数理解析研究所講究録
第 566 巻 1985 年 1-7

グラフの正則成分因子

加納幹雄 (明石工高専)

Mikio KANO Akashi Technological College, Uozumi Akashi 674

1. Introduction

We consider a finite graph G which may have multiple edges but has no loops. We denote by V(G) and E(G) the set of vertices and the set of edges of G, respectively. We write $d_G(x)$ for the degree of a vertex x in G. Let a, b and r be integers such that $0 \le a \le b$ and $r>0$. A spanning subgraph F of G is called an [a,b]-factor of G if $a \le d_F(x) \le b$ for all $x \in V(G)$, and we call an [r,r]-factor an r-factor. An r-regular graph is a graph in which each vertex has degree r.

Tutte [8]([3],p.77) proved that for any odd integer r and any integer k ($0 \le k \le r$), every r-regular graph has a [k-1,k]-factor. It was proved in [5],[9] that every regular graph has a [1,2]-factor each of whose components is regular. Enomoto and Saito [4] gave the following conjecture: Every r-regular graph has a [k-1,k]-factor each of whose components is regular for any k, $0<k<r$. Note that this conjecture is true when r is even by Petersen's 2-factorable theorem (see Lemma 1). So the essential part of this conjecture is the case that r is odd. We obtain the following two theorems.

Theorem 1. Let r and k be positive integers. If $k \le 2(2r+1)/3$, then every (2r+1)-regular graph has a [k-1,k]-factor each of whose components is regular.

Theorem 2. Let k and r be positive integers. If $2r+3 - \sqrt{2r+1} < k \le 2r$, then there exists a simple (2r+1)-regular graph that has no [k-1,k]-factor each component of which is regular.

It seems that there exists a (2r+1)-regular graph that has no [k-1,k]-factor with regular components if $2(2r+1)/3 < k \le 2r$. Some results related to our results can be found in a survey article [1].

2. Proofs of Theorems

Let G be a graph, and g and f be integer-valued functions defined on V(G) such that $g(x) \le f(x)$ for all $x \in V(G)$. A spanning subgraph F of G is called a (g,f)-factor if $g(x) \le d_F(x) \le f(x)$ for all $x \in V(G)$. A (g,f)-factor satisfying $g(x)=f(x)$ for all $x \in V(G)$ is briefly called an f-factor. For a vertex subset X of G, we write G-X for the graph obtained from G by deleting the vertices in X together with their incident edges. Similarly, for an edge subset Y of G, G-Y denotes the graph obtained from G by deleting all the edges in Y. For two disjoint subsets S and T of V(G), we denote by $e_G(S,T)$ the number of edges of G joining S and T.

Lemma 1. (Petersen [7],[2]p.166) Every 2r-regular graph has a 2k-factor for every integer k, $0<k<r$.

Lemma 2 [6] Let G be an n-edge-connected graph ($n \ge 1$), θ be a real number such that $0 \le \theta \le 1$, and f be an integer-valued function defined on V(G). Suppose (1) and (2) hold. Moreover, if one of (3a) and (3b) holds, then G has an f-factor.

(1) $\sum_{x \in V(G)} f(x) \equiv 0 \pmod 2$.

(2) $\varepsilon = \sum_{x \in V(G)} |f(x) - \theta d_G(x)| < 2.$

(3a) $\{f(x) \mid x \in V(G)\}$ consists of even numbers, and $m(1-\theta) \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod 2$.

(3b) $\{d_G(x), f(x) \mid x \in V(G)\}$ consists of odd numbers, and $m\theta \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod 2$.

Lemma 3. Let G be an n-edge-connected graph ($n \geq 1$), θ be a real number such that $0 < \theta < 1$, and g and f be integer-valued functions defined on V(G) such that $g(x) \leq f(x)$ for all $x \in V(G)$. Suppose (1) and (2) hold. Moreover, if one of (3a) and (3b) holds, then G has a (g,f)-factor.

(1) G has at least one vertex v such that $g(v) < f(v)$.

(2) $\varepsilon = \sum_{x \in V(G)} \left(\max\{0, g(x) - \theta d_G(x)\} + \max\{0, \theta d_G(x) - f(x)\}\right) < 1.$

(3a) $\{f(x) \mid f(x) = g(x), x \in V(G)\}$ consists of even numbers, and $m(1-\theta) \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod 2$.

(3b) $\{f(x), d_G(x) \mid f(x) = g(x), x \in V(G)\}$ consists of odd numbers, and $m\theta \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod 2$.

Lemma 4. Let G be a 2-edge-connected (2r+1)-regular graph, and h be a positive integer. If $2h \leq 2(2r+1)/3$, then G has a 2h-factor. If $(2r+1)/3 \leq 2h+1 \leq (2r+1)$, then G has a (2h+1)-factor. In particular, for every integer k, $0 < k < 2r+1$, G has a [k-1,k]-factor each component of which is regular.

Proof Define a function f on V(G) by $f(x) = 2h$ for all $x \in V(G)$, and set $\theta = 2h/(2r+1)$. We show that G, f and θ satisfy conditions (1), (2) and (3a) of Lemma 2. Since G is of even order, (1) holds, and (2) is trivial as $\varepsilon = 0$. Furthermore, (3a) follows from $m=3$ and $2h \leq 2(2r+1)/3$. Hence G has an f-factor, that is, G has a 2h-factor. Similarly, we can prove that G has a (2h+1)-factor if $2h+1\geq(2r+1)/3$ by using (3b) instead of (3a). Since of of $\{k-1,k\}$ is odd and the other is even, the last statement is an easy consequence of the two results proved above.

Lemma 5. Let G be a 2-edge-connected [2r,2r+1]-graph having exactly one vertex w of degree 2r. Then

(1) if $0<2k\leq 2(2r+1)/3$, then G has a 2k-factor; and

(2) if $(2r+1)/3\leq 2k+1\leq 2r+1$, then G has a [2k,2k+1]-factor F such that $d_F(w)=2k$ and $d_F(x)=2k+1$ for all $x\in V(G)\setminus\{w\}$.

Proof We prove only (2). It is clear that we may assume 2r>2k. Define two functions g and f on V(G) by

$$g(x)=\begin{cases} 2k & \text{if } x=w \\ 2k+1 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x)=2k+1 \quad \text{for all} \quad x\in V(G).$$

Put $\theta=(2k+1)/(2r+1)$. We show that G, g, f and θ satisfy conditions (1), (2) and (3b) of Lemma 3. Since $g(w)>f(w)$, (1) holds. It is immediate that $g(w)<\theta d_G(w)<f(w)$. Thus (2) holds. Since $\{d_G(x),\ f(x) \mid f(x)=g(x),\ x\in V(G)\}=\{2r+1,\ 2k+1\}$ and m=3, (3b) follows. Therefore, G has a (g,f)-factor F, which is a [2k,2k+1]-factor. Since G is of odd order, we have $d_F(w)=2k$. Consequently, F is a desired factor.

The next lemma plays an important role in the proof of Theorem 1, and its proof is not so short.

Lemma 6. Let G be a connected (2r+1)-regular graph with at least two bridges, and k be a positive integer. If $(2r+1)/3\leq k\leq 2(2r+1)/3$, then G has a [k-1,k]-factor each component of which is regular.

<u>Proof of Theorem 1</u> We prove the theorem by induction on $2r+1$. Let G be a $(2r+1)$-regular graph and k be an integer such that $2 \le k \le 2(2r+1)/3$. Note that every regular graph has a $[0,1]$-factor with regular components since it has a 0-factor. By Lemma 4, we may assume G is not 2-edge-connected. Suppose G has one bridge vw. Then each component C of $G-vw$ is a 2-edge-connected $[2r,2r+1]$-graph possessing one vertex of degree $2r$. Thus C has a k-factor or a $(k-1)$-factor by Lemma 5. Therefore G has a k-factor or a $(k-1)$-factor , and the theorem holds. Consequently, we may assume G has at least two bridges.

By Lemma 6, a 3-regular graph with at least two bridges has a $[1,2]$-factor with regular components. Hence every 3-regular graph has a $[1,2]$-factor with regular components, and so the theorem is true if $2r+1=3$. Similarly, we can show that every 5-regular graph has a $[2,3]$-factor F_1 with regular components. Since 3-regular components of F_1 has a $[1,2]$-factor with regular components, F_1 has a $[1,2]$-factor with regular components, which is of course a desired $[1,2]$-factor of G. Hence the theorem follows for $2r+1=5$. In general, if a $(2r+1)$-regular graph G has an $[h-1,h]$-factor F_2 with regular components and if each component of F_2 has a $[k-1,k]$-factor with regular components, then G has a $[k-1,k]$-factor with regular components. By this argument, we can verify that if $2r+1 \le 13$, then the theorem holds. Suppose $2r+1 \ge 15$. If $(2r+4)/3 \le k \le 2(2r+1)/3$, then a $(2r+1)$-regular graph G has a $[k-1,k]$-factor with regular components by Lemma 6. Hence we may assume $k<(2r+4)/3$. Let h be the greatest integer not exceeding $2(2r+1)/3$. Then G has an

$[h-1,h]$-factor F with regular components. Since $2(h-1)/3 \geq 2(4r-1)/9$ and $2r+1 \geq 15$, we have $2(h-1)/3 \geq (2r+4)/3 > k$. Hence each component of F has a $[k-1,k]$-factor with regular comoponents by Lemma 1 or by the inductive hypothesis. Therefore G has a $[k-1,k]$-factor with regular components, and we conclude that the proof of Theorem 1 is complete.

Proof of Theorem 2. Let k and r be positive integers such that $2r+2-\sqrt{2r+1} < 2k \leq 2r$. Let k' be an odd integer that is one of $\{2k-1, 2k+1\}$ and not equal to $2r+1$. Let K_{2r+3} denote the complete graph with vertex set $\{a_1, \ldots, a_{2r+3}\}$. We obtain the graph R from K_{2r+3} by deleting edges $a_1a_2, a_1a_3, \ldots, a_1a_{2r-2k+5}$, $a_{2r-2k+6}a_{2r-2k+7}, \ldots, a_{2r+2}a_{2r+3}$. It is clear that $d_R(a_1)=2k-2$ and $d_R(a_i)=2r+1$ for all i, $i \neq 1$. Let $R(1), \ldots, R(2r)$ be copies of R, and let b_i be the vertex of R_i whose degree is $2k-2$ for all i. We construct a graph H with vertex set $V(R(1)) \cup \ldots \cup V(R(2r)) \cup \{c_1, \ldots, c_{2r-2k+2}, v\}$ as follows. Join every b_i to all c_j $(1 \leq j \leq 2r-2k+2)$ and v by new edges, and add new edges c_1c_2, $c_3c_4, \ldots, c_{2r-2k+1}c_{2r-2k+2}$ (see Figure). Then $d_H(v)=2r$ and $d_H(x)=2r+1$ for all $x \in V(H) \setminus \{v\}$.

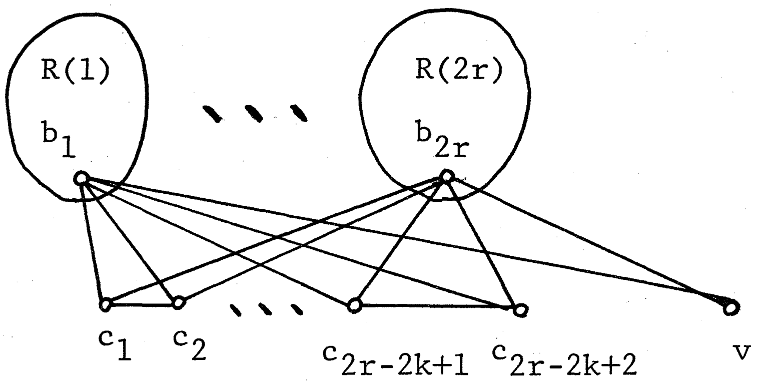

Figure Graph H.

Let $H_1, \ldots, H_{2r+1}$ be copies of H, and let v_i be the vertex of H_i whose degree is 2r for every i. We now construct a (2r+1)-regular graph G as follows, which has the required property. Set $V(G)=V(H_1) \cup \ldots \cup V(H_{2r+1}) \cup \{w\}$, and join each v_i to w by a new edge. We omit the proof of the non-existence of [k-1,k]-factor with regular components in G.

Acknowledgements The author would like to thank Professor H. Enomoto for his helpful suggestions.

References

[1] J. Akiyama and M. Kano, Factors and factorizations of graphs —— A survey, J. Graph Theory 9 (1985) Spring

[2] M. Behzad, G. Chartrand and L. Lesniak-Foster, "Graphs & Digraph", Prindle Weber and Schmidt, Boston (1979). 訳本 (共立出版)

[3] B. Bollobás, "Extremal Graph Theory". Academic Press, London (1978).

[4] H. Enomoto and A. Saito, Private communication.

[5] A. Hajnal, A theorem on k-saturated graphs, Canad. J. Math. 17 (1965) 720 - 772.

[6] M. Kano, [a,b]-factorization of a graph, J. Graph Theory 9 (1985) Spring

[7] J. Petersen, Die Theorie der regulären Graph, Acta Math. 15 (1891) 193 - 220.

[8] W.T. Tutte, The subgraph problem, Ann. Discrete Math. 3 (1978) 289 - 295.

[9] W.T. Tutte, The factors of graphs, Canad. J. Math. 4 (1952), 314 - 328.

本稿で述べた定理の完全な証明は下記の論文にある.

M. Kano, Factors of regular graphs, to appear.