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Cycles in Graphs

Katsuhiro Ota (太田克弘)

and

Akira Saito (斎藤明)

Department of Information Science

Faculty of Science

University of Tokyo

In this paper, we consider only finite simple graphs. A cycle C in G is called *separating* if deletion of its vertices results in a disconnected graph. A cycle is *non-separating* if it is not separating. Lovász[3] remarked that every 3-connected graph has a non-separating cycle. Thomassen and Toft[4] extended his result and proved that every 3-connected graph has a non-separating induced cycle. We prove in this paper that every 3-connected graph has many non-separating induced cycles.

When we prove the property of 3-connected graph, the idea of a contractible edge is very useful. In a 3-connected graph, an edge e is called ***contractible*** if contraction of e results in a 3-connected graph. An edge is ***non-contractible*** if it is not contractible. Obviously, an edge xy in G is non-contractible if and only if G has a cutset of order three which contains both x and y.

We denote the set of vertices and the set of edges of the graph G by $V(G)$ and $E(G)$, respectively. Let $x \in V(G)$. We denote the set of the vertices adjacent to x by $\Gamma_G(x)$, the degree of x by $d_G(x)$ and the order of G by $|G|$. Other notations may be found in [2].

For a 3-connected graph G and $x \in V(G)$, we write

$\Gamma^{(1)}(x) := \{y \in \Gamma_G(x) \mid xy$ is a contractible edge$\}$
$\Gamma^{(2)}(x) := \{y \in \Gamma_G(x) \mid xy$ is a non-contractible edge$\}$

and define $U_i (i=0,1,2,3), W_i (i \geq 0)$, subset of $V(G)$, by

$U_i := \{x \in V(G) \mid d_G(x)=3, |\Gamma^{(1)}(x)| = i\}$ $(i=0,1,2,3)$
$W_i := \{x \in V(G) \mid d_G(x) \geq 4, |\Gamma^{(1)}(x)| = i\}$ $(i \geq 0)$

Ando, Enomoto and Saito[1] proved the following theorems.

Theorem A. *Let G be a 3-connected graph of order at least five and $x \in V(G)$. If $d_G(x)=3$, say $\Gamma_G(x)=\{a,b,c\}$, and both xb and xc are non-contractible, then $d_G(b)=d_G(c)=3$, and b and c are adjacent.*

Theorem B. $U_0 = \phi$ *if* $|G| \geq 5$.

Theorem C. *Suppose* $|G| \geq 5$. *If* $x \in W_0$ *then* $|\Gamma^{(2)}(x) \cap U_2| \geq 3$. *If* $x \in W_1$ *then* $|\Gamma^{(2)}(x) \cap U_2| \geq 2$.

Theorem B is easily deduced from Theorem A.

The following theorem is a slight generalization of the result of Ando, Enomoto and Saito([1, Theorem 3]).

Theorem 1. *Let G be a 3-connected graph of order at least five, $x \in V(G)$ and $X \subset V(G)$. Suppose $\Gamma^{(1)}(x) \subset X$ and $d_G(x) \geq 4$. Moreover, suppose there exists a least cutset S, which contains x, such that $G-S$ has a connected component disjoint from X. Then $(\Gamma^{(2)}(x)-X) \cap U_2 \neq \phi$.*

Proof. Define

$C_x := \{S$; a least cutset such that $x \in S$
and $G-S$ has a conencted component A which is disjoint from $X\}$.

By the assumption $C_x \neq \phi$. For each $S \in C_x$, let A_S be the smallest component of $G-S$ such that A_S is disjoint from X. Moreover, we choose $S \in C_x$ such that $|A_S|$ is minimum. Let $B_S = V(G) - A_S - S$.

Since $A_S \cap X=\phi$, there exists a vertex $\boldsymbol{y}$ in $A_S \cap \Gamma_G(\boldsymbol{x})$ such that $\boldsymbol{xy}$ is a non-contractible edge. Let T be a least cutset containing the edge $\boldsymbol{xy}$, C be one of the connected components of $G-T$ and $D=V(G)-T-C$. Let

$$X_1{:=}(S\cap C)\cup(S\cap T)\cup(A_S\cap T)$$

and

$$X_2{:=}(S\cap C)\cup(S\cap T)\cup(B_S\cap T).$$

First we claim that $S\cap C\neq\phi$. Assume $S\cap C=\phi$. Since $y\in A_S\cap T$, $|X_2|\leq 2$. If $B_S\cap C\neq\phi$ then X_2 is a cutset of order at most two. This contradicts the assumption on connectivity of G. Therefore, $B_S\cap C=\phi$ and it follows $A_S\cap C\neq\phi$ since $C\neq\phi$. Hence X_1 is a cutset and since $X_1\subset T$, $|X_1|\leq 3$. This contradicts either the connectivity of G or the minimality of A_S.

The similar argument leads us to $S\cap D\neq\phi$. Therefore, we know that $|S\cap C| = |S\cap T| = |S\cap D| =1$.

Next we claim $B_S\cap T\neq\phi$. Assume $B_S\cap T=\phi$. Without loss of generality, We may assume that $B_S\cap C\neq\phi$, and it follows that X_2 is a cutset of order at most two. This is a contradiction.

Therefore, we know that $|A_S\cap T| = |S\cap T| = |B_S\cap T| =1$, and $|X_1| =3$. Since A_S is minimal, $A_S\cap C=\phi$ and $A_S\cap D=\phi$. Hence we have $|A_S|=1$ and $A_S=\{y\}$. Then $\Gamma_G(y)=S$ and $d_G(y)=3$.

Let $a\in S-\{x\}$. If ya is a non-contractible edge, then $d_G(x)=3$ by Theorem A. This contradicts the assumption that $d_G(x)\geq 4$. Hence ay is a contractible edge. This implies $y\in U_2$. ▪

For $X\subset V(G)$ and $F\subset E(G)$, we say X *covers* F in case that any edge of F is incident with at least one of the vertices of X.

Theorem 2. *Let G be a 3-connected graph of order at least six. Then, the set of the contractible edges of G cannot be covered with any set of two* *vertices.*

Proof. We assume that there is a 3-connected graph G of order at least six such that a set of contractible edges is covered with two vertices, say u and v. From Theorem B, $V(G)$ can be written as

$$V(G)=U_1\cup U_2\cup W_0\cup W_1\cup W_2\cup\{u,v\}.$$

since $W_i=U_i=0$ if $i\geq 3$.

Claim 1: $W_i-\{u,v\}=\phi$ for $i=0,1$.

Suppose $W_i-\{u,v\}\neq\phi$, say $x\in W_i-\{u,v\}$. Let X be $(\Gamma^{(2)}(x)\cap U_2)\cup\{u,v\}$. By Theorem C, there exists $a\in(\Gamma^{(2)}(x)\cap U_2)-\{u,v\}$. (If $x\in W_1$ and $\Gamma^{(2)}(x)\cap U_2=\{u,v\}$, then xu and xv are non-contractible, which contradicts the assumption that $x\in W_1$). If $a\in(\Gamma^{(2)}(x)\cap U_2)-\{u,v\}$, then $\Gamma_G(a)=\{x,u,v\}$. Hence $S=\{x,u,v\}$ is a least cutset which contains x and $G-S$ has a connected component disjoint from X, if $\Gamma_G(x)\not\subset X$. Assume $\Gamma_G(x)\not\subset X$. Then by applying Theorem C, we have $(\Gamma^{(2)}(x)-X)\cap U_2\neq\phi$. This contradicts the fact that $\Gamma^{(2)}(x)\cap U_2\subset X$. It follows $\Gamma_G(x)\subset X$.

Next assume that $V(G)\neq\{x\}\cup X$, then $\{u,v\}$ is a cutset of G, which is impossible because G is 3-connected.

Thus G is the graph such that

$$V(G)=\{x\}\cup(\Gamma^{(2)}(x)\cap U_2)\cup\{u,v\}\qquad\text{(disjoint)}$$
$$E(G)\supset\{xy,yu,yv\mid y\in\Gamma^{(2)}(x)\cap U_2\}$$

and for every $y\in\Gamma^{(2)}(x)\cap U_2$, xy is a non-contractible edge and yu and yv are contractible edges. In this graph, however, for each vertex $y\in\Gamma^{(2)}(x)\cap U_2$, $G-\{x,y\}$ is 2-connected since $|G|\geq 6$. This contradicts the fact that an edge xy is non-contractible. Hence the claim follows.

Now we have

$$V(G)=U_1\cup U_2\cup W_2\cup\{u,v\}\qquad(1)$$

Claim 2: Let $x \in W_2$, then $d_G(x)=4$, say $\Gamma_G(x)=\{a,b,u,v\}$, and $\{x,a,b\}$ is a cutset separating u and v.

Let X be $\{u,v\}$ ($=\Gamma^{(1)}(x)$) . If there exists a least cutset S such that $x \in S$ and $G-S$ has a connected component disjoint from X, then by Theorem C, $(\Gamma^{(2)}(x)-X) \cap U_2 \neq \phi$. This means that $\{x,u,v\}$ is a cutset, which is a contradiction since edges xu and xv are contractible. Therefore, every least cutset S which contains x, say $\{x,a,b\}$, separates u and v. Let A be a connected component of $G-S$ which contains u, and $B=V(G)-S-A$. If there is a vertex $y \in (A-\{u\}) \cap \Gamma_G(x)$, then by (1) $y \in U_1 \cup U_2 \cup W_2$. However y and v cannot be adjacent, so $y \in U_1$. Since an edge xy is non-contractible, we have $d_G(x)=3$ by Theorem A. This is a contradiction since $x \in W_2$. Since $d_G(x) \geq 4$, $\Gamma_G(x)=\{a,b,u,v\}$, and the claim follows.

Now we consider $G-\{u,v\}$. Since G is 3-connected, $G-\{u,v\}$ is connected and $\delta(G-\{u,v\}) \leq 2$ by (1) and Claim 2. Hence $G-\{u,v\}$ is a path or a cycle.

First assume that $G-\{u,v\}$ is a path, say $G-\{u,v\} \simeq P_n$. Since $|G| \geq 6$, $n \geq 4$. Let x be a internal vertex of P_n. Then $\{x,u,v\}$ is a cutset of G. On the other hand, xu or xv is contractible since $x \in U_1 \cup U_2 \cup W_2$. This is a contradiction. Therefore, $G-\{u,v\}$ is a cycle, say $G-\{u,v\} \simeq C_n$. In this case $U_2-\{u,v\}=\phi$, and hence $V(G)=U_1 \cup W_2 \cup \{u,v\}$. First we claim $U_1-\{u,v\}=\phi$. Assume the contrary and let $x \in U_1-\{u,v\}$. By Theorem A, two neighbors of x in $V(G)-\{u,v\}$ are adjacent. This is impossible since $|G| \geq 6$.

Thus $V(G)=W_2 \cup \{u,v\}$. However, for each $x \in V(G)-\{u,v\}$, $\Gamma^{(2)}(x) \cup \{x\}$ is not a cutset since $|G| \geq 6$. This contradicts Claim 2. This is a final contradiction and the proof is complete. ▪

Theorem 3. *Let G be a 3-connected graph and e be an edge of G. Then there exists a non-separating induced cycle which contains e.*

Proof. We prove the theorem by induction on $|G|$. When $|G| \leq 5$, we can easily check the result. Now we can assume that $|G| \geq 6$, and that all 3-connected graph of order less than $|G|$ has a non-separating induced cycle which contains a specified edge.

Assume G has no non-separating induced cycle which contains a specified edge e. Let a and b be the endvertices of e. By Theorem 2, G has a contractible edge, say xy, which is not incident with a or b. Let G' be the graph obtained from G by contraction of xy. By induction hypothesis, G' has a non-separating induced cycle C' which contains e. If the contracted vertex z is not on C', then C' is also a non-separating induced cycle in G which contains e. Therefore, we can assume that z lies on C'. Let u,v be the vertices adjacent to z on C' and P' be the path obtained by $C'-z$. In G, u and v are adjacent to x or y. Now two cases occur.

Case 1: At least one of $\{x,y\}$ is adjacent to both u and v.

Without loss of generality, we have $\Gamma_G(x) \supset \{u,v\}$. If y is adjacent to the vertex of $V(G)-V(C')$, then $P' \cup \{x\}$ with edges xu and xv is a non-separating cycle in G, a contradiction. Therefore y can be adjacent only to u,v and x, since C' is an induced cycle in G'. This implies $\Gamma_G(y)=\{u,v,x\}$ since the minimum degree of G is at least three. Applying the same argument to y, we have $\Gamma_G(x)=\{u,v,y\}$. Then, degree of z is two in G', which contradicts the fact that xy is a contractible edge.

Case 2: The vertex x is adjacent to one of u , v and y to the other.

We can assume that x is adjacent to u (and not to v) and y to v (and not to u). Then, $P' \cup \{x,y\}$ with edges ux , xy and yv is a non-separating induced cycle in G.

This completes the proof of the theorem. ∎

Corollary 4. *Let G be a 3-connected graph and x and y be vertices of G. Then there exists an induced x,y-path P such that $G-V(P)$ is connected.*

Proof. If x and y are adjacent, then the edge xy is a desired path. Otherwise, in the graph obtained from G by adding an edge xy, there exists a non-separating induced cycle which contains xy, by Theorem 3. This cycle induces a non-separating induced path in G. ▪

The above corollary leads us to the following conjecture.

Conjecture. *For a given integer k $(k \geq 1)$, there exists a minimum number n_k such that every n_k-connected graph G satisfies the following property (*).*

(*) *For every pair of distinct vertices x,y of G, there exists an induced x,y-path P such that $G-V(P)$ is k-connected.*

Theorem 3 indicates that $n_1=3$.

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