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Fock Space Representation of the Virasoro Algebra I.

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§1. The Virasoro algebra $\underline{\underline{L}}$ is the Lie algebra over the complex number field $\mathbb{C}$ of the following form:

$$\underline{\underline{L}} = \sum_{n\in\mathbb{Z}} \mathbb{C}\, e_n \oplus \mathbb{C}\, e_0' ,$$

with the relations: for any $n, m\in\mathbb{Z}$

$$\begin{cases} [e_n, e_m] = (m-n)e_{n+m} + \dfrac{m^3-m}{12}\delta_{n+m,0}\, e_0' , \\ [e_0', e_n] = 0 . \end{cases}$$

This is a unique (up to isomorphisms) central extension of the Lie algebra $\underline{\underline{L}}'$ of trigonometric polynomial vector fields on the circle:

$$\underline{\underline{L}}' = \sum_{n\in\mathbb{Z}} \mathbb{C}\, \ell_n \quad ; \quad [\ell_n, \ell_m] = (m-n)\, \ell_{n+m} \quad (n, m\in\mathbb{Z}).$$

Let $\underline{h} = \mathbb{C}e_0 + \mathbb{C}e_0'$ be the abelian subalgebra of $\underline{\underline{L}}$ of maximal dimension. For each $(h,c) \in \mathbb{C}^2 \simeq \underline{h}^*$ the dual of $\underline{h}$, we can define the Verma module $M(h,c)$ and its dual $M^\dagger(h,c)$ as follows: $M(h,c)$ and $M^\dagger(h,c)$ are the left and right $\underline{\underline{L}}$-module with a cyclic vector $|h,c\rangle$ and $\langle c,h|$ with the following fundamental relations respectively:

$$e_{-n}|h,c\rangle = 0 \quad (n\geq 1); \; e_0|h,c\rangle = h|h,c\rangle \; , \; e_0'|h,c\rangle = c|h,c\rangle .$$

$$\langle c,h|\, e_n = 0 \quad (n\geq 1); \; \langle c,h|e_0 = \langle c,h|h \; , \; \langle c,h|e_0' = \langle c,h|c .$$

V.G.Kac[1979] studied these $\underline{\underline{L}}$-modules and obtained the formula concerning the determinant of the matrices of their vacuum expectation values, and by this Kac's determinant formula, F.L.Feigin and D.B.Fuks[1983] determined the composition series of $M(h,c)$.

§2. Consider the associative algebra $\underline{A}$ over $\mathbb{C}$ generated by $\{p_n$ $(n\in\mathbb{Z}),\ \Lambda\}$ with the following Bose commutation relations:

$$[p_n, p_m] = m\delta_{n+m,0} \quad ; \quad [\Lambda, p_n] = 0 \qquad (n,m\in\mathbb{Z}).$$

And consider the following operators in a completion $\hat{\underline{A}}$ of $\underline{A}$:

$$L_n = (p_0 - n\Lambda)\, p_n + \frac{1}{2}\sum_{j=1}^{n-1} p_j\, p_{n-j} + \sum_{j\geq 1} p_{n+j}\, p_{-j} \quad (n\geq 1);$$

$$L_{-n} = (p_0 + n\Lambda)\, p_{-n} + \frac{1}{2}\sum_{j=1}^{n-1} p_{-j}\, p_{j-n} + \sum_{j\geq 1} p_j\, p_{-n-j} \quad (n\geq 1);$$

$$L_0 = \frac{1}{2}(p_0{}^2 - \Lambda^2) + \sum_{j\geq 1} p_j\, p_{-j} \ ;$$

$$L_0' = (-12\Lambda^2 + 1)\ \mathrm{id}.$$

Then by easy but long calculations, we get

<u>Theorem 1.</u> The operators $L_n (n\in\mathbb{Z})$ and L_0' satisfy the commutation relations of the Virasoro algebra: for $n,m\in\mathbb{Z}$

$$\begin{cases} [L_n,\ L_m] = (m-n)L_{n+m} + \dfrac{m^3 - m}{12}\,\delta_{n+m,0}\, L_0' \ ; \\ [L_0',\ L_n] = 0. \end{cases}$$

§3. For each $(w,\lambda) \in \mathbb{C}^2$, we consider the left and right $\underline{A}$-module $\underline{F}(w,\lambda)$ and $\underline{F}^\dagger(w,\lambda)$ with cyclic vectors $|w,\lambda\rangle$ and $\langle\lambda,w|$ with the following fundamental relations respectively:

$$p_{-n}|w,\lambda\rangle = 0 \ (n\geq 1); \ p_0|w,\lambda\rangle = w|w,\lambda\rangle \ ; \ \Lambda|w,\lambda\rangle = \lambda|w,\lambda\rangle \ .$$

$$\langle\lambda,w|p_n = 0 \ (n\geq 1); \ \langle\lambda,w|p_0 = \langle\lambda,w|w \ ; \ \langle\lambda,w|\Lambda = \langle\lambda,w|\lambda \ .$$

Then by using the canonical homomorphism π (i.e. $\pi(e_n) = L_n\ (n\in\mathbb{Z})$; $\pi(e_0') = L_0'$), we get the left $\underline{L}$-module $(\underline{F}(w,\lambda),\pi_{(w,\lambda)},\underline{L})$ which is called the Fock space reprsentation, and by the explicit formulae of L_n and L_0',

$$\begin{cases} L_0|w,\lambda\rangle = \frac{1}{2}(w^2-\lambda^2)|w,\lambda\rangle \;; \quad L_0'|w,\lambda\rangle = (1-12\lambda^2)|w,\lambda\rangle; \\ L_{-n}|w,\lambda\rangle = 0 \quad \text{for } n \geqq 1 . \end{cases}$$

By the universal property of the Verma module $M(h,c)$ as an $\underline{\underline{L}}$-module, for each $(w,\lambda) \in \mathbb{C}^2$ we get the unique $\underline{\underline{L}}$-module mappping

$$\pi_{w,\lambda}: M(h(w,\lambda),c(\lambda)) \longrightarrow \underline{\underline{F}}(w,\lambda)$$

which sends the vacuum vector $|h(w,\lambda),c(\lambda)\rangle \in M(h(w,\lambda),c(\lambda))$ to the vacuum vector $|w,\lambda\rangle \in \underline{\underline{F}}(w,\lambda)$, where

$$h(w,\lambda) = \frac{1}{2}(w^2-\lambda^2) \quad \text{and} \quad c(\lambda) = 1 - 12\lambda^2.$$

Then by constructing intertwining operators(Theorem 3) and by showing their nontriviality (Theorem 6), we get the following.

<u>Theorem 2.</u> For each $(w,\lambda)\in\mathbb{C}^2$, let $s_\pm$ be the roots of the equation $\lambda = \frac{1}{s} - \frac{s}{2}$.

(1) The canonical $\underline{\underline{L}}$-module mapping

$$\pi_{w,\lambda}:M(h(w,\lambda),c(\lambda)) \longrightarrow \underline{\underline{F}}(w,\lambda)$$

is isomorphic, if and only if the equation

$$(*) \qquad w + \frac{a}{2}s_+ + \frac{b}{2}s_- = 0$$

has no integral solutions $(a,b)\in\mathbb{Z}^2$ with $a\geqq 1$ and $b\geqq 1$.

(2) The $\underline{\underline{L}}$-module mapping $\pi^\dagger_{w,\lambda}:M^\dagger(h(w,\lambda),c(\lambda)) \longrightarrow \underline{\underline{F}}^\dagger(w,\lambda)$ is isomorphic, if and only if the equation (*) has no integral solutions $(a,b)\in\mathbb{Z}^2$ with $a\leqq -1$ and $b\leqq -1$.

(3) $\underline{\underline{F}}(w,\lambda)$ is irreducible as an $\underline{\underline{L}}$-module, if and only if the equation (*) has no integral solutions $(a,b) \in \mathbb{Z}^2$ with $ab\geqq 1$.

And this condition(3) is equivalent to the fact that the corresponding Verma module $M(h(w,\lambda),c(\lambda))$ is irreducible.

§3. To construct intertwining operators between Fock spaces, we introduce the operators of following type acting on $\underline{\underline{F}}(w,\lambda)$. Fix $s \in \mathbb{C}^*$, and consider

$$X(s,\zeta) = \exp(s \sum_{n=1}^{\infty} \zeta^n \frac{p_n}{n}) \exp(-s \sum_{n=1}^{\infty} \zeta^{-n} \frac{p_{-n}}{n}) \zeta^{sp_0 - \frac{s^2}{2}} T_s ,$$

and for any $a \geq 1$

$$Z(s;\zeta_1,\cdots,\zeta_a) = F(\frac{s^2}{2};\zeta_1,\cdots,\zeta_a) \exp(s \sum_{n=1}^{\infty} (\zeta_1^n+\cdots+\zeta_a^n)\frac{p_n}{n})$$

$$\exp(-s \sum_{n=1}^{\infty} (\zeta_1^{-n}+\cdots+\zeta_a^{-n})\frac{p_{-n}}{n}) T_{as} ,$$

where

$$T_s : \underline{\underline{F}}(w,\lambda) \longrightarrow \underline{\underline{F}}(w+s,\lambda)$$

is the operator such that

$$T_s|w,\lambda\rangle = |w+s,\lambda\rangle \ ; \quad [T_s,p_n] = 0 \ (n \neq 0) \ ; \ [T_s,\Lambda] = 0 ,$$

and

$$F(\alpha;\zeta_1,\cdots,\zeta_a) = \prod_{j=1}^{a} \zeta_j^{-(a-1)\alpha} \prod_{1 \leq i<j \leq a} (\zeta_i - \zeta_j)^{2\alpha} .$$

Operators of this type are called Vertex Operators.

Then $X(s;\zeta)$ and $Z(s;\zeta_1,\cdots,\zeta_a)$ are multi-valued holomorphic functions of $\zeta \in \mathbb{C}^*$ and $(\zeta_1,\cdots,\zeta_a) \in M_a$ repectively with valued in the operators acting on $\underline{\underline{F}}(w,\lambda)$'s, where M_a is the manifold defined by

$$M_a = \{(\zeta_1,\cdots,\zeta_a) \in (\mathbb{C}^*)^a; \quad \zeta_i \neq \zeta_j \ (1 \leq i<j \leq a) \} .$$

In order to get intertwining operators, we want integrate these vertex operators $Z(s;\zeta_1,\cdots\zeta_a)$. For the guarantee of the convergence of these integrals, we introduce the homology theory associated to the monodromy structure of the multi-valued function $F(\alpha;\zeta_1,\cdots,\zeta_a)$.

For each $\alpha \in \mathbb{C}^*$, denote by $\underline{\underline{S}}_\alpha^*$ the local coefficient system with values in $\mathbb{C}$ which is determined by the monodromy of the multi-valued holomorphic function $F(\alpha;\zeta_1,\cdots,\zeta_a)$ on M_a, and denote by $\underline{\underline{S}}_\alpha$ the dual local system of $\underline{\underline{S}}_\alpha^*$.

Fix $s \in \mathbb{C}^*$ and an integer $a \geq 1$, and take an element $\Gamma \in H_a(M_a;\underline{\underline{S}}_\alpha)$. For each integer $b \in \mathbb{Z}$, we consider the operator

$$O(s,\Gamma;a,b) = \int_\Gamma Z(s;\zeta_1,\cdots,\zeta_a)\, \zeta_1^{-b-1}\cdots\zeta_a^{-b-1} d\zeta_1\cdots d\zeta_a .$$

Then we get the following.

Theorem 3.

1) For each $(w,\lambda)\in \mathbb{C}^2$, the operator $O(s,\Gamma;a,b)$ acts as

$$O(s,\Gamma;a,b) : \underline{\underline{F}}(w,\lambda) \longrightarrow \underline{\underline{F}}(w+as,\lambda) .$$

2) Take $s \in \mathbb{C}^*$ and $a,b \in \mathbb{Z}$ with $a \geq 1$. Put $\lambda = \lambda(s) = \frac{1}{s} - \frac{s}{2}$, then the operator

$$O(s,\Gamma;a,b) : \underline{\underline{F}}(-\tfrac{a}{2}s - \tfrac{b}{s}, \lambda) \longrightarrow \underline{\underline{F}}(\tfrac{a}{2}s - \tfrac{b}{s}, \lambda) .$$

commutes with the action of $\underline{\underline{L}}$.

For suitable $s \in \mathbb{C}^*$ and $w \in \mathbb{C}$, the equation (*) $w = \frac{a}{2}s - \frac{b}{s}$ in Theorem 2 has a countable number of integral solutions $(a,b) \in \mathbb{Z}^2$, if and only if $\alpha = s^2/2$ is a rational number. This index α characterizes the property of the monodromy of the function

$$F(\alpha;\zeta_1,\cdots,\zeta_a) = \prod_{j=1}^{a} \zeta_j^{-(a-1)\alpha} \prod_{1\leq i<j\leq a} (\zeta_i - \zeta_j)^{2\alpha} .$$

If α is irrational, then the monodromy of the function F is of logarithmic type, and if α is rational, then the monodromy of F is of algebraic type.

<u>Sketch of the Proof of Theorem 3.</u> The essential points to prove Theorem 3 are the following two propositions which are obtained by rather long calculations:

<u>Proposition 4.</u> For any $(\zeta_1,\cdots,\zeta_a)\in M_a$ and $s\in\mathbb{C}^*$,

$$X(s,\zeta_1)\cdots X(s,\zeta_a) = Z(s;\zeta_1,\cdots,\zeta_a)\,(\zeta_1\cdots\zeta_a)^{sp_0+\frac{a}{2}s^2}.$$

<u>Proposition 5.</u> (Conformal Covariance)

$$[L_m, X(s,\zeta)] = \zeta^{-m}(\zeta\frac{d}{d\zeta} - m\,(s\Lambda + \frac{s^2}{2}))\,X(s,\zeta)$$

for each $m\in\mathbb{Z}$ and $s,\zeta\in\mathbb{C}^*$, and

$$[L_m, Z(s;\zeta_1,\ldots,\zeta_a)] =$$

$$= \sum_{j=1}^{a}\zeta_j^{-m}\,[\zeta_j\frac{\partial}{\partial\zeta_j} + \{sp_0 - \frac{a}{2}s^2 - n(s\Lambda + \frac{s^2}{2})\}]\,Z(s;\zeta_1,\cdots,\zeta_a)$$

for each $m\in\mathbb{Z}$, $s\in\mathbb{C}$ and $(\zeta_1,\cdots,\zeta_a)\in M_a$.

Introduce the new coordinates $(k_1,\cdots,k_{a-1},\zeta)$ on the manifold M_a as the product manifold $M_a = Y_{a-1}\times\mathbb{C}^*$, where

$$Y_{a-1} = \{(k_1,\cdots,k_{a-1})\in(\mathbb{C}^*)^{a-1};\ k_i\neq k_j\ (i\neq j),\ k_i\neq 1\},$$

and

$$\zeta_i = k_i\zeta \quad (1\leqq i\leqq a-1) \text{ and } \zeta_a = \zeta.$$

Then the function $F(\alpha;\zeta_1,\cdots,\zeta_a)$ is independent of ζ, and we get

<u>Proposition 6.</u> (Total Momemtum Conservation) The integral

$$\int_\Gamma \zeta_1^{-\ell_1-1}\cdots\zeta_a^{-\ell_a-1}\,F(s;\zeta_1,\zeta_2,\cdots,\zeta_a)\,d\zeta_1\cdots d\zeta_a$$

vanishes unless $\ell_1+\ell_2+\cdots+\ell_a = 0$.

This assures the statement 1) of the theorem. And by Proposition 5 and by using integration by parts, we get the following formula.

$$[L_{-m}, O(s,\Gamma;a,b)] =$$

$$= \sum_{j=1}^{a} \{ b + s\, p_0 - \frac{a\, s^2}{2} + m\,(s\,\Lambda + \frac{s^2}{2} - 1)\}\; O(s,\Gamma;b,\ldots,\underset{\hat{j}}{b-n},\ldots,b).$$

From this formula, we know when $O(s,\Gamma,a,b)$ is intertwining.

§5. We must construct a cycle $\Gamma(\alpha) \in H_a(M_a;\underline{\underline{S}}_\alpha)$ which gives a nontrivial intertwining operator $O(s,\Gamma;a,b)$.

If we expand

$$\exp[s \sum_{n=1}^{\infty} (\zeta_1^n + \cdots + \zeta_a^n)\, \frac{p_n}{n}]\; \exp[-s \sum_{n=1}^{\infty} (\zeta_1^{-n} + \cdots + \zeta_a^{-n})\, \frac{p_{-n}}{n}]$$

as a Laurent series of $(\zeta_1,\cdots,\zeta_a)$, then the coefficient of the each term of the operator

$$\int_\Gamma Z(s;\zeta_1,\cdots,\zeta_a)\; \zeta_1^{-b-1}\cdots\zeta_a^{-b-1} d\zeta_1\cdots d\zeta_a$$

is written as

$$\int_\Gamma F(\frac{s^2}{2};\zeta_1,\cdots,\zeta_a)\; \zeta_1^{-\ell_1-1} \cdots \zeta_a^{-\ell_a-1}\; d\zeta_1\cdots d\zeta_a ,$$

and this integral is reduced to

$$\int_{\Gamma_2} k_1^{-\ell_1-1} \cdots k_{a-1}^{-\ell_{a-1}-1}\; G(\alpha;k_1,\cdots,k_{a-1})\; dk_1\cdots dk_{a-1},$$

where $\Gamma = \Gamma_1\times\Gamma_2 \in H_a(M_a;\underline{\underline{S}}_\alpha) \cong H_1(\mathbb{C}^*;\mathbb{C}) \otimes H_{a-1}(Y_{a-1};\underline{\underline{S}}'_\alpha)$, Γ_1 is a generator of $H_1(\mathbb{C}^*;\mathbb{C})$, and $\underline{\underline{S}}'_\alpha$ is the local system on Y_{a-1} similarly associated with the function $G(\alpha;k_1,\cdots,k_{a-1})$ as $\underline{\underline{S}}_\alpha$, and

$$G(\alpha;k_1,\cdots,k_{a-1}) = \prod_{j=1}^{a-1} k_j^{-(a-1)\alpha}(1-k_j)^{2\alpha} \prod_{1\leqq i<j\leqq a-1} (k_i-k_j)^{2\alpha} .$$

Now we must construct the cycle $\Gamma_2(\alpha) \in H_{a-1}(Y_{a-1};\underline{\underline{S}}_\alpha^{\vee})$ which regularizes the divergent integral

$$\int_{\Delta(a-1)} k_1^{-\ell_1-1} \cdots k_{a-1}^{-\ell_{a-1}-1} \; G(\alpha;k_1,\cdots,k_{a-1})\, dk_1\cdots dk_{a-1}.$$

where $\Delta(a-1)$ is the open $(a-1)$-simplex defined by

$$\Delta(a-1) = \{(k_1,\cdots,k_{a-1}) \in \mathbb{R}^{a-1};\ 1>k_1>\cdots>k_{a-1}>0\ \}.$$

Let $m=a-1$, and define the set

$$\Omega_m = \{\alpha\in\mathbb{C};\ d(d+1)\alpha\notin\mathbb{Z},\ d(a-d)\alpha\notin\mathbb{Z}\ \ (1\leq d\leq m)\}.$$

Then by using the technique of resolutions of singularities, we get

<u>Theorem 7.</u> There exist cycles $\Gamma_2(\alpha) \in H_m(Y_m;\underline{\underline{S}}_\alpha^{\vee})$ defined on Ω_m such that

1) $\Gamma_2(\alpha)$ is holomorphic on Ω_m .

2) If $\alpha\in\Omega_m$ and $(\ell_1,\cdots,\ell_m)\in\mathbb{Z}^m$ satisfy the inequalities

$$\mathrm{Re}\alpha > 0 \quad \text{and} \quad \mathrm{Re}2\alpha > -\min_j \ell_j ,$$

then the following equality of integrals holds:

$$\int_{\Gamma_2(\alpha)} G(\alpha;k_1,\cdots,k_m)\, k_1^{\ell_1}\cdots k_m^{\ell_m}\, dk_1\cdots dk_m =$$

$$= \int_{\Delta(m)} G(\alpha;k_1,\cdots,k_m)\, k_1^{\ell_1}\cdots k_m^{\ell_m}\, dk_1\cdots dk_m$$

And the latter integral is known explicitly as follows:

<u>Theorem 8.</u> (A.Selberg[1944])

Let $\alpha,\beta,\gamma \in \mathbb{C}$ satisfy the inequalities

$$\mathrm{Re}\beta > -1,\quad \mathrm{Re}\gamma > -1\ ,\quad \mathrm{Re}\alpha > -\min\{\tfrac{1}{m},\tfrac{\mathrm{Re}\beta+1}{m-1},\tfrac{\mathrm{Re}\gamma+1}{m-1}\}\ ,$$

then the improper integral (**) converges absolutely and is explicitly expressed as

$$(**) \quad \int_{\Delta(m)} \prod_{1\leq i<j\leq m} (k_i-k_j)^{2\alpha} \prod_{j=1}^{m} k_j^{\beta} (1-k_j)^{\gamma}\, dk_1\cdots dk_m =$$

$$= \frac{1}{m!} \prod_{j=1}^{m} \frac{\Gamma(j\alpha+1)\,\Gamma((j-1)\alpha+\beta+1)\,\Gamma((j-1)\alpha+\gamma+1)}{\Gamma(\alpha+1)\,\Gamma((m+j-2)\alpha+\beta+\gamma+2)} .$$

Finally we get

Theorem 9. There exists $\Gamma(\alpha) \in H_a(M_a;\underline{\underline{S}}_\alpha)$ which depends holomorphically on $\alpha \in \Omega_{a-1}$ such that the operator

$$O(s;a,b) = O(s,\Gamma(\tfrac{s^2}{2});a,b) : \underline{\underline{F}}(w-as,\tfrac{1}{s}-\tfrac{s}{2}) \longrightarrow \underline{\underline{F}}(w,\tfrac{1}{s}-\tfrac{s}{2})$$

is nontrivial in the sense that for any $w\in\mathbb{C}$

1) for $b\geq 0$, the image $O(s;a,b)|w-as,\frac{1}{s}-\frac{s}{2}\rangle$ is a nonzero vector.

2) for $b<0$, there is a vector $|v\rangle \in \underline{\underline{F}}(w-as,\frac{1}{s}-\frac{s}{2})$ such that $O(s;a,b)|v\rangle = |w,\frac{1}{s}-\frac{s}{2}\rangle$.

§6. Case that $\lambda=0$. Assume that $\lambda=0$, i.e., $c=1$, then the solutions $s_\pm$ of $\lambda(s)=0$ are $s_\pm=\pm\sqrt{2}$, and $\alpha = \frac{1}{2}s^2 = 1$. For any $m\in\mathbb{Z}$, denote $q_m = \sqrt{2}\, p_m$, then

$$[q_m, q_n] = 2n\delta_{n+m,0}\, L_0' \qquad \text{(note that } L_0' = \text{id in this case)},$$

and the operators L_n's are explicitly written as

$$\begin{cases} L_n = \frac{1}{2} q_0 q_n + \frac{1}{4} \sum_{j=1}^{n-1} q_j q_{n-j} + \frac{1}{2} \sum_{j\geq 1} q_{n+j} q_{-j} & \\ & (n\geq 1) \\ L_{-n} = \frac{1}{2} q_0 q_{-n} + \frac{1}{4} \sum_{j=1}^{n-1} q_{-j} q_{j-n} + \frac{1}{2} \sum_{j\geq 1} q_j q_{-n-j} & \\ L_0 = \frac{1}{4} q_0^2 + \frac{1}{2} \sum_{j\geq 1} q_j q_{-j} & . \end{cases}$$

And denote by $|m\rangle$ and $\langle m|$ the vacuum vectors of the Fock spaces $\underline{\underline{F}}(m) = \underline{\underline{F}}(m/\sqrt{2},0)$ and $\underline{\underline{F}}^\dagger(m) = \underline{\underline{F}}^\dagger(m/\sqrt{2},0)$ respectively, that is,

$$q_0|m\rangle = m|m\rangle \;,\; \Lambda|m\rangle = 0 \quad ; \quad \langle m|q_0 = \langle m|\, m \;,\; \langle m|\Lambda = 0 \quad .$$

Let

$$\hat{\underline{h}} := \mathbb{C}q_0 \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_0' \supset \underline{h}' := \mathbb{C}q_0 \oplus \mathbb{C}L_0 ,$$

then $\hat{\underline{h}}$ is the maximal abelian subalgebra of the Lie subalgebra $\underline{\underline{L}} \oplus \mathbb{C}q_0 \subset \underline{\underline{A}}$. The sums $\underline{\underline{F}}$ and $\underline{\underline{F}}^{\dagger}$ of Fock spaces $\underline{\underline{F}}(m)$ and $\underline{\underline{F}}^{\dagger}(m)$ can be considered as $(\underline{\underline{L}}\oplus\mathbb{C}q_0)$-modules, and have the weight space decompositions w.r.t. the abelian subalgebra $\underline{h}'$:

$$\underline{\underline{F}} = \sum_{m\in\mathbb{Z}} \sum_{d\geq 0} \underline{\underline{F}}_d(m) \quad \text{and} \quad \underline{\underline{F}}^{\dagger} = \sum_{m\in\mathbb{Z}} \sum_{d\geq 0} \underline{\underline{F}}_d^{\dagger}(m) ,$$

where $\underline{\underline{F}}_d(m)$ and $\underline{\underline{F}}_d^{\dagger}(m)$ are the weight spaces belonging to the same weight $(m, \frac{m^2}{4} + d)$.

Now decompose the vertex operators $X(\pm\sqrt{2},\zeta)$ as

$$X(\pm\sqrt{2};\zeta) = \exp(\pm \sum_{n\geq 1} \frac{q_n}{n} \zeta^n) \exp(\mp \sum_{n\geq 1} \frac{q_{-n}}{n} \zeta^{-n}) T_{\pm\sqrt{2}} \zeta^{\pm q_0 + 1}$$

$$= \sum_{m\in\mathbb{Z}} X_m^{\pm} \zeta^m \qquad (\zeta\in\mathbb{C}^*) .$$

Then as operators on the Fock spaces $\underline{\underline{F}}$ and $\underline{\underline{F}}^{\dagger}$, $X_m^{\pm}$ are well-defined and satisfy the following commutation relations : for $n,m \in \mathbb{Z}$,

$$(\#) \quad \begin{cases} [q_n, X_m^{\pm}] = \pm 2 X_{n+m}^{\pm} \ ; \quad [q_n, q_m] = 2m\delta_{n+m,0} L_0' , \\ [X_n^+, X_m^+] = [X_n^-, X_m^-] = 0 \ ; \ [X_n^+, X_m^-] = m\,\delta_{n+m,0} L_0' + q_{n+m} , \end{cases}$$

$$(\#2) \quad [L_n, X_m^{\pm}] = m X_{n+m}^{\pm} \ ; \quad [L_n, q_m] = m q_{n+m} ,$$

in particular

$$(\#3) \quad [q_0, X_0^{\pm}] = \pm 2 X_0^{\pm} \ ; \quad [X_0^+, X_0^-] = q_0 ,$$

$$(\#4) \quad [L_n, X_0^{\pm}] = [L_n, q_0] = 0 .$$

Hence the space $\underline{g}$ spanned by above operators

$$\underline{g} = \sum_{n\in\mathbb{Z}} \mathbb{C}X_n^+ \oplus \sum_{n\in\mathbb{Z}} \mathbb{C}q_n \oplus \sum_{n\in\mathbb{Z}} \mathbb{C}X_n^- \oplus \mathbb{C}L_0' \oplus \mathbb{C}L_0$$

is a Lie algebra isomorphic to the affine Lie algebra of type $A_1^{(1)}$

In fact, $A_1^{(1)}$ is realized as

$$A_1^{(1)} = \underline{sl}(2,\mathbb{C})\otimes\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

$$= \mathrm{span}\{e\otimes t^n, h\otimes t^n, f\otimes t^n \ (n\in\mathbb{Z})\} \oplus \mathbb{C}c \oplus \mathbb{C}d ,$$

where $\{e,h,f\}$ is a canonical basis of $\underline{sl}(2,\mathbb{C})$ with the relations:

$$[e,f] = h \ ; \ [h,e] = 2e \ ; \ [h,f] = -2f.$$

The isomorphism φ of $\underline{g}$ to $A_1^{(1)}$ is given by

$$\begin{cases} \varphi(X_n^+) = e\otimes t^n \ ; \ \varphi(\alpha_n) = h\otimes t^n \ ; \ \varphi(X_n^-) = f\otimes t^n \quad (n\in\mathbb{Z}) \\ \varphi(L_0') = - c \ ; \quad \varphi(L_0) = d = t\frac{d}{dt} \ . \end{cases}$$

Under this identification φ, fix the notations of the following Lie algebras

$$(\#\#) \qquad \begin{aligned} &\mathring{\underline{g}} := \mathbb{C}X_0^+ + \mathbb{C}\alpha_0 + \mathbb{C}X_0^- \cong \underline{sl}(2,\mathbb{C}) \\ &\cap \\ &\underline{g} = \mathring{\underline{g}} \otimes \mathbb{C}[t,t^{-1}] + \mathbb{C}L_0' + \mathbb{C}L_0 = A_1^{(1)} \\ &\cap \\ &\tilde{\underline{\underline{L}}} := \underline{g} + \sum_{n\neq 0} \mathbb{C}L_n = \mathring{\underline{g}} \otimes \mathbb{C}[t,t^{-1}] + \underline{\underline{L}} \ . \end{aligned}$$

Note that $L_0' = -c$ belongs to the center of $\tilde{\underline{\underline{L}}}$, and $L_n = t^{n+1}\frac{d}{dt}$ is the derivation of the loop algebra $\underline{g}/(\mathbb{C}L_0+\mathbb{C}L_0')$ of $\underline{sl}(2,\mathbb{C})$.

By (#4), $\mathring{\underline{g}} = \underline{sl}(2,\mathbb{C})$ commutes with the Virasoro algebra $\underline{\underline{L}}$:

$$[\mathring{\underline{g}}, \underline{\underline{L}}] = 0 \ ; \quad \text{hence} \quad [U(\mathring{\underline{g}}), \underline{\underline{L}}] = 0 \ .$$

The operators $X_0^{\pm} : \underline{\underline{F}}(m) \longrightarrow \underline{\underline{F}}(m\pm 2)$ are intertwining operators for any $m\in\mathbb{Z}$, and q_0 acts on $\underline{\underline{F}}(m)$ as the scalar operator m id.

Decompose the Fock space $\underline{\underline{F}}$ as

$$\underline{\underline{F}} = \underline{\underline{F}}^{even} \oplus \underline{\underline{F}}^{odd} ,$$

where

$$\underline{\underline{F}}^{even} = \sum_{m\in\mathbb{Z}} \underline{\underline{F}}(2m) \quad ; \quad \underline{\underline{F}}^{odd} = \sum_{m\in\mathbb{Z}} \underline{\underline{F}}(2m+1) \ .$$

Then these two components $\underline{\underline{F}}^{even}$ and $\underline{\underline{F}}^{odd}$ are irreducible $\underline{\underline{\check{L}}}$-modules, furthermore these are also irreducible as $A_1^{(1)}$-modules. Remark that $\underline{\underline{F}}^{even}$ is isomorphic to the basic representation of the affine algebra $A_1^{(1)}$.

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