Cycles in Graphs

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In this paper, we consider only finite simple graphs. A cycle $C$ in $G$ is called separating if deletion of its vertices results in a disconnected graph. A cycle is non-separating if it is not separating. Lovász[3] remarked that every 3-connected graph has a non-separating cycle. Thomassen and Toft[4] extended his result and proved that every 3-connected graph has a non-separating induced cycle. We prove in this paper that every 3-connected graph has many non-separating induced cycles.

When we prove the property of 3-connected graph, the idea of a contractible edge is very useful. In a 3-connected graph, an edge $e$ is called contractible if contraction of $e$ results in a 3-connected graph. An edge is non-contractible if it is not contractible. Obviously, an edge $xy$ in $G$ is non-contractible if and only if $G$ has a cutset of order three which contains both $x$ and $y$.

We denote the set of vertices and the set of edges of the graph $G$ by $V(G)$ and $E(G)$, respectively. Let $x \in V(G)$. We denote the set of the vertices adjacent to $x$ by $\Gamma_G(x)$, the degree of $x$ by $d_G(x)$ and the order of $G$ by $|G|$. Other notations may be found in [2].

- 1 -
For a 3-connected graph $G$ and $x \in V(G)$, we write

\[ \Gamma^{(1)}(x) := \{ y \in \Gamma_G(x) \mid xy \text{ is a contractible edge} \} \]
\[ \Gamma^{(2)}(x) := \{ y \in \Gamma_G(x) \mid xy \text{ is a non-contractible edge} \} \]

and define $U_i (i=0,1,2,3), W_i (i \geq 0)$, subset of $V(G)$, by

\[ U_i := \{ x \in V(G) \mid d_G(x) = 3, |\Gamma^{(1)}(x)| = i \} \quad (i=0,1,2,3) \]
\[ W_i := \{ x \in V(G) \mid d_G(x) \geq 4, |\Gamma^{(1)}(x)| = i \} \quad (i \geq 0) \]

Ando, Enomoto and Saito[1] proved the following theorems.

**Theorem A.** Let $G$ be a 3-connected graph of order at least five and $x \in V(G)$. If $d_G(x) = 3$, say $\Gamma_G(x) = \{a,b,c\}$, and both $xb$ and $xc$ are non-contractible, then $d_G(b) = d_G(c) = 3$, and $b$ and $c$ are adjacent.

**Theorem B.** $U_0 = \emptyset$ if $|G| \geq 5$.

**Theorem C.** Suppose $|G| \geq 5$. If $x \in W_0$ then $|\Gamma^{(2)}(x) \cap U_2| \geq 3$. If $x \in W_1$ then $|\Gamma^{(2)}(x) \cap U_2| \geq 2$.

Theorem B is easily deduced from Theorem A.

The following theorem is a slight generalization of the result of Ando, Enomoto and Saito([1, Theorem 3]).

**Theorem 1.** Let $G$ be a 3-connected graph of order at least five, $x \in V(G)$ and $X \subset V(G)$. Suppose $\Gamma^{(1)}(x) \subset X$ and $d_G(x) \geq 4$. Moreover, suppose there exists a least cutset $S$, which contains $x$, such that $G-S$ has a connected component disjoint from $X$. Then $(\Gamma^{(2)}(x)-X) \cap U_2 \neq \emptyset$.

**Proof.** Define

\[ C_x := \{ S \mid \text{a least cutset such that } x \in S \}
\text{ and } G-S \text{ has a connected component } A \text{ which is disjoint from } X \} \]

By the assumption $C_x \neq \emptyset$. For each $S \in C_x$, let $A_S$ be the smallest component of $G-S$ such that $A_S$ is disjoint from $X$. Moreover, we choose $S \in C_x$ such that $|A_S|$ is minimum. Let $B_S = V(G)-A_S-S$. 

\[ -2 - \]
Since \( A_S \cap X = \phi \), there exists a vertex \( y \) in \( A_S \cap \Gamma_G (x) \) such that \( xy \) is a non-contractible edge. Let \( T \) be a least cutset containing the edge \( xy \). \( C \) be one of the connected components of \( G-T \) and \( D = V(G)-T-C \). Let

\[
X_1 := (S \cap C) \cup (S \cap T) \cup (A_S \cap T)
\]

and

\[
X_2 := (S \cap C) \cup (S \cap T) \cup (B_S \cap T).
\]

First we claim that \( S \cap C \neq \phi \). Assume \( S \cap C = \phi \). Since \( y \in A_S \cap T \), \( |X_2| \leq 2 \).

If \( B_S \cap C \neq \phi \) then \( X_2 \) is a cutset of order at most two. This contradicts the assumption on connectivity of \( G \). Therefore, \( B_S \cap C = \phi \) and it follows \( A_S \cap C \neq \phi \) since \( C \neq \phi \). Hence \( X_1 \) is a cutset and since \( X_1 \subset T \), \( |X_1| \leq 3 \). This contradicts either the connectivity of \( G \) or the minimality of \( A_S \).

The similar argument leads us to \( S \cap D \neq \phi \). Therefore, we know that \( |S \cap C| = |S \cap T| = |S \cap D| = 1 \).

Next, we claim \( B_S \cap T \neq \phi \). Assume \( B_S \cap T = \phi \). Without loss of generality, we may assume that \( B_S \cap C \neq \phi \), and it follows that \( X_2 \) is a cutset of order at most two. This is a contradiction.

Therefore, we know that \( |A_S \cap T| = |S \cap T| = |B_S \cap T| = 1 \), and \( |X_1| = 3 \).

Since \( A_S \) is minimal, \( A_S \cap C = \phi \) and \( A_S \cap D = \phi \). Hence we have \( |A_S| = 1 \) and \( A_S = \{y\} \). Then \( \Gamma_G (y) = S \) and \( d_G (y) = 3 \).

Let \( a \in S - \{x\} \). If \( ya \) is a non-contractible edge, then \( d_G (x) = 3 \) by Theorem A. This contradicts the assumption that \( d_G (x) = 4 \). Hence \( ay \) is a contractible edge. This implies \( y \in U_2 \).

For \( X \subset V(G) \) and \( F \subset E(G) \), we say \( X \) covers \( F \) in case that any edge of \( F \) is incident with at least one of the vertices of \( X \).

**Theorem 2.** Let \( G \) be a 3-connected graph of order at least six. Then, the set of the contractible edges of \( G \) cannot be covered with any set of two
vertices.

Proof. We assume that there is a 3-connected graph $G$ of order at least six such that a set of contractible edges is covered with two vertices, say $u$ and $v$. From Theorem B, $V(G)$ can be written as

$$V(G) = U_1 \cup U_2 \cup W_0 \cup W_1 \cup W_2 \cup \{u, v\}.$$ 

since $W_i = U_i = 0$ if $i \geq 3$.

Claim 1: $W_i - \{u, v\} = \emptyset$ for $i = 0, 1$.

Suppose $W_i - \{u, v\} \neq \emptyset$, say $x \in W_i - \{u, v\}$. Let $X = (\Gamma^{(2)}(x) \cap U_2) - \{u, v\}$. By Theorem C, there exists $a \in (\Gamma^{(2)}(x) \cap U_2) - \{u, v\}$. (If $x \in W_1$ and $\Gamma^{(2)}(x) \cap U_2 = \{u, v\}$, then $xu$ and $xv$ are non-contractible, which contradicts the assumption that $x \in W_1$. If $a \in (\Gamma^{(2)}(x) \cap U_2) - \{u, v\}$, then $\Gamma_G(a) = \{x, u, v\}$. Hence $S = \{x, u, v\}$ is a least cutset which contains $x$ and $G - S$ has a connected component disjoint from $X$, if $\Gamma_G(x) \cap X$. Assume $\Gamma_G(x) \cap X$. Then by applying Theorem C, we have $(\Gamma^{(2)}(x) - X) \cap U_2 \neq \emptyset$. This contradicts the fact that $\Gamma^{(2)}(x) \cap U_2 \subseteq X$. It follows $\Gamma_G(x) \cap X$.

Next assume that $V(G) \neq \{x\} \cup X$, then $\{u, v\}$ is a cutset of $G$, which is impossible because $G$ is 3-connected.

Thus $G$ is the graph such that

$$V(G) = \{x\} \cup (\Gamma^{(2)}(x) \cap U_2) \cup \{u, v\} \quad \text{(disjoint)}$$

$$E(G) = \{xy, yu, yv \mid y \in \Gamma^{(2)}(x) \cap U_2\}$$

and for every $y \in \Gamma^{(2)}(x) \cap U_2$, $xy$ is a non-contractible edge and $yu$ and $yv$ are contractible edges. In this graph, however, for each vertex $y \in \Gamma^{(2)}(x) \cap U_2$, $G - \{x, y\}$ is 2-connected since $|G| \geq 6$. This contradicts the fact that an edge $xy$ is non-contractible. Hence the claim follows.

Now we have

$$V(G) = U_1 \cup U_2 \cup W_2 \cup \{u, v\}$$

(1)
Claim 2: Let $x \in W_2$, then $d_G(x)=4$, say $\Gamma_G(x) = \{a,b,u,v\}$, and $\{x,a,b\}$ is a cutset separating $u$ and $v$.

Let $X$ be $\{u,v\}$ ( = $\Gamma^{(1)}(x)$ ). If there exists a least cutset $S$ such that $x \in S$ and $G-S$ has a connected component disjoint from $X$, then by Theorem C, $(\Gamma^{(2)}(x) - X) \cap U_2 \neq \emptyset$. This means that $\{x,u,v\}$ is a cutset, which is a contradiction since edges $xu$ and $xv$ are contractible. Therefore, every least cutset $S$ which contains $x$, say $\{x,a,b\}$, separates $u$ and $v$. Let $A$ be a connected component of $G-S$ which contains $u$, and $B = V(G)-S-A$. If there is a vertex $y \in (A - \{u\}) \cap \Gamma_G(x)$, then by (1) $y \in U_1 \cup U_2 \cup W_2$. However $y$ and $v$ cannot be adjacent, so $y \in U_1$. Since an edge $xy$ is non-contractible, we have $d_G(x)=3$ by Theorem A. This is a contradiction since $x \in W_2$. Since $d_G(x)=4$, $\Gamma_G(x) = \{a,b,u,v\}$, and the claim follows.

Now we consider $G-\{u,v\}$. Since $G$ is 3-connected, $G-\{u,v\}$ is connected and $\delta(G-\{u,v\}) \leq 2$ by (1) and Claim 2. Hence $G-\{u,v\}$ is a path or a cycle.

First assume that $G-\{u,v\}$ is a path, say $G-\{u,v\} \simeq P_n$. Since $|G| \geq 6$, $n \geq 4$. Let $x$ be an internal vertex of $P_n$. Then $\{x,u,v\}$ is a cutset of $G$. On the other hand, $xu$ or $xv$ is contractible since $x \in U_1 \cup U_2 \cup W_2$. This is a contradiction.

Therefore, $G-\{u,v\}$ is a cycle, say $G-\{u,v\} \simeq C_n$. In this case $U_2 - \{u,v\} = \emptyset$, and hence $V(G) = U_1 \cup W_2 \cup \{u,v\}$. First we claim $U_1 - \{u,v\} = \emptyset$. Assume the contrary and let $x \in U_1 - \{u,v\}$. By Theorem A, two neighbors of $x$ in $V(G)-\{u,v\}$ are adjacent. This is impossible since $|G| \geq 6$.

Thus $V(G) = W_2 \cup \{u,v\}$. However, for each $x \in V(G)-\{u,v\}$, $\Gamma^{(3)}(x) \cup \{x\}$ is not a cutset since $|G| \geq 6$. This contradicts Claim 2. This is a final contradiction and the proof is complete. *

Theorem 3. Let $G$ be a 3-connected graph and $e$ be an edge of $G$. Then there exists a non-separating induced cycle which contains $e$. 

- 5 -
Proof. We prove the theorem by induction on $|G|$. When $|G| \leq 5$, we can easily check the result. Now we can assume that $|G| \geq 6$, and that all 3-connected graph of order less than $|G|$ has a non-separating induced cycle which contains a specified edge.

Assume $G$ has no non-separating induced cycle which contains a specified edge $e$. Let $a$ and $b$ be the endvertices of $e$. By Theorem 2, $G$ has a contractible edge, say $xy$, which is not incident with $a$ or $b$. Let $G'$ be the graph obtained from $G$ by contraction of $xy$. By induction hypothesis, $G'$ has a non-separating induced cycle $C'$ which contains $e$. If the contracted vertex $z$ is not on $C'$, then $C'$ is also a non-separating induced cycle in $G$ which contains $e$. Therefore, we can assume that $z$ lies on $C'$. Let $u,v$ be the vertices adjacent to $z$ on $C'$ and $P'$ be the path obtained by $C' - z$. In $G$, $u$ and $v$ are adjacent to $z$ or $y$. Now two cases occur.

Case 1: At least one of $\{x,y\}$ is adjacent to both $u$ and $v$.

Without loss of generality, we have $\Gamma_G(x) \supset \{u,v\}$. If $y$ is adjacent to the vertex of $V(G) - V(C')$, then $P' \cup \{z\}$ with edges $zu$ and $zu$ is a non-separating cycle in $G$, a contradiction. Therefore $y$ can be adjacent only to $u$, $v$ and $z$, since $C'$ is an induced cycle in $G'$. This implies $\Gamma_G(y) = \{u,v,x\}$ since the minimum degree of $G$ is at least three. Applying the same argument to $y$, we have $\Gamma_G(x) = \{u,v,y\}$. Then, degree of $z$ is two in $G'$, which contradicts the fact that $xy$ is a contractible edge.

Case 2: The vertex $z$ is adjacent to one of $u$, $v$ and $y$ to the other.

We can assume that $x$ is adjacent to $u$ (and not to $v$) and $y$ to $v$ (and not to $u$). Then, $P' \cup \{z,y\}$ with edges $ux$, $xy$ and $yu$ is a non-separating induced cycle in $G$.

This completes the proof of the theorem. •
Corollary 4. Let $G$ be a 3-connected graph and $x$ and $y$ be vertices of $G$. Then there exists an induced $x,y$-path $P$ such that $G-V(P)$ is connected.

Proof. If $x$ and $y$ are adjacent, then the edge $xy$ is a desired path. Otherwise, in the graph obtained from $G$ by adding an edge $xy$, there exists a non-separating induced cycle which contains $xy$, by Theorem 3. This cycle induces a non-separating induced path in $G$. •

The above corollary leads us to the following conjecture.

Conjecture. For a given integer $k$ ($k \geq 1$), there exists a minimum number $n_k$ such that every $n_k$-connected graph $G$ satisfies the following property (*).

(*) For every pair of distinct vertices $x,y$ of $G$, there exists an induced $x,y$-path $P$ such that $G-V(P)$ is $k$-connected.

Theorem 3 indicates that $n_1=3$.

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References.


