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Symmetries of complex projective spaces

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Let X be a topological space and G a group. In this article G will be either S^1 or a finite cyclic group. We say that X is symmetric with respect to G if X admits an effective action of G. This concept can be considered in many categories. In the smooth category we consider smooth (C^∞) manifolds with smooth symmetries (smooth actions). Similarly we can consider topological or locally smooth symmetries. We discuss the effect of certain algebraic invariants, like Pontrjagin classes, on the symmetries of a space.

An easy, but nontrivial, example is as follows. The space $X = S^1$ is symmetric with respect to every subgroup G of S^1. The space $Y = S^1 \vee S^1$ has only very few finite cyclic symmetries. The only finite cyclic groups which act effectively on Y are $\mathbb{Z}_2$ and $\mathbb{Z}_4$.

In some sense Petrie's conjecture [P1] is the starting point for the questions discussed in this paper. One formulation of this conjecture is:

Suppose X is a closed smooth homotopy $\mathbb{C}P^n$ and $t \in H^2(X,\mathbb{Z})$ is a generator. If X is smoothly S^1 symmetric then $p(X) = (1+t^2)^{n+1}$. Here p stands for the Pontrjagin class.

This conjecture has been verified for $n = 3$ by Dejter [De] and for $n = 4$ by James [J]. It has also been verified in many other special cases by Hattori [H], Masuda [M1], Petrie [P2], Wang [Wa], and Yoshida [Y]. These references are only a sample.

The significance of the Pontrjagin class in this context is that it determines the diffeomorphism type up to finite ambiguity.

That is a result of Sullivan [S]. Observe also that for $n \geq 3$ there are infinitely many distinct differentiable manifolds homotopy equivalent to $\mathbb{C}P^n$. In case $n = 3$, the first Pontrjagin class determines the diffeomorphism type [MY,W]. A 1-1 correspondence between the diffeomorphism types and the integers is given through the assignment $X_k \longleftrightarrow k$ where $p_1(X_k) = (4+24k)t^2$.

The question we are asking is:

Q: How do the Pontrjagin classes of a homotopy complex projective space restrict its symmetries.

We begin with some results on $\mathbb{Z}_2$ symmetries. A homotopy equivalence $f\colon X \longrightarrow Y$ between smooth manifolds is called tangential if TX and f^*TY are stably isomorphic. In this case $p(X) = f^*p(Y)$. We say than an involution ($\mathbb{Z}_2$ action) on a homotopy complex projective space is of conjugation type if the $\mathbb{Z}_2$ fixed point set is $\mathbb{Z}_2$ cohomology equivalent to a real projective space. The main result is

Theorem (Kakutani [K], Dovermann-Masuda-Schultz [DMS], Stolz [St]) Suppose n is not of the form 2^j-1 and X is a smooth closed manifold homotopy equivalent to $\mathbb{C}P^n$. There exists a manifold Y tangentially homotopy equivalent to X which has a smooth conjugation type involution.

The proof is based on extensive $\mathbb{Z}_2$ homotopy theory calculations, $\mathbb{Z}_2$ surgery theory and the calculation of surgery obstructions, i.e., Arf invariants in terms of Sullivan's characteristic variety formula and signatures. For a result on free involutions on homotopy $\mathbb{C}P^3$'s compare also Petrie [P3].

Let X be a closed smooth manifold homotopy equivalent to $\mathbb{C}P^n$. Let $t \in H^2(X,\mathbb{Z})$ denote the cohomology generator. If $p(X)$ is not of the form $(1+t^2)^{n+1}$ we say that its Pontrjagin classes are not standard. Masuda and Tsai constructed smooth $\mathbb{Z}_m$ action on homotopy complex projective spaces with nonstandard Pontrjagin classes. Here m is an odd prime in [MT] and an odd integer in [T]. The actions considered in these references have isolated fixed points. The basic results are

Theorem (Masuda-Tsai [MT], Tsai [T]): There are infinitely many homotopy complex projective spaces with nonstandard Pontrjagin classes which admit smooth cyclic symmetries.

Theorem [MT]: Every homotopy $\mathbb{C}P^3$ admits a $\mathbb{Z}_p$ symmetry for infinitely many primes p.

Masuda and I refined the technique of [MT]. A corollary of this work is:

Theorem [DM]: Let X_{-4} be the homotopy complex projective space with $p_1(X_{-4}) = (4-4\cdot 24)t^2$. This space X_{-4} admits a smooth $\mathbb{Z}_m$ action for every prime m and for every integer which is prime to 30.

These last three theorems are proved using elementary number theory, equivariant transversality theory and equivariant surgery theory.

One important difference between the study of S^1 actions and the study of $\mathbb{Z}_m$ actions is that the Atiyah-Singer Index and Signature Theorems are much stronger in the presence of an S^1 action. In our next results we add some homological (dimension) assumptions. The result will be that a symmetry again imposes very strong restrictions on the algebraic invariants of the manifold and that such an action almost has to look like a linear action.

We now consider $\mathbb{Z}_p$ actions where p is an odd prime. Let X be a closed manifold of dimension $2n$ whose cohomology ring is $\mathbb{Z}[t]/(t^{n+1})$, where t is a generator of $H^2(X,\mathbb{Z})$. Such manifolds are called cohomology complex projective spaces. We consider smooth and locally smooth actions of $\mathbb{Z}_p$ whose fixed point set $F(X)$ contains a $2n-2$ dimensional component F_0. We denote the inclusion of F_0 by $j: F_0 \longrightarrow X$ and we set $t_0 = j^*t$. A result of Bredon [Bl] is

<u>Proposition</u> (i) $F(X) = F_0 \amalg$ point.

(ii) $H^*(F_0,\mathbb{Z})$ contains the polynomial ring $\mathbb{Z}[t_0]/(t_0^n)$.

Let X be as in the proposition. We say that the action is <u>algebraically standard</u> if:

(i) $p(X) = (1+t^2)^{n+1}$

(ii) The inclusion $\mathbb{Z}[t_0]/(t_0^n) \longrightarrow H^*(F_0,\mathbb{Z})$ is an isomorphism after dividing out torsion.

(iii) $p(F_0) \equiv (1+t_0^2)^n$ mod torsion.

(iv) $c(\nu(F_0,X)) \equiv 1 \pm t_0$ mod torsion.

(v) As a real representation $T_{pt}X = n\nu_x(F_0,X)$ where pt is the isolated fixed point and $x \in F_0$.

Here T denotes the tangent bundle, ν the normal bundle, c stands for Chern class and p for Pontrjagin class. The meaning of all the expressions is clear in the smooth category. If the action is locally smooth some care is required. Based on results of Kirby-Siebenmann [KS, p. 254] and Kneser [Kn] one can show that F_0 has an equivariant vector bundle neighborhood in X. So (iv) makes sense. One may have only rational Pontrjagin classes but in our next theorem the dimension assumptions will imply that these classes are in fact integral. The proof of the next theorem is based on extensive computations. First one applies the Atiyah Singer G Signature Theorem to translate the problem into a number theoretical one. One obtains an equation $P(\xi) = 0$ where ξ is a primitive p^{th} root of unity and $P(\xi)$ is a polynomial of degree $\leq p-1$. The coefficients of P are functions of $p(F_0)$, $c(\nu(F_0,X))$ and the representations $T_{pt}X$ and $\nu_x(F_0,X)$. Because $1 + \xi + \ldots + \xi^{p-1}$ is the minimal polynomial it divides $P(\xi)$, so all coefficients of $P(\xi)$ must be equal. From this one draws conclusions and shows:

<u>Theorem</u> [D] Let $\mathbb{Z}_p$, p odd prime, act on a closed manifold X with the integral cohomology ring of $\mathbb{C}P^n$. Suppose that the $\mathbb{Z}_p$ fixed point set of X has a $2n-2$ dimensional component.

(i) If the action is locally smooth and $n = 1, 2, 3$, the action is algebraically standard.

(ii) If the action is smooth and $n = 4$, $p \geq 5$, the action is algebraically standard.

(iii) If the action is smooth, $n = 5$, and (α) $p \geq 59$ and $h_1(p) \equiv 1(2)$ (where h_1 is the relative class number of $\mathbb{Z}[\xi]$ with $\xi = \exp(2\pi i/p)$) or (β) $7 \leq p \leq 53$, $p \neq 31$, then the action is algebraically standard.

Because of the low dimensional evidence we make this:

<u>Conjecture</u>: With the assumptions as in the previous theorem, actions in the smooth category are algebraically standard for all odd primes p and all n.

Various modifications of this conjecture may be formulated. For example, one can replace smooth actions by locally smooth actions. Also, for any given dimension one can conjecture that $\mathbb{Z}_p$ actions are algebraically standard for almost all (or infinitely many) primes p.

The concept of an action which is algebraically standard also makes sense for $\mathbb{Z}_2$ action. In particular Masuda [M2] and Petrie [P3] showed that a cohomology $\mathbb{C}P^n$ with $\mathbb{Z}_2$ action as above must have standard Pontrjagin class if $n = 3$ or $n = 4$. So $p(X)$ is $(1+t^2)^4$ or $(1+t^2)^5$ in these cases.

Actions of the tori $(S^1)^n$ and $(\mathbb{Z}_p)^n$ on manifolds X cohomology equivalent to $\mathbb{C}P^n$ (with appropriate coefficients) have been studied by W. Y. Hsiang [Hs]. So one can ask whether the existence of an effective torus action implies an algebraic rigidity of the action. At least one hopes to show that the Pontrjagin classes of X has to be standard. This has been shown in many cases by Petrie [P4] and Masuda [M2]. Next we discuss the assumptions and conclusions of our previous theorem.

Even for smooth actions of $\mathbb{Z}_p$ one cannot expect a stronger statement than stated in (ii) of the property 'algebraic standard'. Here is one way to see this. Bredon [B2, Chapter I, Section 7 or B3] constructed smooth $\mathbb{Z}_p$ actions on 5-dimensional spheres with lens spaces as fixed point set. Remove an open disk around a fixed point and cross the resulting disk with a second disk on which $\mathbb{Z}_p$ acts trivially. Round corners and take the boundary of this product. The result is a sphere Σ^m with smooth $\mathbb{Z}_p$ action and a codimension 2 fixed point set. Here m can be any integer greater or equal to 5. Choose $m = 2n$. Now take a connected sum of an appropriate linear action on $\mathbb{C}P^n$ with Σ^{2n}

at a fixed point. The result is a smooth action which is algebraically standard but $H^*(F_0,\mathbb{Z})$ contains torsion.

R. Schultz pointed out to me that there is little hope to expect any further rigidity results without local smoothness of the action. Based on results of Cappell-Shaneson [CS1, CS2] one can show:

<u>Proposition</u> Let X be a smooth homotopy $\mathbb{C}P^n$ such that $48 \mid (p_1(X)-(n+1)t^2)$. For every odd prime p there exists an action of $\mathbb{Z}_p$ on $\mathbb{C}P^{n+1}$ such that the fixed point set is $X \amalg$ point.

Obviously such actions are not algebraically standard. These actions are also not locally smooth.

One may ask whether the concept of being algebraically standard depends on the fact that one has a fixed point set which consists of a codimension 2 fixed point component and a point. We might just suppose that the fixed point set consists of two components. Then one has to modify the definition of being algebraically standard in the obvious way. In this more general setting we do not expect any strong result. In fact, M. Masuda and I showed

<u>Theorem</u> Let p be an odd prime. There exist infinitely many homotopy complex projective spaces X with a smooth $\mathbb{Z}_p$ action such that $p(X)$ is nonstandard and the fixed point set consists of two components.

The examples constructed specifically are homotopy equivalent to $\mathbb{C}P^{2k+1}$ and the fixed point set consists of two copies of $\mathbb{C}P^k$.

Finally, there is a non-smoothable manifold Ch homotopy equivalent to $\mathbb{C}P^2$[F]. Kwasik-Vogel [Kw] showed

<u>Theorem</u> The manifold Ch has a locally smoothable $\mathbb{Z}_p$ action for all primes $p \geq 3$ but not for $p = 2$.

For $p \geq 3$ the actions are constructed fairly explicitly. For $p = 2$ one uses the Kirby Siebenmann obstruction and the non-smoothability of Ch to exclude an involution.

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