2r+2}a_{2r+3}$. It is clear that $d_R(a_i) = 2k-2$ and $d_R(a_i) = 2r+1$ for all $i, i \neq 1$. Let $R(1), \ldots, R(2r)$ be copies of $R$, and let $b_i$ be the vertex of $R_i$ whose degree is $2k-2$ for all $i$. We construct a graph $H$ with vertex set $V(R(1)) \cup \ldots \cup V(R(2r)) \cup \{c_1, \ldots, c_{2r-2k+2}, v\}$ as follows. Join every $b_i$ to all $c_j$ ($1 \leq j \leq 2r-2k+2$) and $v$ by new edges, and add new edges $c_1c_2, c_3c_4, \ldots, c_{2r-2k+1}c_{2r-2k+2}$ (see Figure). Then $d_H(v) = 2r$ and $d_H(x) = 2r+1$ for all $x \in V(H) \setminus \{v\}$.

![Graph H](image_url)
Let $H_1, \ldots, H_{2r+1}$ be copies of $H$, and let $v_i$ be the vertex of $H_i$ whose degree is $2r$ for every $i$. We now construct a $(2r+1)$-regular graph $G$ as follows, which has the required property. Set $V(G)=V(H_1) \cup \ldots \cup V(H_{2r+1}) \cup \{w\}$, and join each $v_i$ to $w$ by a new edge. We omit the proof of the non-existence of $[k-1,k]$-factor with regular components in $G$.

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References

本稿で述べた定理の完全な証明は下記の論文に見られる。