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Nilpotent Orbits and Cayley Transform

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§1. INTRODUCTION.

Let $\underline{g}$ be a real semisimple Lie algebra and let $\underline{g} = \underline{k} + \underline{p}$ be its Cartan decomposition. Let $\underline{g}_c$, $\underline{k}_c$ and $\underline{p}_c$ be the complexifications of $\underline{g}$, $\underline{k}$ and $\underline{p}$, respectively. Put $G = \mathrm{Int}\ \underline{g}$, $G_c = \mathrm{Int}\ \underline{g}_c$ and let K_c be the analytic subgroup of G_c corresponding to $\underline{k}_c$. Then K_c acts on the vector space $\underline{p}_c$. If $\underline{\underline{N}}(\underline{p}_c)$ denotes the totality of the nilpotent elements of $\underline{p}_c$, K_c also acts on $\underline{\underline{N}}(\underline{p}_c)$. On the other hand, if $\underline{\underline{N}}(\underline{g})$ denotes the totality of the nilpotent elements of $\underline{g}$, G acts on $\underline{\underline{N}}(\underline{g})$. Then B. Kostant proposed the following.

CONJECTURE (cf. [K]): Does there exist a bijective correspondence between the set of K_c-orbits of $\underline{\underline{N}}(\underline{p}_c)$ and that of G-orbits of $\underline{\underline{N}}(\underline{g})$.

It is easy to generalize this conjecture to the case of the nilpotent variety of the tangent space of a semisimple symmetric space. The purpose of this note is to formulate this generalization and explain the outline of its proof. For the details, refer to [S].

§2. EXAMPLE.

First we give a typical example.

Take $\underline{g} = \underline{sl}(2,\mathbb{R})$ and put $\theta(X) = -{}^tX$ for each $X \in \underline{g}$. Then θ is a Cartan involution of $\underline{g}$. Let $\underline{g} = \underline{k} + \underline{p}$ be the corresponding Cartan decomposition. In this case

$$\underline{\underline{N}}(\underline{g}) = \{ \begin{pmatrix} t & x \\ y & -t \end{pmatrix} ; t, x, y \in \mathbb{R}, t^2 + xy = 0 \}$$

and there are three G-orbits of $\underline{\underline{N}}(\underline{g})$ and we can choose the representatives as follows:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

On the other hand,

$$\underline{\underline{N}}(\underline{p}_c) = \{ \begin{pmatrix} t & x \\ x & -t \end{pmatrix} ; t, x \in \mathbb{C}, t^2 + x^2 = 0 \}.$$

There are three K_c-orbits of $\underline{\underline{N}}(\underline{p}_c)$ and we can choose the respresntatives of them as follows:

$$\frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For any $X \in \underline{\underline{N}}(\underline{g})$, it is easy to show that there exist A, $Y \in \underline{g}$ such that

(I) $[A, X] = 2X, \quad [A, Y] = -2Y, \quad [X, Y] = A.$

For such a triple (A, X, Y), we define

(II) $A^d = i(X-Y), \quad X^d = \frac{1}{2}(X+Y+iA), \quad Y^d = \frac{1}{2}(X+Y-iA).$

Then

(III) $[A^d, X^d] = 2X^d, \quad [A^d, Y^d] = -2Y^d, \quad [X^d, Y^d] = A^d.$

At this stage, we assume that

(IV) $\theta(X) = -Y, \quad \theta(A) = -A.$

Then it follows that

(V) $\theta(A^d) = A^d, \quad \theta(X^d) = -X^d, \quad \theta(Y^d) = -Y^d.$

This means that $A^d \in \underline{k}_c$, X^d, $Y^d \in \underline{p}_c$.

Now take a G-orbit $\underline{\underline{O}}$ of $\underline{\underline{N}}(\underline{g})$. Then one can show the following by direct calculation:

<u>CLAIM</u>. There exists a triple (A, X, Y) such that $X \in \underline{\underline{O}}$ and that this satisfies the conditions (I) and (IV). Define A^d, X^d and Y^d by (II). Then the K_c-orbits of X^d and Y^d only depend on $\underline{\underline{O}}$.

Take a G-orbit $\underline{\underline{O}}$ of $\underline{\underline{N}}(\underline{g})$. Using the notation in CLAIM, we define the K_c-orbit of X^d in $\underline{\underline{N}}(\underline{p}_c)$. Then CLAIM implies that this map defines a required bijection.

For example, we take $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. If we put $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the triple (A, X, Y) satisfies (I), (IV). Then we find that $X^d = \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$ and $Y^d = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$.

§3. A GENERALIZATION.

Let $\underline{g}$ be a semisimple Lie algebra of the non-compact type and let σ be its involution. Then we obtain the direct sum $\underline{g} = \underline{h} + \underline{q}$, where $\underline{h}$ and $\underline{q}$ are the 1- and (−1)-eigenspaces of σ, respectively. The pair $(\underline{g}, \underline{h})$ is called a semisimple symmetric pair.

It is known that there is a Cartan involution θ of $\underline{g}$ commuting with σ. Let $\underline{g} = \underline{k} + \underline{p}$ be the corresponding Cartan involution. Since $\theta\sigma$ is also an involution of $\underline{g}$, we obtain the direct sum $\underline{g} = \underline{h}^a + \underline{q}^a$, where $\underline{h}^a$ and $\underline{q}^a$ are the 1- and (−1)-eigenspaces of $\theta\sigma$, respectively. Putting $\underline{g}^a = \underline{g}$, we obtain a symmetric pair $(\underline{g}^a, \underline{h}^a)$ $(= (\underline{g}, h)^a)$. This is called the associated symmetric pair.

Let $\underline{g}_c$ be the complexification of $\underline{g}$ and we extent θ, σ to $\underline{g}_c$ as complex linear involutions. Put

$$\underline{g}^d = \underline{k} \cap \underline{h} + i(\underline{k} \cap \underline{q}) + i(\underline{p} \cap \underline{h}) + \underline{p} \cap \underline{q}.$$

Then $\underline{g}^d$ defines a real form of $\underline{g}_c$. Since θ is an involution of $\underline{g}^d$, we obtain the direct sum $\underline{g}^d = \underline{h}^d + \underline{q}^d$ for θ and a symmetric pair $(\underline{g}^d, \underline{h}^d)$ $(= (\underline{g}, \underline{h})^d)$, which is called dual to $(\underline{g}, \underline{h})$.

The following diagram holds:

$$\begin{tikzcd}
(\underline{g}, \underline{h}) \arrow[r, leftrightarrow, "\text{associated}"] \arrow[d, leftrightarrow, "\text{dual}"] & (\underline{g}, \underline{h})^a \arrow[r, leftrightarrow, "\text{dual}"] & (\underline{g}, \underline{h})^{ad} \arrow[d, leftrightarrow, "\text{associated}"] \\
(\underline{g}, \underline{h})^d \arrow[r, leftrightarrow, "\text{associated}"] & (\underline{g}, \underline{h})^{da} \arrow[r, leftrightarrow, "\text{dual}"] & (\underline{g}, \underline{h})^{dad}
\end{tikzcd}$$

Let $\underline{\underline{N}}(\underline{q})$ be the totality of the nilpotent elements of $\underline{q}$ and let H be the analytic subgroup of Int $\underline{g}$ corresponding to $\underline{h}$. Since H acts on $\underline{\underline{N}}(\underline{q})$, we denote by $[\underline{\underline{N}}(\underline{q})]$ the set of H-orbits of $\underline{\underline{N}}(\underline{q})$. It is known that $[\underline{\underline{N}}(\underline{q})]$ is a finite set. Similarly, we define $[\underline{\underline{N}}(\underline{q}^a)]$ and $[\underline{\underline{N}}(\underline{q}^d)]$ for the pairs $(\underline{g}^a, \underline{q}^a)$ and $(\underline{g}^d, \underline{h}^d)$, respectively.

<u>§4</u>. <u>THE MAIN THEOREM</u>.

Let $(\underline{g}, \underline{h})$ be a symmetric pair as above.

<u>Lemma 1</u>. For any $X \in \underline{\underline{N}}(\underline{q})$, there exist $A \in \underline{h}$ and $Y \in \underline{q}$ such that $[A, X] = 2X$, $[A, Y] = -2Y$, $[X, Y] = A$.

<u>Definition 2</u>. A triple (A, X, Y) satisfying the condition of Lemma 1 is called a normal S-triple.

Definition 3. A triple (A, X, Y) is called a strictly normal S-triple if (A, X, Y) is a normal S-triple such that $\theta(A) = -A$, $\theta(X) = -Y$.

Lemma 4. If (A, X, Y) is a normal S-triple, there is $h \in H$ such that $(h\cdot A, h\cdot X, h\cdot Y)$ is a strictly normal S-triple.

Lemma 5. Let (A_i, X_i, Y_i) $(i = 1, 2)$ be strictly normal S-triples. If X_1 and X_2 are H-conjugate, there is $k \in H \cap K$ such that $(k\cdot A_1, k\cdot X_1, k\cdot Y_1) = (A_2, X_2, Y_2)$.

For the details of the proof of the above lemmas, refer to [S].

We are now going to formulate our main result. Let $\underline{\underline{O}}$ be an H-orbit of $\underline{\underline{N}}(\underline{q})$. Then it follows from Lemmas 1 and 4 that there exist $X \in \underline{\underline{O}}$, $A \in \underline{h}$ and $Y \in \underline{q}$ such that (A, X, Y) is a strictly normal S-triple. Put

$$A^d = i(X-Y), \quad X^d = \frac{1}{2}(X+Y+iA), \quad Y^d = \frac{1}{2}(X+Y-iA).$$

Then (A^d, X^d, Y^d) is a strictly normal S-triple for the pair $(\underline{g}^d, \underline{h}^d)$. Moreover it follows from Lemma 5 that the H^d-orbits $H^d\cdot X^d$ and $H^d\cdot Y^d$ only depend on X. Noting this, we define maps

$$\Phi_\pm : [\underline{\underline{N}}(\underline{q})] \to [\underline{\underline{N}}(\underline{q}^d)]$$

by $\Phi_+(\underline{\underline{O}}) = H^d\cdot X^d$ and $\Phi_-(\underline{\underline{O}}) = H^d\cdot Y^d$. By a similar argument, we also define maps $\Phi^d_\pm : [\underline{\underline{N}}(\underline{q}^d)] \to [\underline{\underline{N}}(\underline{q})]$. Then we find that

$$\Phi^d_-(\Phi_+(H\cdot X)) = H\cdot X, \quad \Phi^d_+(\Phi_+(H\cdot X)) = H\cdot Y.$$

This, in particular, implies the bijectivity of $\Phi_\pm$.

Let (A, X, Y) be a strictly normal S-triple. Put

$$A' = X+Y, \quad X' = \frac{1}{2}(X-Y-A), \quad Y' = \frac{1}{2}(-X+Y-A).$$

Then it follows that (A', X', Y') is a strictly normal S-triple for the pair $(\underline{g}^a, \underline{h}^a)$. Noting this, we also obtain a bijection $[\underline{\underline{N}}(\underline{q})] \to [\underline{\underline{N}}(\underline{q}^a)]$ by an argument similar to the above one.

Hence we obtain the following theorem.

THEOREM 5. $[\underline{\underline{N}}(\underline{q})] \simeq [\underline{\underline{N}}(\underline{q}^a)] \simeq [\underline{\underline{N}}(\underline{q}^d)] \simeq [\underline{\underline{N}}(\underline{q}^{ad})]$
$\simeq [\underline{\underline{N}}(\underline{q}^{da})] \simeq [\underline{\underline{N}}(\underline{q}^{ada})]$.

§5. PROOF OF THE CONJECTURE.

We return to the situation in §1. By definition, $(\underline{g}_c, \underline{k}_c)$ and $(\underline{g}_c, \underline{g})$ are symmetric pairs. Moreover, if we define an involution of $\underline{g} \oplus \underline{g}$ by $(X, Y) \to (Y, X)$, we find that $(\underline{g} \oplus \underline{g}, \underline{g})$ is also a symmetric pair. In this case, the following diagram holds:

$$\begin{tikzcd}
(\underline{g} \oplus \underline{g}, \underline{g}) \arrow[r, leftrightarrow, "\text{dual}"] \arrow[loop below, "\text{associated}"] & (\underline{g}_c, \underline{k}_c) \arrow[r, leftrightarrow, "\text{associated}"] & (\underline{g}_c, \underline{g}) \arrow[loop below, "\text{dual}"]
\end{tikzcd}$$

Let $[\underline{\underline{N}}(\underline{g})]$ be the set of G-orbits of $\underline{\underline{N}}(\underline{g})$ and let $[\underline{\underline{N}}(\underline{p}_c)]$ be that of K_c-orbits of $\underline{\underline{N}}(\underline{p})$. Then we find that the conjecture stated in §1 is a special case of THEOREM 5.

Corollary 6. $[\underline{\underline{N}}(\underline{p}_c)] \simeq [\underline{\underline{N}}(\underline{g})]$

Let $\tilde{K}_c$ and $\tilde{G}$ be the normalizers of $\underline{k}_c$ and $\underline{g}$ in G_c, respectively. Let $[\underline{\underline{N}}(\underline{g})]_\theta$ be the set of $\tilde{G}$-orbits of $\underline{\underline{N}}(\underline{g})$ and let $[\underline{\underline{N}}(\underline{p}_c)]_\theta$ be that of $\tilde{K}_c$-orbits of $\underline{\underline{N}}(\underline{p}_c)$. Then an analogy of Corollary holds:

<u>Theorem 7</u> (B. Kostant). $[\underline{\underline{N}}(\underline{p}_c)]_\theta \simeq [\underline{\underline{N}}(\underline{g})]_\theta$

<u>Remark 8</u>. Assume that $\underline{g}_c$ is simple of the classical type.

(i) The G-orbital structure of $\underline{\underline{N}}(\underline{g})$ is determined by Bourgoyne and Cushman ([BC]).

(ii) D. King proves Corollary 6 in this case by using the classification ([K]).

References

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