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Embeddings of Graphs in the 3-Sphere

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1. Introduction

Throughout this paper, we work in the piecewise- linear category, consisting of simplicial complexes and piecewise-linear maps.

By a graph with μ components $(P \subset S^3) = (K_1 \cup \ldots \cup K_\mu \subset S^3)$ will be meant a pair of the 3-sphere S^3 with a fixed orientation and its finite 1-dimensional subpolyhedron $P = K_1 \cup \ldots \cup K_\mu$ with μ connected components $K_1, \ldots, K_\mu$. A graph with μ components $(K_1 \cup \ldots \cup K_\mu \subset S^3)$ is a link with μ components iff each component K_i is homeomorphic to the 1-sphere S^1, and especially a link with one component is a socalled knot. A graph $(P \subset S^3)$ is called trivial (or unknotted) iff there exists a 2-sphere S^2 in S^3 such that $S^2 \supset P$.

Two graphs $(P \subset S^3)$ and $(P' \subset S^3)$ are said to be equivalent (or of the same knot type), denoted by $(P \subset S^3) \cong (P' \subset S^3)$, iff there exists an orientation preserving homeomorphism $\psi : S^3 \to S^3$ such that $\psi(P) = P'$. We call the equivalence class of a graph $(P \subset S^3)$ the knot type of $(P \subset S^3)$. In the paper, we do not clearly distinguish a graph $(P \subset S^3)$ and its knot type.

H.Schubert [7] showed that every non-trivial knot is decomposable into prime knots in a unique way up to equivalence, and Y.Hashizume [4] extended this unique prime decomposition theorem from knots to links. In the previous paper [8], we also formulated and prove a unique prime decomposition theorem for special graphs which were called n-leafed-roses ([8, Theorem 3.7]). In this paper, we shall formulate another prime decompositions for graphs combining above two concepts, and prove the existence and the uniqueness of the decompositions.

2. Prime Decompositions for Graphs

In the paper, ∂M and $°M$ denote the boundary and the interior of a manifold M, respectively. For a subpolyhedron X of a manifold M, by $N(X;M)$ we denote a regular neighborhood of X in M; that is, we construct its second derived and take the closed star of X in M. For a finite 1-dimensional polyhedron P, $\mu(P)$ and $\beta(P)$ stand for the number of connected components of P and the 1-dimensional Betti number of P, respectively.

For a graph $(P \subset S^3)$, by the exterior $E(P)$ we mean the closure of $S^3 - N(P;S^3)$. $E(P)$ is a compact, connected and oriented 3-manifold with boundary $\partial E(P) = \partial N(P;S^3)$. By Δ_Σ and ∇_Σ we sall denote the closures of connected components of $S^3 - \Sigma$ for a 2-sphere Σ in S^3; it will be noticed that Δ_Σ and ∇_Σ are 3-balls by Alexander[1], and $\Delta_\Sigma \cup \nabla_\Sigma = S^3$, $\Delta_\Sigma \cap \nabla_\Sigma = \partial\Delta_\Sigma = \partial\nabla_\Sigma = \Sigma$.

A graph $(P \subset S^3)$ is said to be splittable, iff there exists a 2-sphere Σ in $S^3 - P$ such that $°\Delta_\Sigma \cap P \neq \emptyset$ and $°\nabla_\Sigma \cap P \neq \emptyset$.

2.1. Definition. (1) A 2-sphere Σ in S^3 will be called admissible of type I for a graph $(P \subset S^3)$, iff

(i) $\Sigma \cap P$ consists of a single point, say ω, and

(ii) $(P - \omega) \cap \Delta_\Sigma \neq \emptyset$, $(P - \omega) \cap \nabla_\Sigma \neq \emptyset$.

(2) In this case, we have two graphs $(P_1 \subset S^3) \equiv (P \cap \Delta_\Sigma \subset S^3)$ and $(P_2 \subset S^3)$

$\equiv (P \cap \nabla_\Sigma \subset S^3)$, and we say that $(P \subset S^3)$ is decomposed into $(P_1 \subset S^3)$ and $(P_2 \subset S^3)$ by Σ, and denoted by

$$(P \subset S^3) \cong (P_1 \subset S^3) \vee_\Sigma (P_2 \subset S^3), \text{ or simply by}$$
$$(P \subset S^3) \cong (P_1 \subset S^3) \vee (P_2 \subset S^3).$$

2.2. Proposition. *If* $(P \subset S^3) \cong (P_1 \subset S^3) \vee (P_2 \subset S^3)$, *then*

$$\mu(P) = \mu(P_1) + \mu(P_2) - 1, \quad \beta(P) = \beta(P_1) + \beta(P_2). \quad \square$$

2.3. Definition. (1) A 2-sphere Σ in S^3 will be called <u>admissible of type</u> II for a graph $(P \subset S^3)$, iff

(iii) $\Sigma \cap P$ consists of two points, say a and b, and

(iv) the annulus $A = \Sigma - {}^\circ N(P;S^3) = \Sigma - {}^\circ N(a \cup b\,;\Sigma)$ is incompressible in $E(P)$.

(2) In this case, we choose a simple arc, say α, on Σ such that $\partial\alpha = \{a,b\}$, and then we have two graphs $(Q_1 \subset S^3) \equiv (P \cap \Delta_\Sigma \cup \alpha \subset S^3)$ and $(Q_2 \subset S^3) \equiv (P \cap \nabla_\Sigma \cup \alpha \subset S^3)$. We say that $(P \subset S^3)$ is decomposed into $(Q_1 \subset S^3)$ and $(Q_2 \subset S^3)$ by Σ, and denoted by

$$(P \subset S^3) \cong (Q_1 \subset S^3) \#_\Sigma (Q_2 \subset S^3), \text{ or simply by}$$
$$(P \subset S^3) \cong (Q_1 \subset S^3) \# (Q_2 \subset S^3).$$

It should be noted that the knot types of $(Q_1 \subset S^3)$ and $(Q_2 \subset S^3)$ do not depend on the choice of the simple arc α.

2.4. Proposition. *Let* $\Sigma \subset S^3$ *be an admissible* 2*-sphere of type* II *for a graph* $(P \subset S^3)$ *giving a decomposition* $(P \subset S^3) \cong (Q_1 \subset S^3) \# (Q_2 \subset S^3)$, *and we suppose that* $\Sigma \cap P = \{a,b\}$.

(1) *If* a *and* b *belong to different components of* P, *then*

$$\mu(P) = \mu(Q_1) + \mu(Q_2), \quad \beta(P) = \beta(Q_1) + \beta(Q_2).$$

(2) *If* a *and* b *belong to the same component of* P, *then*

$$\mu(P) = \mu(Q_1) + \mu(Q_2) - 1, \quad \beta(P) = \beta(Q_1) + \beta(Q_2) - 1. \quad \square$$

2.5. Definition. A graph $(P \subset S^3)$ is said to be prime, iff it satisfies the following three conditions :

(0) $(P \subset S^3)$ is non-trivial and non-splittable,

(1) there are no admissible 2-spheres of type I for $(P \subset S^3)$, and

(2) for any decomposition $(P \subset S^3) \cong (Q_1 \subset S^3) \# (Q_2 \subset S^3)$ of type II, at least one of $(Q_1 \subset S^3)$ and $(Q_2 \subset S^3)$ is a trivial knot.

We can now formulate our prime decomposition theorem, to fix ideas :

2.6. Theorem. *Every non-trivial and non-splittable graph* $(P \subset S^3)$ *can be decomposed into a finite number of prime graphs, say* $(P_1 \subset S^3), \ldots,$ $(P_u \subset S^3)$, *and some trivial graphs, by some admissible* 2*-spheres of type* I *and* II, *such that* $(P_1 \subset S^3), \ldots, (P_u \subset S^3)$ *are unique up to order and equivalence.*

The proof of the existence of such a decomposition will be given in the next Section 3 and of the uniqueness will be given in Section 4.

3. Proof of Existence of Prime Decompositions.

In order to prove the existence of prime decompositions for a graph, we use the following Haken's finiteness theorem on incompressible surfaces in a 3-manifold [3]. We refer the reader to Jaco [5, pp.42-50].

3.1. Haken's Finiteness Theorem. *For a compact, connected and orientable manifold* M, *there exists a non-negative integer* $n_0(M)$ *such that if* $\{F_1, \ldots, F_n\}$ *is any collection of mutually disjoint incompressible closed surfaces in* $°M$, *then either* (i) $n < n_0(M)$, (ii) *for some* i, F_i *is parallel to a component of* ∂M, *or* (iii) *for some* $i \neq j$, F_i *is parallel to* F_j *in* M. □

The collection of non-negative integers $n_0(M)$ satisfying the conclusion f Theorem 3.1 is not empty. The minimal of such integers is denoted by (M) and called the <u>closed Haken number</u> of M (Jaco[5, p.49]).

Let Σ be an admissible 2-sphere of type II for a graph $(P \subset S^3)$ giving decomposition $(P \subset S^3) \cong (Q_1 \subset S^3) \# (Q_2 \subset S^3)$, and we assume that $(Q_1 \subset S^3)$ nd $(Q_2 \subset S^3)$ are obtained from $P \cap \Delta_\Sigma$ and $P \cap \nabla_\Sigma$, respectively. From efinition 2.3(iv), the annulus $A = \Sigma - {}^\circ N(P;S^3)$ is incompressible in $E(P)$. et B and C be components of $\partial N(P;S^3) \cap \Delta_\Sigma$ and $\partial N(P;S^3) \cap \nabla_\Sigma$, respective-y, such that $B \cap A = \partial B \cap \partial A = \partial B = \partial A$ and $C \cap A = \partial C \cap \partial A = \partial C = \partial A$. Then e have two closed connected orientable surfaces $A \cup B$ and $A \cup C$ in $E(P)$, nd after a suitable slight modification of $A \cup B$ and $A \cup C$, we have two losed connected orientable surfaces, say A_Δ and A_∇, in ${}^\circ E(P)$ such that oth A_Δ and A_∇ are of positive genus. In particular, if both $(Q_1 \subset S^3)$ nd $(Q_2 \subset S^3)$ are non-trivial, we may assume that A_Δ and A_∇ are incom-ressible in $E(P)$. (In fact, if B (resp. C) is compressible in $E(P)$, hen we apply some surgery for B (resp. C) so that A_Δ (resp. A_∇) is now ncompressible.) If $\beta(Q_1) = 1$, then A_Δ is of genus 1. From this, we ave the following :

3.2. Proposition. *Let* $(P \subset S^3)$ *be a non-trivial and non-splittable graph.*

(A) *If there are no admissible 2-spheres of type* I *for* $(P \subset S^3)$ *and* $\bar{h}(E(P)) = 1$, *then* $(P \subset S^3)$ *is prime.*

(B) *If* $(P \subset S^3)$ *has a decomposition* $(P \subset S^3) \cong (Q_1 \subset S^3) \# (Q_2 \subset S^3)$ *such that both* $(Q_1 \subset S^3)$ *and* $(Q_2 \subset S^3)$ *are non-trivial and* $\beta(Q_1) = 1$ *and* $\beta(Q_2) \geq 1$, *then it holds that*

$$\bar{h}(E(P)) \geq \bar{h}(E(Q_1)) + \bar{h}(E(Q_2)) + 1. \qquad \square$$

We are going to prove the existence assertion of Theorem 2.6, that is, the following lemma :

3.3. Lemma. *Let* $(P \subset S^3)$ *be a non-trivial and non-splittable graph. Then* $(P \subset S^3)$ *can be decomposed into a finite number of prime graphs* $(P_1 \subset S^3)$, ... , $(P_u \subset S^3)$ *and some trivial graphs by some admissible* 2*-spheres of type* I *and* II.

Proof. We shall prove Lemma 3.3 by induction on the 1-dimensional Betti number $\beta(P)$ and the closed Haken number $\overline{h}(E(P))$. We may assume, without loss of generality, that there is no vertex v of P with the degree $\deg(v) \leq 1$, and so $\beta(P) \geq 1$.

If $\beta(P) = 1$, then $(P \subset S^3)$ is a knot since it is non-splittable, and Lemma follows from the Schubert's result [7]. (In fact, if there exists an admissible 2-sphere Σ for $(P \subset S^3)$, then it must be of type II. If Σ gives a non-trivial decomposition $(P \subset S^3) \cong (Q_1 \subset S^3) \# (Q_2 \subset S^3)$, then we can deduce that $\overline{h}(E(Q_i)) < \overline{h}(E(P))$ for $i=1,2$.)

Now we wish to make the induction step and accordingly suppose that $\beta(P) \geq 2$ and every non-trivial and non-splittable graph $(P' \subset S^3)$ with $\beta(P') \leq \beta(P)$ has a prime decomposition provided that $\beta(P') < \beta(P)$ or $\beta(P') = \beta(P)$ and $\overline{h}(E(P')) < \overline{h}(E(P))$.

If $(P \subset S^3)$ is prime, then there is nothing to prove. So we assume that $(P \subset S^3)$ is not prime. Hence, there exists an admissible 2-sphere $\Sigma \subset S^3$ for $(P \subset S^3)$, which gives a decomposition either

$(P \subset S^3) \cong (P_1 \subset S^3) \vee (P_2 \subset S^3)$ with $\beta(P_1) \geq 1$ and $\beta(P_2) \geq 1$, or

$(P \subset S^3) \cong (Q_1 \subset S^3) \# (Q_2 \subset S^3)$ such that both $(Q_1 \subset S^3)$ and $(Q_2 \subset S^3)$ are non-trivial knots, according as Σ is of type I or type II. We distinguish three cases :

(I) Σ is of type I : From our assumption, we can easily deduce that

$0 < \beta(P_i) < \beta(P)$ $(i=1,2)$ by Proposition 2.2. From the induction hypothesis, $(P_i \subset S^3)$ has a prime decomposition $(i=1,2)$, and so $(P \subset S^3)$ has a prime decomposition.

(II)-(1) Σ is of type II and Σ intersects with P in different components : By Proposition 2.4(1), we can deduce that $0 < \beta(Q_i) < \beta(P)$ for $i=1,2$, and so Lemma follows from the induction hypothesis.

(II)-(2) Σ is of type II and Σ intersects with P in one component : It will be noticed that $\beta(Q_i) \geq 1$ for $i=1,2$. If $\beta(Q_i) > 1$ for $i=1,2$, then we can also deduce that $\beta(Q_i) < \beta(P)$ by Proposition 2.4(2), and so Lemma follows from the induction hypothesis as the same way as that of above two cases. We have therefore only to consider the case of $\beta(Q_1) = 1$ and $\beta(Q_2) = \beta(P)$. Now $(Q_1 \subset S^3)$ has a prime decomposition from the induction hypothesis. By Proposition 3.2(B), $\overline{h}(E(Q_2)) < \overline{h}(E(P))$, and then we can deduce that $(Q_2 \subset S^3)$ also has a prime decomposition from the induction hypothesis, and so $(P \subset S^3)$ has a prime decomposition.

This completes the proof of Lemma 3.3.

4. Proof of Uniqueness of Prime Decompositions.

The uniqueness assertion of Theorem 2.6 will clearly follow from the following four lemmas 4.1, 4.2, 4.3 and 4.4. (We refer to Fox[2,§7].)

4.1. Lemma. *Let* $(P \subset S^3)$ *be a non-trivial and non-splittable graph, and we suppose that there are two admissible* 2*-spheres* Σ *and* Σ' *of type* I *for* $(P \subset S^3)$ *giving decompositions*

$$(P \subset S^3) \cong (P_1 \subset S^3) \vee_{\Sigma} (P_2 \subset S^3), \textit{ and}$$
$$(P \subset S^3) \cong (Q \subset S^3) \vee_{\Sigma'} (Q' \subset S^3), \textit{ respectively}.$$

If $(Q \subset S^3)$ *is prime, then either* $(P_1 \subset S^3)$ *or* $(P_2 \subset S^3)$ *has* $(Q \subset S^3)$ *as a prime component.*

Proof. Let $\omega = \Sigma \cap P$ and $\omega' = \Sigma' \cap P$, and we may assume that $Q = P \cap \Delta_{\Sigma'}$ and $Q' = P \cap \nabla_{\Sigma'}$. If $\Sigma \cap \Sigma' = \emptyset$, then we are finished ; and if $\omega = \omega'$ and $\Sigma \cap \Sigma' = \omega$, then we are also finished. We may assume, after a slight modification, that $\Sigma \cap \Sigma'$ consists of a finite number of simple loops. We distinguish two cases :

Case 1. $\omega \neq \omega'$: We may assume that $\Sigma \cap \Sigma'$ consists of a finite number of mutually disjoint simple loops, say $c_1, \dots, c_\nu$. Let $\tau_1, \dots, \tau_\nu$ be disks on Σ' bounded by $c_1, \dots, c_\nu$, respectively, such that $\tau_i \not\ni \omega'$. Let c_1 be an innermost loop; that is, $\Sigma \cap \tau_1 = \partial\tau_1 = c_1$. Let σ_1 be the disk on Σ bounded by c_1 with $\sigma_1 \not\ni \omega$. Then the 2-sphere $\sigma_1 \cup \tau_1$ bounds a 3-ball, say B_1^3, in S^3 such that $B_1^3 \not\ni \omega$ by Alexander's Theorem [1]. Since $(P \subset S^3)$ is non-splittable and $\sigma_1 \cap P = \emptyset$ and $\tau_1 \cap P = \emptyset$, it holds that $B_1^3 \cap P = \emptyset$. Now we have a new admissible 2-sphere $(\Sigma - \sigma_1) \cup \tau_1$ of type I which gives the decomposition $(P \subset S^3) \cong (P_1 \subset S^3) \vee (P_2 \subset S^3)$. After deforming $(\Sigma - \sigma_1) \cup \tau_1$ slightly away from Σ', we may obtain a new admissible 2-sphere, again denote it by Σ, which intersects Σ' in a subcollection of $c_2, \dots, c_\nu$.

By the repetition of the procedure, we can get rid of all intersections $c_1, \dots, c_\nu$ of $\Sigma \cap \Sigma'$; thus Lemma 4.1 is established for Case 1.

Case 2. $\omega = \omega'$: This case is the same as that of Suzuki[8, Lemma 3.9]. We may assume that $\Sigma \cap \Sigma'$ consists of a finite number of simple loops, say $c_1, \dots, c_\nu, d_1, \dots, d_\lambda$, such that $c_i \cap c_j = \emptyset$ $(i \neq j)$, $d_i \cap d_j = \omega$ $(i \neq j)$ and $(c_1 \cup \dots \cup c_\nu) \cap (d_1 \cup \dots \cup d_\lambda) = \emptyset$. By the same way as that of Case 1, we can remove the loops $c_1, \dots, c_\nu$, and now we may assume that $\Sigma \cap \Sigma' = d_1 \cup \dots \cup d_\lambda$. At least one of these loops, say d_1, bounds a disk, say t_1, on Σ' which contains no point of Σ in its interior; $\Sigma \cap t_1 = \partial t_1 = d_1$. Let s_1 and s_1' be disks on Σ bounded by d_1 with $s_1 \cup s_1' = \Sigma$. Then we two admissible 2-spheres $\Sigma_1 = s_1 \cup t_1$ and $\Sigma_2 = s_1' \cup t_1$ of type I for $(P \subset S^3)$. We can deform $\Sigma_1 \cup \Sigma_2$ in S^3 so that $\Sigma_1 \cap \Sigma_2 = \omega$ and $(\Sigma_1 \cup \Sigma_2) \cap \Sigma' = d_2 \cup \dots \cup d_\lambda$. Moreover, we can deduce easily that $\Sigma_1 \cup \Sigma_2$ decomposes $(P \subset S^3)$ into three graphs, say $(P_1' \subset S^3)$, $(P_2' \subset S^3)$ and $(P_3' \subset S^3)$, such that

$$\begin{cases} (P_1 \subset S^3) \cong (P_1' \subset S^3) \vee (P_2' \subset S^3) \text{ and } (P_2 \subset S^3) \cong (P_3' \subset S^3) \text{ if } t_1 \subset \Delta_\Sigma, \\ (P_1 \subset S^3) \cong (P_1' \subset S^3) \text{ and } (P_2 \subset S^3) \cong (P_2' \subset S^3) \vee (P_3' \subset S^3) \text{ if } t_1 \subset \nabla_\Sigma. \end{cases}$$

Repeating the procedure, we have $\lambda + 1$ admissible 2-spheres, say $\Sigma_1, \dots, \Sigma_{\lambda+1}$, of type I for $(P \subset S^3)$ having the one point ω in common. In particular, these 2-spheres decompose $(P_1 \subset S^3)$ and $(P_2 \subset S^3)$ into $\lambda + 2$ graphs and $(\Sigma_1 \cup \dots \cup \Sigma_{\lambda+1}) \cap \Sigma' = \omega$. Since $(Q \subset S^3)$ is prime, we can take an admissible 2-sphere of type I, again denote it by Σ', in S^3 such that Σ' gives the decomposition $(P \subset S^3) \cong (Q \subset S^3) \vee (Q' \subset S^3)$ and $(\Sigma_1 \cup \dots \cup \Sigma_{\lambda+1}) \cap \Delta_{\Sigma'} = (\Sigma_1 \cup \dots \cup \Sigma_{\lambda+1}) \cap \Sigma' = \omega$.

Thus, we can conclude that $(Q \subset S^3)$ is a prime component of either $(P_1 \subset S^3)$ or $(P_2 \subset S^3)$. This completes the proof of Lemma 4.1. □

4.2. **Lemma.** *Let* $(P \subset S^3)$ *be a non-trivial and non-splittable graph, and we suppose that there are two admissible 2-spheres* Σ *of type* I *and* Σ' *of type* II *for* $(P \subset S^3)$ *giving decompositions*

$$(P \subset S^3) \cong (P_1 \subset S^3) \vee_\Sigma (P_2 \subset S^3), \text{ and}$$
$$(P \subset S^3) \cong (Q \subset S^3) \#_{\Sigma'} (Q' \subset S^3), \text{ respectively.}$$

If $(Q \subset S^3)$ *is prime, then either* $(P_1 \subset S^3)$ *or* $(P_2 \subset S^3)$ *has* $(Q \subset S^3)$ *as a prime component.*

Proof. We set $\Sigma \cap P = \omega$ and $\Sigma' \cap P = \{a,b\}$. If $\Sigma \cap \Sigma' = \emptyset$, then we are finished ; and if $\omega = a$ (resp. $\omega = b$) and $\Sigma \cap \Sigma' = \omega$, then we are also finished. We may assume, after a slight deformation, that $\Sigma \cap \Sigma'$ consists of a finite number of simple loops. We distinguish two cases :

Case 1. $\omega \cap \{a,b\} = \emptyset$ (i.e. $\omega \neq a$, $\omega \neq b$): We may assume that $\Sigma \cap \Sigma'$ consists of a finite number of mutually disjoint simple loops, say $c_1, \ldots, c_\nu$. Let $\sigma_1, \ldots, \sigma_\nu$ be disks on Σ bounded by $c_1, \ldots, c_\nu$, respectively, such that $\sigma_i \not\ni \omega$. If there exists a loop, say c_i, in $c_1, \ldots, c_\nu$ such that c_i bounds a disk τ_i on Σ' with $\tau_i \cap P = \tau_i \cap \{a,b\} = \emptyset$, then we can choose an innermost one, say c_1, in these loops ; that is, c_1 bounds a disk τ_1 on Σ' with $\tau_1 \cap P = \emptyset$ and $\Sigma \cap \tau_1 = \partial\tau_1 = c_1$. Now we can remove c_1 from $\Sigma \cap \Sigma'$ by changing Σ as the same way as that of Case 1 in the proof of Lemma 4.1. Therefore, we may assume that every c_i $(i=1,\ldots,\nu)$ bounds disks τ_i and τ_i' on Σ' such that $\tau_i \cap P = a$ and $\tau_i' \cap P = b$; and so c_i is essential on the annulus $A' = \Sigma' - {}^\circ N(P;S^3)$. Let c_1 be an innermost one on Σ ; that is, $\sigma_1 \cap \Sigma' = \partial\sigma_1 = c_1$. Then σ_1 is a disk in ${}^\circ E(P)$ such that $\sigma_1 \cap A' = \partial\sigma_1 = c_1$. This contradicts to Definition 2.3(iv), and so we deduce that $\Sigma \cap \Sigma' = \emptyset$; thus Lemma 4.2 is established for Case 1.

Case 2. $\omega \cap \{a,b\} = \omega$: In this case, we may assume that $\omega = a$ $(\omega \neq b)$. Now we may assume that $\Sigma \cap \Sigma'$ consists of a finite number of simple loops, say $c_1, \ldots, c_\nu, d_1, \ldots, d_\lambda$, such that $c_i \cap c_j = \emptyset$ $(i \neq j)$, $d_i \cap d_j = \omega$ $(i \neq j)$ and $(c_1 \cup \ldots \cup c_\nu) \cap (d_1 \cup \ldots \cup d_\lambda) = \emptyset$. By the same way as that of Case 1, we can remove $c_1 \cup \ldots \cup c_\nu$ from $\Sigma \cap \Sigma'$, and then we may assume that $\Sigma \cap \Sigma' = d_1 \cup \ldots \cup d_\lambda$. It follows from this condition that every d_i bounds a disk, say t_i, on Σ' such that $t_i \not\ni b$ $(i=1,\ldots,\lambda)$. Now Lemma follows by the same argument as that of Case 2 in the proof of Lemma 4.1. □

4.3. Lemma. *Let* $(P \subset S^3)$ *be a non-trivial and non-splittable graph, and we suppose that there are two admissible 2-spheres* Σ *of type* II *and* Σ' *of type* I *for* $(P \subset S^3)$ *giving decompositions*

$$(P \subset S^3) \cong (P_1 \subset S^3) \#_\Sigma (P_2 \subset S^3), \text{ and}$$

$$(P \subset S^3) \cong (Q \subset S^3) \vee_{\Sigma'} (Q' \subset S^3), \text{ respectively.}$$

If $(Q \subset S^3)$ *is prime, then either* $(P_1 \subset S^3)$ *or* $(P_2 \subset S^3)$ *has* $(Q \subset S^3)$ *as a prime component.*

Proof. The proof of Lemma 4.3, which is omitted here, is very similar to that of Lemma 4.2 except for obvious modifications. □

4.4. Lemma. *Let* $(P \subset S^3)$ *be a non-trivial and non-splittable graph, and we suppose that there are two admissible 2-spheres* Σ *and* Σ' *of type* II *for* $(P \subset S^3)$ *giving decompositions*

$$(P \subset S^3) \cong (P_1 \subset S^3) \#_{\Sigma} (P_2 \subset S^3), \text{ and}$$
$$(P \subset S^3) \cong (Q \subset S^3) \#_{\Sigma'} (Q' \subset S^3), \text{ respectively.}$$

If $(Q \subset S^3)$ *is prime, then either* $(P_1 \subset S^3)$ *or* $(P_2 \subset S^3)$ *has* $(Q \subset S^3)$ *as a prime component.*

Proof. We set $\Sigma \cap P = \{a,b\}$ and $\Sigma' \cap P = \{a',b'\}$, and we may assume that $(Q \subset S^3)$ is obtained from $P \cap \Delta_{\Sigma'}$. In the following three cases, there is nothing to prove :

(i) $\Sigma \cap \Sigma' = \emptyset$,

(ii) $\{a,b\} \cap \{a',b'\}$ consists of one point, say $a = a'$, and $\Sigma \cap \Sigma' = a$,

(iii) $\{a,b\} = \{a',b'\}$ and $\Sigma \cap \Sigma' = \{a,b\}$.

We now assume, after a slight modification, that $\Sigma \cap \Sigma'$ consists of a finite number of simple loops. We distinguish three cases :

Case 1. $\{a,b\} \cap \{a',b'\} = \emptyset$: If $\Sigma \cap \Sigma' \neq \emptyset$, then we may assume that $\Sigma \cap \Sigma'$ consists of a finite number of mutually disjoint simple loops, say $c_1, \dots,$ c_ν. For clarity, we divide the proof into three steps.

(1) We suppose that there exists a loop, say c_i, in $\Sigma \cap \Sigma'$, such that c_i bounds a disk, say τ_i, on Σ' with $\tau_i \cap \{a',b'\} = \emptyset$. Then, we can

choose an innermost one, say c_1, in such loops ; that is, c_1 bounds a disk τ_1 on Σ' such that $\Sigma \cap \tau_1 = \partial\tau_1 = c_1$. Let σ_1 and σ_1' be the disks on Σ bounded by c_1.

If $\sigma_1 \cap \{a,b\} = \emptyset$ or $\sigma_1' \cap \{a,b\} = \emptyset$, we can remove c_1 from $\Sigma \cap \Sigma'$ by changing Σ as the same way as that of Case 1 in the proof of Lemma 4.1.

If $\sigma_1 \cap \{a,b\} = a$ (or $\sigma_1 \cap \{a,b\} = b$), then it follows from the same argument as that of Case 1 in the proof of Lemma 4.2, that the annulus $A = \Sigma - {}^\circ N(P;S^3)$ is compressible in $E(P)$, which contradicts to Definition 2.3(iv). We see that such a loop c_i does not exist.

(2) If there exists a loop, say c_i, in $\Sigma \cap \Sigma'$ such that c_i bounds a disk, say σ_i, on Σ with $\sigma_i \cap \{a,b\} = \emptyset$, then we can also remove c_i from $\Sigma \cap \Sigma'$ by the same way as that of above (1).

(3) Now we may assume that every c_i separates a and b on Σ and a' and b' on Σ' $(i=1,\dots,\nu)$ and so $c_1, \dots, c_\nu$ are concentric on both Σ and Σ'. A single one of these loops, say c_1, bounds a disk, say σ_1, on Σ which contains no point of Σ' and contains a. Let τ_1 and τ_1' be the disks on Σ' bounded by c_1 with $\tau_1 \ni a'$ and $\tau_1' \ni b'$. If $\sigma_1 \subset \Delta_{\Sigma'}$, then we have two admissible 2-spheres $\Sigma_1 = \sigma_1 \cup \tau_1$ and $\Sigma_2 = \sigma_1 \cup \tau_1'$ of type II for $(P \subset S^3)$ and also for $(Q \subset S^3)$ with $\Sigma_1 \cap P = \{a,a'\}$, $\Sigma_2 \cap P = \{a,b'\}$. Since $(Q \subset S^3)$ is prime, one of $(Q \cap \Delta_{\Sigma_1} \subset \Delta_{\Sigma_1}) = (P \cap \Delta_{\Sigma_1} \subset \Delta_{\Sigma_1})$ and $(Q \cap \Delta_{\Sigma_2} \subset \Delta_{\Sigma_2}) = (P \cap \Delta_{\Sigma_2} \subset \Delta_{\Sigma_2})$ represents a trivial knot (i.e. is equivalent to the standard disk-pair $(D^1 \subset D^3)$), provided that $\Delta_{\Sigma_1} \subset \Delta_{\Sigma'}$ and $\Delta_{\Sigma_2} \subset \Delta_{\Sigma'}$, and so we can deform Σ so that $\Sigma \cap \Sigma' \subset c_2 \cup \dots \cup c_\nu$. This argument implies that $a \in \sigma_1 \subset \nabla_{\Sigma'}$ (and also $b \in \nabla_{\Sigma'}$ as well), and so $\Sigma \cap \Sigma'$ consists of even number of loops.

Now in the loops $\Sigma \cap \Sigma' = c_1 \cup \dots \cup c_\nu$, we choose adjacent loops on Σ, say

c_1 and c_2, such that the annulus $B \subset \Sigma$ bounded by $c_1 \cup c_2$ lies in $\Delta_{\Sigma'}$. Let B' be the annulus on Σ' bounded by $c_1 \cup c_2$. Then the torus $B \cup B'$ bounds a socalled solid-torus, say T, in $\Delta_{\Sigma'}$ as $B \cup B'$ is unknotted in S^3 (Alexander [1]). Since $(Q \subset S^3)$ is prime, it is easy to check that $T \cap P = \emptyset$ from Definition 2.3. Therefore, we can deform Σ along T ambient isotopically in S^3 keeping P fixed so that $\Sigma \cap \Sigma' \subset c_3 \cup \dots \cup c_\nu$.

Repeating the procedure, we can deduce that $\Sigma \cap \Sigma' = \emptyset$ and also $\Sigma \cap \Delta_{\Sigma'} = \emptyset$; and completing the proof of Lemma for Case 1.

Case 2. $\{a,b\} \cap \{a',b'\}$ consists of a point : We can assume, without loss of generality, that $a \neq a'$ and $b = b'$; and that $\Sigma \cap \Sigma'$ consists of a finite number of simple loops, say $c_1, \dots, c_\nu, d_1, \dots, d_\lambda$, such that $c_i \cap c_j = \emptyset$ $(i \neq j)$, $d_i \cap d_j = b$ $(i \neq j)$ and $(c_1 \cup \dots \cup c_\nu) \cap (d_1 \cup \dots \cup d_\lambda) = \emptyset$. We also divide the proof into three steps.

(1) If there exists a loop, say c_i, in $\Sigma \cap \Sigma'$ such that c_i bounds a disk τ_i on Σ' with $\tau_i \cap \{a',b'\} = \emptyset$ or c_i bounds a disk σ_i on Σ with $\sigma_i \cap \{a,b\} = \emptyset$, then we can remove c_i from $\Sigma \cap \Sigma'$ by the same way as that of Case 1(1) and (2). Therefore, we assume that every c_i separates a and b on Σ and a' and $b' = b$ on Σ' $(i=1,\dots,\nu)$.

(2) Now among loops $d_1, \dots, d_\lambda$, there must be at least one, say d_1, that bounds a disk, say t_1, on Σ' whose interior contains no other loops $c_1, \dots, c_\nu, d_2, \dots, d_\lambda$ and the point a' ; $\Sigma \cap t_1 = \partial t_1 = d_1$ and $t_1 \not\ni a'$. Let s_1 and s_1' be the disks on Σ bounded by d_1 with $s_1 \ni a$. Then we have two admissible 2-spheres $\Sigma_0 = s_1 \cup t_1$ of type II and $\Sigma_1 = s_1' \cup t_1$ of type I for $(P \subset S^3)$ with $\Sigma_0 \cap P = \{a,b\}$ and $\Sigma_1 \cap P = b$. We can deform $\Sigma_0 \cup \Sigma_1$ in S^3 so that $\Sigma_0 \cap \Sigma_1 = b$ and $(\Sigma_0 \cup \Sigma_1) \cap \Sigma' = c_1 \cup \dots \cup c_\nu \cup d_2 \cup \dots \cup d_\lambda$. Moreover, we can easily deduce that $\Sigma_0 \cup \Sigma_1$ decomposes $(P \subset S^3)$ into three graphs, say $(P_1' \subset S^3)$, $(P_2' \subset S^3)$ and $(P_3' \subset S^3)$, such that

$$\begin{cases} (P_1 \subset S^3) \cong (P_1' \subset S^3) \vee (P_2' \subset S^3),\ (P_2 \subset S^3) \cong (P_3' \subset S^3) & \text{if } t_1 \subset \Delta_\Sigma, \\ (P_1 \subset S^3) \cong (P_1' \subset S^3),\ (P_2 \subset S^3) \cong (P_2' \subset S^3) \vee (P_3' \subset S^3) & \text{if } t_1 \subset \nabla_\Sigma. \end{cases}$$

Repeating the procedure, we have $\lambda + 1$ admissible 2-spheres Σ_0 of type II and $\Sigma_1, \dots, \Sigma_\lambda$ of type I for $(P \subset S^3)$ having the point b in common such that $(\Sigma_0 \cup \Sigma_1 \cup \dots \cup \Sigma_\lambda) \cap \Sigma' = b \cup c_1 \cup \dots \cup c_\nu$, $\Sigma_i \cap \Sigma' = b$ $(i=1,\dots,\nu)$, $\Sigma_0 \cap \Sigma' = b \cup c_1 \cup \dots \cup c_\nu$ and $\Sigma_0 \cup \Sigma_1 \cup \dots \cup \Sigma_\lambda$ decomposes $(P_1 \subset S^3)$ and $(P_2 \subset S^3)$ into $\lambda + 2$ graphs. The argument here is very similar to that of Case 2 in the proof of Lemma 4.1.

(3) Now it is easy to check that every c_i separates a and b on Σ_0 and a' and b' on Σ' $(i=1,\dots,\nu)$, and so we can remove $c_1, \dots, c_\nu$ by the same way as that of above Case 1(3) ; proving Lemma for Case 2.

Case 3. $\{a,b\} \cap \{a',b'\} = \{a,b\} = \{a',b'\}$: We can assume that $a = a'$ and $b = b'$, and that $\Sigma \cap \Sigma'$ consists of a finite number of simple loops, say $c_1, \dots, c_\nu, d_1, \dots, d_\lambda, d_1', \dots, d_\kappa'$, and even number of simple arcs, say $e_1, \dots, e_{2m}$, such that $c_i \cap c_j = \emptyset$ $(i \neq j)$, $d_i \cap d_j = a$ $(i \neq j)$, $d_i' \cap d_j' = b$ $(i \neq j)$, $e_i \cap e_j = \partial e_i = \partial e_j = \{a,b\}$ $(i \neq j)$, $(c_1 \cup \dots \cup c_\nu) \cap (d_1 \cup \dots \cup d_\lambda \cup d_1' \cup \dots \cup d_\kappa' \cup e_1 \cup \dots \cup e_{2m}) = \emptyset$, $(d_1 \cup \dots \cup d_\lambda) \cap (d_1' \cup \dots \cup d_\kappa') = \emptyset$, $(d_1 \cup \dots \cup d_\lambda) \cap (e_1 \cup \dots \cup e_{2m}) = a$ and $(d_1' \cup \dots \cup d_\kappa') \cap (e_1 \cup \dots \cup e_{2m}) = b$. If there exists a loop, say c_i, in $\Sigma \cap \Sigma'$ such that c_i bounds a disk τ_i on Σ' with $\tau_i \cap \{a,b\} = \emptyset$ or a disk σ_i on Σ with $\sigma_i \cap \{a,b\} = \emptyset$, then we can remove c_i from $\Sigma \cap \Sigma'$ by the same way as that of Case 1(1) and (2). Therefore, we assume that every c_i separates a and b on both Σ and Σ' $(i=1,\dots,\nu)$. It should be noted that if $\nu > 0$ then $m = 0$, and if $m > 0$ then $\nu = 0$. There are two subcases to consider :

Case 3.1. $\Sigma \cap \Sigma' = c_1 \cup \dots \cup c_\nu \cup d_1 \cup \dots \cup d_\lambda \cup d_1' \cup \dots \cup d_\kappa'$: In this case, Lemma follows from the quite similar argument to that of Case 2(2) and (3), and we omit the proof.

Case 3.2. $\Sigma \cap \Sigma' = d_1 \cup \ldots \cup d_\lambda \cup d'_1 \cup \ldots \cup d'_\kappa \cup e_1 \cup \ldots \cup e_{2m}$: By the same way as that of Case 2(2), we have $\lambda+\kappa+1$ admissible 2-spheres Σ_0 of type II and $\Sigma_1, \ldots, \Sigma_\lambda, \Sigma^*_1, \ldots, \Sigma^*_\kappa$ of type I for $(P \subset S^3)$ such that $\Sigma_0 \cap \Sigma_i = a$ $(i=1,\ldots,\lambda)$, $\Sigma_0 \cap \Sigma^*_k = b$ $(k=1,\ldots,\kappa)$, $\Sigma_i \cap \Sigma_j = a$ $(i \neq j)$, $\Sigma^*_k \cap \Sigma^*_h = b$ $(k \neq h)$, $\Sigma_i \cap \Sigma' = a$ $(i=1,\ldots,\lambda)$, $\Sigma^*_k \cap \Sigma' = b$ $(k=1,\ldots,\kappa)$, $\Sigma_0 \cap \Sigma' = e_1 \cup \ldots \cup e_{2m}$ and $\Sigma_0 \cup \Sigma_1 \cup \ldots \cup \Sigma_\lambda \cup \Sigma^*_1 \cup \ldots \cup \Sigma^*_\kappa$ decomposes $(P_1 \subset S^3)$ and $(P_2 \subset S^3)$ into $\lambda+\kappa+2$ graphs.

We take in $\Sigma \cap \Sigma'$ adjacent arcs on Σ', say e_1 and e_2, then the simple loop $e_1 \cup e_2$ bounds a disk, say ε, on Σ' such that $\Sigma_0 \cap \varepsilon = \partial\varepsilon = e_1 \cup e_2$. Let δ and δ' be the disks on Σ_0 bounded by $e_1 \cup e_2$. Then we have two admissible 2-spheres $\Sigma^0_1 = \delta \cup \varepsilon$ and $\Sigma^0_2 = \delta' \cup \varepsilon$ of type II for $(P \subset S^3)$ with $\Sigma^0_1 \cap P = \Sigma^0_2 \cap P = \{a,b\}$. We can deform $\Sigma^0_1 \cup \Sigma^0_2$ in S^3 so that $\Sigma^0_1 \cap \Sigma^0_2 = \{a,b\}$ and $(\Sigma^0_1 \cup \Sigma^0_2) \cap \Sigma' = e_3 \cup \ldots \cup e_{2m}$, and in particular $\Sigma^0_1 \cup \Sigma^0_2 \cup \Sigma_1 \cup \ldots \cup \Sigma_\lambda \cup \Sigma^*_1 \cup \ldots \cup \Sigma^*_\kappa$ decomposes $(P_1 \subset S^3)$ and $(P_2 \subset S^3)$ into $\lambda+\kappa+3$ graphs.

Repeating the procedure, finally we have $\lambda+\kappa+m+1$ admissible 2-spheres $\Sigma^0_1, \ldots, \Sigma^0_{m+1}$ of type II and $\Sigma_1, \ldots, \Sigma_\lambda, \Sigma^*_1, \ldots, \Sigma^*_\kappa$ of type I for $(P \subset S^3)$ such that $\Sigma^0_\xi \cap \Sigma^0_\rho = \{a,b\}$ $(\xi \neq \rho)$, $\Sigma^0_\xi \cap \Sigma_i = a$ $(\xi=1,\ldots,m+1\,;\, i=1,\ldots,\lambda)$, $\Sigma^0_\xi \cap \Sigma^*_k = b$ $(\xi=1,\ldots,m+1\,;\, k=1,\ldots,\kappa)$, $\Sigma_i \cap \Sigma_j = a$ $(i \neq j)$, $\Sigma^*_k \cap \Sigma^*_h = b$ $(k \neq h)$, $\Sigma^0_\xi \cap \Sigma' = \{a,b\}$ $(\xi=1,\ldots,m+1)$, $\Sigma_i \cap \Sigma' = a$ $(i=1,\ldots,\lambda)$, $\Sigma^*_k \cap \Sigma' = b$ $(k=1,\ldots,\kappa)$ and $\Sigma^0_1 \cup \ldots \cup \Sigma^0_{m+1} \cup \Sigma_1 \cup \ldots \cup \Sigma_\lambda \cup \Sigma^*_1 \cup \ldots \cup \Sigma^*_\kappa$ decomposes $(P_1 \subset S^3)$ and $(P_2 \subset S^3)$ into $\lambda+\kappa+m+2$ graphs. Since $(Q \subset S^3)$ is prime, we conclude Lemma for Case 3.2 as the same way as that of Case 2 in the proof of Lemma 4.1.

In every cases, we see that at least one of $(P_1 \subset S^3)$ and $(P_2 \subset S^3)$ has $(Q \subset S^3)$ as a prime component, and completing the proof of Lemma 4.4. □

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