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Proper subanalytic transformation groups and unique triangulation of the orbit spaces

by

Takao Matumoto and Masahiro Shiota

松本堯生　　　　塩田昌弘

§ 1. Introduction

Let G be a transformation group of a topological space X. Triangulation of the orbit space X/G was treated by several people (e.g. [4], [11] and [12]) in some cases of compact differentiable transformation groups. The authors showed in [6] a unique triangulation of X/G, provided that G is a compact Lie group, X is a real analytic manifold and the action is analytic. Moreover, the uniqueness was extended to the case of differentiable G-manifolds and played an important role in defining the equivariant simple homotopy type of a compact differentiable G-manifolds when G is a compact Lie group. Let us explain what the uniqueness means here. Under the above conditions we can give naturally X/G a subanalytic structure. On the other hand we know a combinatorially unique subanalytic triangulation of a locally compact subanalytic set ([3] and [10]). Hence X/G comes to admit a unique subanalytic triangulation.

Now we consider a problem under what weaker condition X/G has a natural subanalytic structure. Of course we may assume that X,G and the action are subanalytic; as subanalytic set is Hausdorff, it is natural to assume a condition that the action is proper in the sense of [5] and [8] (see §2); moreover, in order to simplify the description we assume that X is locally compact. In this paper we shall show that these conditions are sufficient (Corollary 3.4) and hence we obtain a unique subanalytic triangulation of the orbit space of a proper subanalytic transformation group of a locally compact subanalytic set (Corollary 3.5).

We shall see that a subanalytic group is homeomorphic to a Lie group. But we shall not use properties of Lie group except for the Montgomery-Zippin neighboring subgroups theorem [7].

See [6] for more references and our terminology.

§ 2. Subanalytic transformation group

Let G be a topological group contained in a real analytic manifold M. If G is subanalytic in M then we call G a <u>subanalytic group</u> in M.

Remark 2.1. A subanalytic group in an analytic manifold is homeomorphic to a Lie group. It is possible that G may be subanalytically homeomorphic to a Lie group.

Proof. As the Hilbert's fifth problem is affirmative [7] it suffices to see that G is locally Euclidean at some point of G. But this is clear by the fact that a subanalytic set admits a subanalytic stratification (see Lemma 2.2, [6]).

Let G be a subanalytic group in M_1 and X a subanalytic set in M_2. If G is a topological transformation group of X and the action $G \times X \ni (g, x) \to gx \in X$ is subanalytic (i.e. the graph is subanalytic in $M_1 \times M_2$) then we call (G, M_1) a <u>subanalytic transformation group</u> of (X, M_2).

A transformation group G of a topological space X is called <u>proper</u> if for any $x, y \in X$, there exist neighborhoods U of x and V of y such that $\{h \in G: hU \cap V \neq \phi\}$ is relatively compact in G ([5] and [8]). This is equivalent to say that $G \times X \ni (g, x) \to (gx, x) \in X \times X$ is proper when G is locally compact and X is Hausdorff.

Remark 2.2. Let G be a locally compact proper transformation group of a completely regular space X. Then X/G is completely regular [8].

Lemma 2.3. Let (G, M_1) be a subanalytic proper transformation group of a subanalytic set (X, M_2) and $\{X_i\}$ be the decomposition of X by orbit types. Then $\{X_i\}$ is locally finite in U and each X_i is subanalytic in U for some open neighborhood U of X in M_2.

Proof. For each $x \in X$ let G_x denote the isotropy subgroup of G at x. Put

$$A = \bigcup_{x \in X} G_x \times x = \{(g, x) \in G \times X: gx = x\}$$

and let $\pi: M_1 \times M_2 \to M_2$ be the projection. Then A is subanalytic in $M_1 \times M_2$. Moreover, we can choose an open neighborhood U of X in M_2 so that $\pi|_{A'}: A' \to U$ is proper from the fact that a subanalytic set is σ-compact and the assumption of properness that $\pi|_{A}:A \to X$ is proper, where A' is the closure of A in $G \times U$. We may consider the problem in U and an open neighborhood of G in M_1 in place of M_2 and M_1 respectively, and this U will satisfy the requirements in the lemma. Hence we can assume from the beginning that G is closed in M_1 and $\pi|_{\bar{A}}:\bar{A} \to M_2$ is proper where $\bar{A}$ is the closure of A in $M_1 \times M_2$. Let $\bar{X}$ also denote the closure of X in M_2. We remark $\bar{A} \cap G \times X = A$ because A is closed in $G \times X$.

Assertion: $\bar{A}$ and $\bar{X}$ have subanalytic stratification $\mathcal{A} = \{A_i\}$ and $\mathcal{Y} = \{Y_j\}$ respectively such that $\pi|_{\bar{A}}:\mathcal{A} \to \mathcal{Y}$ is a stratified map: i.e.,

(i) For each stratum A_i of $\mathcal{A}$, $\pi(A_i)$ is contained in some Y_j.

(ii) For such i and j, $\pi|_{A_i}:A_i \to Y_j$ is a C^∞ submersion.

(iii) For each j, $\mathcal{A}_j = \{A_i \in \mathcal{A}: \pi(A_i) \subset Y_j\}$ is a Whitney stratification ([2] or [9]).

The authers do not know an apt reference to Assertion. So we give a proof. Let p be the dimension of $\bar{A}$. We prove by induction on $k = \dim X$. If $k = 0$ Assertion is the same as that $\bar{A}$ admits a subanalytic Whitney stratification. But this is well-known (e.g. Theorem 4.8 [2]). Hence assume that Assertion is true for $\dim < k$. Choose a subanalytic stratification of $\bar{A}$ and let Z_1 be the union of all strata of dimention $< p$. Then Z_1 is a subanalytic set in $M_1 \times M_2$ of dimention $< p$ and $\bar{A} - Z_1$ is a subanalytic analytic manifold in $M_1 \times M_2$ of dimension p. Now we remark that the connected components of a subanalytic set are subanalytic (Lemma 2.2, [6]). Apply 2.14 of [9] to the restriction of π to each connected component of $\bar{A} - Z_1$. Then there exists a subanalytic set $Z_2(\subset \bar{A} - Z_1)$ in $M_1 \times M_2$ of dimension $< p$ such that Z_2 is closed in $\bar{A} - Z_1$ and the differential $d(\pi|_{\bar{A} - Z_1 - Z_2})$ is of constant rank on each connected component of $\bar{A} - Z_1 - Z_2$. The last property implies that the restriction of π to each connected component of $\bar{A} - Z_1 - Z_2$ is a submersion to the image.

Next consider π on $Z_1 \cup Z_2$. Then we obtain in the same way as above a subanalytic set $Z_3(\subset Z_1 \cup Z_2)$ in $M_1 \times M_2$ of dimension $< p - 1$ such that Z_3 is closed in $Z_1 \cup Z_2$, $(Z_1 \cup Z_2) - Z_3$ is a subanalytic analytic manifold of dimension $p - 1$ (possibly empty), and $d(\pi|_{(Z_1 \cup Z_2) - Z_3})$ is of constant rank on each connected component of $(Z_1 \cup Z_2) - Z_3$. Moreover enlarging Z_3 if necessary, we can assume {connected components of $\bar{A} - Z_1 - Z_2$ and $(Z_1 \cup Z_2) - Z_3$} is a Whitney stratification (by Prop. 4.7, [2]). Repeat this argument. Then we obtain a subanalytic Whitney stratification $\{B_i\}$ of $\bar{A}$ such that for each i the restriction $\pi|_{B_i}$ is a submersion to the image.

As we assumed that $\pi|_{\bar{A}}:\bar{A} \to M_2$ is proper, the image under π of any subanalytic set in M_1 contained in $\bar{A}$ is subanalytic in M_2 ((2.6), [9]). In particular $\pi(B_i)$ are subanalytic in M_2. It also follows from the properness that $\{\pi(B_i)\}$ is locally finite in M_2. Hence we have a subanalytic stratification of $\bar{X}$ compatible with $\{\pi(B_i)\}$ (i.e. $\pi(B_i)$ is a union of some strata) [(2.11), [9]). Let $\mathcal{Y}_1 = \{Y_{1i}\}$ denote the family of all the strata of dimension k, X_2 the union of strata of dimension $< k$, and $\{A_{1\ell}\}_\ell = \{$connected components of $\pi^{-1}(Y_{1j}) \cap B_i\}_{(i,\ j)}$. Then for each ℓ, $\pi(A_{1\ell})$ is contained in some Y_{1j}. For such ℓ and j, $\pi|_{A_{1\ell}}: A_{1\ell} \to Y_{1j}$ is a C^∞ submersion; $\mathcal{A}_1 = \{A_{1\ell}\}$ is a subanalytic Whitney stratification and

$$\pi^{-1}(\bigcup_j Y_{1j}) \cap \bar{A} = \bigcup_\ell A_{1\ell}.$$

Consider the subanalytic sets $A_2 = \pi^{-1}(X_2) \cap \bar{A}$ and X_2 and the proper map $\pi|_{A_2}: A_2 \to X_2$. Then by induction hypothesis we have subanalytic stratifications $\mathcal{A}_2 = \{A_{2i}\}$ and $\mathcal{Y}_2 = \{Y_{2j}\}$ of A_2 and X_2 respectively such that $\pi|_A: \mathcal{A}_2 \to \mathcal{Y}_2$ is a stratified map because of $\dim X_2 < k$. Moreover we can choose $\mathcal{A}_2$ and $\mathcal{Y}_2$ so that $\mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{Y}_1 \cup \mathcal{Y}_2$ are subanalytic stratifications, which is clear by the method of construction of $\mathcal{A}_1$ and $\mathcal{Y}_1$. Then $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2$ are what we wanted. Assertion is thus proved. We can also choose $\mathcal{Y}$ to be compatible with X (i.e., X is a union of some strata of $\mathcal{Y}$).

Apply the Thom's first isotopy lemma to $\pi|_{\bar{A}}: \mathcal{A} \to \mathcal{Y}$ (e.g. 5.2, Chapter II, [1]). Then for each Y_j and $x_1, x_2 \in Y_j$, $\pi^{-1}(x_1) \cap \bar{A}$ and $\pi^{-1}(x_2) \cap \bar{A}$ are homeomorphic. Here it is important that Y_j are connected. Now if $x \in X$ then

$$\pi^{-1}(x) \cap \bar{A} = \pi^{-1}(x) \cap A = G_x \times x.$$

Hence for $x_1, x_2 \in Y_j \subset X$, G_{x_1} and G_{x_2} are homeomorphic. Furthermore, for such x_1 and x_2, G_{x_1} and G_{x_2} will be conjugate. To see this recall the Montgomery-Zippin neighboring subgroups theorem [7, p.216], which states that each compact subgroup H

of G has a neighborhood O in G such that any compact subgroup of G included in O is conjugate to a subgroup of H. Hence, by the properness assumption, each $x \in X$ has a neighborhood V in X such that for any $y \in V$ G_y is conjugate to a subgroup of G_x. But a proper subgroup of G_x is never homeomorphic to G_x as G_x is compact. Therefore if $y \in V$ is in the same stratum as x then G_y is conjugate to G_x. Thus we have proved that for $x_1, x_2 \in Y_j \subset X$, G_{x_1} and G_{x_2} are conjugate. Hence each of X_i in the lemma is a union of some $Y_j \subset X$. Therefore $\{X_i\}$ satisfies the requirements in the lemma, which completes the proof.

Remark 2.4. In Lemma 2.3 and Lemma 3.1 below we can replace the properness condition by a weaker condition that X is a Cartan G-space in the sense of [8], which is clear by their proofs.

In Lemma 2.3 if X is closed in M_2 we can put $U = M_2$ for the following reason (Lemma 2.1, [6]). A subset Y of an analytic manifold M is subanalytic in M if each $x \in M$ has an open neighborhood W in M such that $Y \cap W$ is subanalytic in W.

§ 3. Subanalytic structure on an orbit space and its triangulation

Let X be a topological space. A <u>subanalytic structure</u> on X is a proper continuous map $\varphi: X \to M$ to an analytic manifold such that $\varphi(X)$ is subanalytic in M and $\varphi: X \to \varphi(X)$ is a homeomorphism. Let X_1, X_2 be topological space with subanalytic structures (φ_1, M_1) and (φ_2, M_2) respectively. A subanalytic map $f: X_1 \to X_2$ is a continuous map such that the graph of $\varphi_2 \circ f \circ \varphi_1^{-1}: \varphi_1(X_1) \to \varphi_2(X_2)$ is subanalytic in $M_1 \times M_2$. Subanalytic structures (φ_1, M_1) and (φ_2, M_2) on X are equivalent if the identity map of X is subanalytic with respect to the structures (φ_1, M_1) on the domain and (φ_2, M_2) on the target. We shall regard equivalent subanalytic structures as the same.

If X is a locally compact subanalytic set in an analytic manifold M from the outset, then X is regarded as equipped with the subanalytic structure inclusion : $X \to U$ where U is some open neighborhood of X in M such that X is closed in U. We give every polyhedron a subanalytic structure by PL embedding it in a Euclidean space so that the image is closed in the space. Then a PL map between polyhedra is subanalytic with such subanalytic structures and hence the subanalytic structure on a polyhedron is unique.

Let X be a subanalytic set or a topological space with a subanalytic structure. Then a subanalytic triangulation of X is the pair of a simplicial complex K and a subanalytic homeomorphism $\tau: |K| \to X$. For a family $\{X_i\}$ of subsets of X, a triangulation (K, τ) of X is compatible with $\{X_i\}$ if each X_i is a union of some $\tau(\text{Int}\ \sigma)$, $\sigma \in K$.

We remark that when we consider a subanalytic structure on a topological space or a subanalytic triangulation of the space we shall treat only a locally compact space. Of course we can define a subanalytic structure and a subanalytic 'triangulation' (in this case a subanalytic 'triangulation' consists of open subanalytic simplices and may not contain the boundary of the simplices) without the locally compact assumption. But the description, e.g. the definition of equivalence relation of subanalytic structures, will be complicated, because the composition of two subanalytic maps is not necessarily subanalytic in the usual sense (but always "locally subanalytic" [10]); and to make matters worse a subanalytic finite 'triangulation' (=a decomposition into finite open subanalytic simplices) of a subanalytic set is not unique in general.

Let $q:X\to X/G$ be the natural map for a transformation group G of a topological space X. The following is the key lemma to the main theorems.

Lemma 3.1. Let (G, M_1) be a subanalytic proper transformation group of a subanalytic set (X, M_2) and x_0 a point of X. Assume that X is locally compact. Then there exist a neighborhood U of x_0 in X and a G-invariant subanalytic map $f:GU\to\mathbb{R}^{2k+1}$, $k=\dim x$, such that the induced map $\bar{f}:GU/G\to f(U)$ is a homeomorphism.

Proof. Properly embedding M_2 in a Euclidean space we can assume $M_2=\mathbb{R}^n$ and $x_0=0$. It is sufficient to define a G-invariant subanalytic map $f:GU\to\mathbb{R}^{2k+1}$ so that $\bar{f}:GU/G\to\mathbb{R}^{2k+1}$ is one-to-one, because GU/G is locally compact. Put

$$Z = \{(x, y)\in X\times X: q(x) = q(y)\}.$$

Then Z is the image of the projection to $X\times X$ of the graph of the action $G\times X\to X$. As the problem is local at 0 we can assume by the properness condition that the projection to $\mathbb{R}^n \times \mathbb{R}^n$ of the closure of the above graph is proper and hence by (2.6),[9] Z is subanalytic in $\mathbb{R}^n \times \mathbb{R}^n$. Let $B(\varepsilon, a)$ and $S(\varepsilon, a)$ for $\varepsilon > 0$ and $a \in \mathbb{R}^n$ or $\in \mathbb{R}^n \times \mathbb{R}^n$ denote the open ε-ball and ε-sphere with center at a respectively.

We shall construct open neighborhoods $V_0 \supset \cdots \supset V_{2k+1}$ of 0 in X and G-invariant bounded subanalytic maps $f_i : V_i \to \mathbb{R}^i$, $i = 0, \ldots, 2k+1$, such that

$$f_{i+1} = (f_i|_{V_{i+1}}, g_{i+1}), \; V_i = X \cap B(\varepsilon_i, 0)$$

for some subanalytic function g_{i+1} and some $\varepsilon_i > 0$, and

$$Z_i = \{(x, y) \in V_i \times V_i - Z : f_i(x) = f_i(y)\}$$

is of dimension $\leqq 2k - i$. If we construct these and put $U = V_{2k+1}$ and $f =$ the extension of f_{2k+1} to GU then $\bar{f} : GU/G \to \mathbb{R}^{2k+1}$ will be one-to-one, because $\dim Z_{2k+1} = -1$ means that if $x, y \in U$ belong to distinct orbits then $f(x) \neq f(y)$.

We proceed the above construction by induction on i. For $i = 0$ we put trivially $V_0 = X \cap B(1, 0)$ and $f_0 = 0$. So assume that we have already constructed V_i and f_i. Clearly Z_i is subanalytic in $\mathbb{R}^n \times \mathbb{R}^n$. Assume $\dim Z_i = 2k - 1$, otherwise it suffices to put $V_{i+1} = V_i$ and $g_{i+1} = 0$. Then using a subanalytic stratification of Z_i in the same way as Lemma 2.3 we obtain a subnalytic set Y_{i+1} $(\subset Z_i)$ in $\mathbb{R}^n \times \mathbb{R}^n$, closed in $V_i \times V_i - Z$ and of dimension $\leqq 2k - i - 1$ such that $Z_i - Y_{i+1}$ is an analytic manifold of dimension $2k - i$. For every large integer m we put $W_m = (Z_i - Y_{i+1}) \cap S(1/m, 0)$. Then W_m is an anlytic manifold of dimension $2k-i-1$ since $(Z_i - Y_{i+1}, 0)$ satisfies the Whiteny condition (Prop. 4.7, [7]). Choose a sequence of points $\{a_j\}_{j=1,2,...}$ in $\cup W_m$ so that for any large m and $x \in W_m$, $B(\exp(-m), x)$ contains at least one a_j. Write $a_j = (a_j', a_j'')$. Then $Ga_j' \cap Ga_j'' = \phi$. Put

$$G_0 = \{g \in G: g\bar{V}_0 \cap \bar{V}_0 \neq \phi\}$$

where $\bar{V}_0$ denotes the closure of V_0. Then we have $G_0^{-1} = G_0$, G_0 is compact by the properness condition, and hence $X_0 = G_0\bar{V}_0$ is compact. Let $\{P_\alpha\}$ be the decomposition of X_0 such that x and y in X_0 are contained in the same P_α if and only if there exsits a finite sequence $x = x_0, x_1, \ldots, x_\ell = y$ in X_0 with $g_i x_i = x_{i+1}$ for some g_i of G_0. Here $\ell=3$ is sufficient for the following reason. Let $x_0, \ldots, x_\ell$ be a sequence in X_0 chained by $g_0, \ldots, g_{\ell-1}$ in G_0 as above. Then by definition of X_0 there are $y_0, \ldots, y_\ell$ in $\bar{V}_0$ and $h_0, \ldots, h_\ell$ in G_0 such that $x_i = h_i y_i$. Hence we have

$$y_\ell = h_\ell^{-1} g_{\ell-1} \cdots g_0 h_0 y_0.$$

Therefore, by definition of G_0, $h_\ell^{-1} g_{\ell-1} \cdots g_0 h_0 \in G_0$. Hence the sequence x_0, y_0, y_ℓ, x_ℓ is chained by the elements h_0^{-1}, $h_\ell^{-1} g_{\ell-1} \cdots g_0 h_0$, h_ℓ of G_0, which proves that $\ell=3$ is sufficient.

The above proof shows also that (i) for each α and $x \in P_\alpha \cap \bar{V}_0$, $P_\alpha = G_0(G_0 x \cap \bar{V}_0)$ and $P_\alpha \cap \bar{V}_0 = Gx \cap \bar{V}_0$ (i.e. $\{P_\beta \cap \bar{V}_0\}$ is the family of intersections of G-orbits with $\bar{V}_0$). From the first equality in (i) it follows that each P_α is compact and subanalytic, because $G_0 x \cap \bar{V}_0$ is compact and subanalytic. Moreover $\ell=3$ shows the following. (ii) Let α_1,

$\alpha_2, \ldots$ be a sequence such that there exist $b_1 \in P_{\alpha_1}$, $b_2 \in P_{\alpha_2}$, ... converging to a point b. Then $\cap_{r=1}^{\infty} \overline{\cup_{i=r}^{\infty} P_{\alpha_i}}$ is identical with P_α which contains b.

Define a map $A: C^0(X_0) \to C^0(\bar{V}_0)$ by

$$Ah(x) = \sup\{h(y): y \in P_\alpha \text{ for } \alpha \text{ with } x \in P_\alpha\} \text{ for } x \in \bar{V}_0.$$

Then, by (ii) and by the fact that X_0 is compact, (iii) A is well-defined (i.e. $Ah \in C^0(\bar{V}_0)$ for $h \in C^0(X_0)$) and continuous with respect to the uniform C^0 topology on $C^0(X_0)$ and $C^0(\bar{V}_0)$; (iv) by (i) Ah are G-invariant for $h \in C^0(X_0)$; and (v) if h is subanalytic then Ah is subanalytic for the following reason. Let h be subanalytic. By (i) the set

$$D = \{(x, y) \in X_0 \times X_0: x, y \in P_\alpha \text{ for some } \alpha\}$$

is the image under the proper projection $X_0^2 \times \bar{V}_0^2 \times G_0^3 \to X_0^2$ of the subanalytic set

$$\{(x_1, y_1, x_2, y_2, g_1, g_2, g) \in X_0^2 \times \bar{V}_0^2 \times G_0^3: x_1 = g_1 x_2,\ y_1 = g_2 y_2,\ x_2 = g y_2\}.$$

Hence D is subanalytic. Now by definition $Ah(x) = \sup\{h(y): (x, y) \in D\}$, and the graph of Ah is the boundary of the image by the proper projection $\bar{V}_0 \times X_0 \times \mathbb{R} \ni (x, y, t) \to (x, t) \in V_0 \times \mathbb{R}$ of the subanalytic set

$$\{(x, y, t) \in \bar{V}_0 \times X_0 \times \mathbb{R}: (x, y) \in D,\ t \geq h(y)\}.$$

Therefore, Ah is subanalytic.

Assertion: Let $\varphi_j \in C^0(X_0)$, $j = 1, 2, \ldots$, be sequences satisfying $A\varphi_j(a_j') \neq A\varphi_j(a_j'')$. Given also $b_j > 0$. Then there exist $c_j \geq 0$, $j = 1, 2, \ldots$, such that $c_j \leq b_j$, $\sum_j c_j \varphi_j$ uniformly converges to some $\varphi \in C^0(X_0)$ and $A\varphi(a_j') \neq A\varphi(a_j'')$ for all j.

Proof of Assertion: We define c_j inductively as follows. Put $c_1 = b_1$. Assume we have already defined $c_1, \ldots, c_j$ so that if we put $\psi_\ell = c_1\varphi_1 + \cdots + c_\ell\varphi_\ell$ for $\ell \leq j$ then

$(1)_\ell$ $$A\psi_\ell(a_\ell') \neq A\psi_\ell(a_\ell'') \quad \text{and}$$

$(2)_{\ell p}$ $$c_\ell(|A\varphi_\ell(a_p')| + |A\varphi_\ell(a_p'')|) \leq |A\psi_p(a_p') - A\psi_p(a_p'')|/2^{\ell-p+1} \text{ for } p < \ell.$$

We want c_{j+1} satisfying $(1)_{j+1}$ and $(2)_{j+1p}$, $p \leq j$. If $A\psi_j(a_{j+1}') \neq A\psi_j(a_{j+1}'')$, it suffices to put $c_{j+1} = 0$. If $A\psi_j(a_{j+1}') = A\psi_j(a_{j+1}'')$, then we choose positive c_{j+1} so that $(2)_{j+1p}$, $p \leq j$, hold. In this case

$$A\psi_{j+1}(a_{j+1}') - A\psi_{j+1}(a_{j+1}'') = c_{j+1}(A\varphi_{j+1}(a_{j+1}') - A\varphi_{j+1}(a_{j+1}'')) \neq 0,$$

hence $(1)_{j+1}$ holds. Thus we obtain a sequence $c_1, c_2, \ldots,$ with $(1)_\ell$ and $(2)_{\ell p}$ for $p < \ell$. Then for any integer $p > p' > 0$

(3) $$|A\psi_p(a_{p'}') - A\psi_p(a_{p'}'')| \geq |A\psi_{p'}(a_{p'}') - A\psi_{p'}(a_{p'}'')|/2.$$

Furthermore, diminishing c_j if necessary we can assume ψ_j uniformly converges to some φ. Then it follows from (3) that

$$A\varphi(a_j') \neq A\varphi(a_j'') \text{ for all } j,$$

Which proves Assertion.

For every a_j the polynomial approximation theorem assures the existence of a polynomial φ_j on $\mathbb{R}^n$ such that

$$A(\varphi_j|_{X_0})(a_j') \neq A(\varphi_j|_{X_0})(a_j'').$$

Let $b_1, b_2, \ldots$ be small positive numbers such that the power series $\sum_j b_j\tilde{\varphi}_j$ is of convergence radius ∞ where $\tilde{\varphi}_j(x)$ means $\sum_\alpha |d_\alpha| x^\alpha$ when we write $\varphi_j(x) = \sum_\alpha d_\alpha x^\alpha$.

Apply Assertion to these $\varphi_j|_{\bar{X}_0}$ and b_j. Then we obtain $c_j \geq 0$ such that $\Sigma_{j=1}^\infty c_j\varphi_j$ converges to an analytic function φ on $\mathbb{R}^n$ and

$$A(\varphi|_{X_0})(a_j') \neq A(\varphi|_{X_0})(a_j'') \text{ for all } j.$$

Put $g_{i+1}' = A(\varphi|_{X_0})$ on V_i. Then we have already seen that g_{j+1}' is subanalytic. Hence we only need to see that

$$Z_{i+1}' = \{(x, y) \in Z_i : g_{i+1}'(x) = g_{i+1}'(y)\}$$

is of dimension $\leq 2k - i - 1$ in some small neighborhood $V_{i+1} \times V_{i+1}$ of 0. In fact $g_{i+1} = g_{i+1}'|_{V_{i+1}}$ is what we wanted.

Assume the dimension of Z_{i+1}' at 0 is $2k - i$. Then there is a subanalytic analytic manifold $N_i (\subset Z_{i+1}' \cap (Z_i - Y_{i+1}))$ of dimension $2k - i$ whose closure in $\mathbb{R}^n$ contains 0. Recall the subanalytic version (Prop. 3.9, [2]) of a theorem of Bruhat-Whitney which states that there exists a real analytic map $\rho : [0, 1] \to N_i \cup \{0\}$ such that $\rho(0) = 0$ and $\rho((0, 1]) \subset N_i$. Define a continuous function χ on [0, 1] by

$$\chi(t) = \mathrm{dist}(\rho(t), Z_i - N_i).$$

Then it is easy to see that χ is subanalytic and positive outside 0 and hence that

$$\chi(t) \geq C|t|^d, \quad t \in [0, 1]$$

for some $C, d > 0$ (the Łojasiewicz' inequality). These imply

$$B(C|t|^d, \rho(t)) \cap Z_i \subset N_i$$

in other words

$$g'_{i+1}(x) = g'_{i+1}(y) \text{ for } (x, y) \in B(C|t|^d, \rho(t)) \cap Z_i.$$

On the other hand, by definition of g'_{i+1}

$$g'_{i+1}(a'_j) \neq g'_{i+1}(a''_j) \text{ for all } j.$$

Hence

(4) $$a_j \notin B(C|t|^d, \rho(t)) \text{ for all } j.$$

Consider now the Łojasiewicz' inequality to the inverse function of $|\rho(t)| = \mathrm{dist}(0, \rho(t))$. Then, we have

$$|\rho(t)| \leqq C''|t|^{d''} \text{ for some } C'' \text{ and } d'' > 0.$$

Hence it follows from (4) that for some C' and $d' > 0$

$$a_j \notin B(C'|\rho(t)|^{d'}, \rho(t)) \text{ for all } j.$$

But this contradicts the fact that for any large m and $x \in W_m$, $B(\exp(-m)\ x)$ contains at least one a_j. Hence Z'_{i+1} is of dimension $\leqq 2k - i - 1$ in some neighborhood of 0. Thus we have proved that $\bar{f}$ is one-to-one.

Remark 3.2. In Lemma 3.1 we can choose f to be extensible to X as a G-invariant subanalytic map by retaking $U = V_{2k+1} = X \cap B(\varepsilon_{2k+2}, 0)$ with $\varepsilon_{2k+2} < \varepsilon_{2k+1}$. Indeed let θ be a subanalytic function on X with support in V_0 such that $0 \leqq \theta \leqq 1$ and $\theta^{-1}(1)$ is a neighborhood of $\bar{U}$. Put

$$h(x) = \begin{cases} A(\theta|_{X_0})(y) & \text{on } G\bar{V}_0 \\ 0 & \text{on } G - G\bar{V}_0. \end{cases}$$

Then φ_{2k+2} is a G-invariant subanalytic function, and $F = (f, \varphi_{2k+2}) : X \to \mathbb{R}^{2k+2}$ satisfies moreover

(3.2.1) $$F(GU) \cap F(X - GU) = \phi$$

For such F it follows from (2.6), [9] that

(3.2.2) $F(X)$ is subanalytic in $\mathbb{R}^{2k+2}$

because of $F(X) = F(X \cap B(1, 0))$ and because the closure of graph $F|_{X \cap B(1, 0)}$ is bounded and subanalytic.

Theorem 3.3. Let (G, M_1) be a subanalytic proper transformation group of a locally compact subanalytic set (X, M_2). Then there exist an open neighborhood M_2' of X in M_2 and a G-invariant subanalytic map $\varphi : X \to \mathbb{R}^{2k+1}$ with respect to subanalytic structures (inclusion, M_2') and (identity, $\mathbb{R}^{2k+1}$) such that $\varphi(X)$ is closed and subanalytic in $\mathbb{R}^{2k+1}$ and that the induced map $\bar{\varphi} : X/G \to \varphi(X)$ is a homeomorphism, where $k = \dim X$.

Proof. For each point x of X let U_x be an open neighborhood of x in M_2 such $U_x \cap X$ is contained in a neighborhood of x in X which satisfies the requirements in Lemma 3.1 and Remark 3.2. Let M_2' be the union of all U_x. Properly embedding M_2' in a Euclidean space, we can assume $M_2' = \mathbb{R}^n$ and we give always X a subanalytic sturcture (inclusion, $\mathbb{R}^n$).

The case where $X = G(K \cap X)$ for some compact set K in $\mathbb{R}^n$: As K is covered by a finite number (say s) of U_x, there exists a G-invariant subanalytic map $\psi : X \to \mathbb{R}^{2s(k+1)}$ by Lemma 3.1 and Remark 3.2 such that the induced map $\bar{\psi} : X/G \to \psi(X)$ is a homeomorphism. Here we used (3.2.1) for the existence of $\bar{\psi}^{-1}$, and we see that $\psi(X)$ is subanalytic in $\mathbb{R}^{2s(k+1)}$ for the same reason as (3.2.3), because we can choose subanalytic K, e.g. $\overline{B(\varepsilon, 0)}$ for some large ε, so that $\psi(X) = \psi(K \cap X)$. We note also that $\psi(X)$ is closed in $\mathbb{R}^{2s(k+1)}$ by the compactness of $K \cap X$. Let (K, τ) be a subanalytic triangulation of $\mathbb{R}^{2s(k+1)}$ compatible with $\psi(X)$ (see Lemma 2.3, [6]), K' the family of $\sigma \in K$ whose interior is mapped by τ into $\psi(X)$ and $\pi : |K'| \to \mathbb{R}^{2k+1}$ be a PL embedding. Then $\varphi = \pi \circ \tau^{-1} \circ \psi : X \to \mathbb{R}^{2k+1}$ is what we want.

The case where there is not a compact set K in $\mathbb{R}^n$ such that $X = G(K \cap X)$: Let θ be a G-invariant subanalytic function on X such that for any compact set H in $\mathbb{R}$ there exists a compact K in $\mathbb{R}^n$ such that $\theta^{-1}(H) = G(K \cap X)$

$$(\text{e.g. } \theta(x) = \inf\{|gx| : g \in G\}).$$

and let α be a subanalytic function on $\mathbb{R}$ such that for each integer i

$$\alpha = \begin{cases} 1 & \text{on } [2i, 2i+1] \\ 0 & \text{on } [2i - 2/3, 2i - 1/3]. \end{cases}$$

For each i consider the G-invariant subspace

$$X_i = \theta^{-1}([2i - 1/3, 2i + 4/3])$$

of X. By the property of θ, (X_i, G) corresponds to the first case. Hence there exists a G-invariant subanalytic map $\varphi_i : X_i \to \mathbb{R}^{2k+1}$ such that $\bar{\varphi}_i : X_i/G \to \varphi_i(X_i)$ is a homeomorphism. Define $\Phi : X \to \mathbb{R}^{2k+2}$ by

$$\Phi(x) = \begin{cases} (\alpha \circ \theta(x)\varphi_i(x),\ \theta(x) & \text{for } x \in X_i \\ (0,\ \theta(x)) & \text{for } x \notin \bigcup_{i=1}^{\infty} X_i. \end{cases}$$

Then Φ is G-invariant and subanalytic, $\bar{\Phi}|_{(\cup\theta^{-1}((2i-1/3,2i+4/3)))/G}$ is a homeomorphism onto the image, and for any integers $j \neq j'$

$$\mathrm{dist}(\Phi(\theta^{-1}([j+1/3,\ j+2/3])),\ \Phi(\theta^{-1}([j'+1/3,\ j'+2/3]))) > 0.$$

In the same way we obtain a G-invariant subanalytic map $\Phi' : X \to \mathbb{R}^{2k+2}$ such that $\bar{\Phi}'|_{(\bigcup_i \theta^{-1}((2i-4/3,\ 2i+1/3)))/G}$ is a homeomorphism onto the image. Hence $\psi = (\Phi,\ \Phi') : X \to \mathbb{R}^{4k+4}$ is a G-invariant subanalytic map whose induced map $\bar{\psi} : X/G \to \psi(X)$ is a homoemorphism. Recalling the property of θ, we have a closed neighborhood U of x and a compact set K in $\mathbb{R}^n$ such that $\psi(K \cap X) = \psi(X) \cap U$ for any point x in $\mathbb{R}^{4k+4}$. From this it follows that $\psi(X)$ is closed and subanalytic in $\mathbb{R}^{4k+4}$, since we can choose a subanalytic K. Moreover we can diminish $4k+4$ to $2k+1$ in the same way as the first case. Therefore the theorem is proved.

Corollary 3.4. Let $(G,\ M_1)$ be a subanalytic proper transformation group of a subanalytic set $(X,\ M_2)$. Assume X is locally compact. Then X/G admits a unique subanalytic structure such that $q : X \to X/G$ is subanalytic.

Proof. Trivial by Theorem 3.3.

Corollary 3.5. Let $(G,\ M_1)$ and $(X,\ M_2)$ be as above and give X/G the above subanalytic structure. Then there exists a subanalytic triangulation of X/G compatible with the orbit type stratification and uniquely in the following sense. If there are two subanalytic triangulations (K, τ) and (K', τ'), we have subanalytic triangulation isotopies (K, τ_t) and (K', τ'_t) of X/G such that $\tau_0 = \tau$, $\tau'_0 = \tau'$ and $(\tau'_1)^{-1} \circ \tau_1 : |K| \to |K'|$ is a PL map (see [6] for the definition of subanalytic triangulation isotopy).

Proof. Follows immediately from Lemma 2.4 in [6], Corollary 3.4 and the next fact. Let $\{X_i\}$ be the decomposition of X by orbit types. Then Lemma 2.3 tells us that $\{q(X_i)\}$ is a locally finite family of subanalytic subsets of X/G.

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広島大学理学部

Department of Mathematics,
Faculty of Science,
Hiroshima University,
Hiroshima 730, Japan

and

名古屋大学教養部

Department of Mathematics,
College of General Education,
Nagoya University,
Nagoya 464, Japan