

On puncture variation

AKIRA YAMADA (山田 陽)

Department of Mathematics  
Tokyo Institute of Technology

1. Introduction.

Let  $\Delta$  be the unit disc and let  $D (\neq \mathbb{C})$  be a plane domain containing the origin. Set  $D_p = D \setminus \{p\}$  for  $p \in D \setminus \{0\}$ . Then there exists unique holomorphic universal covering  $f_p: \Delta \rightarrow D_p$  satisfying

$$f_p(0) = 0, \quad f_p'(0) > 0.$$

Our aim is to derive the variation of the covering  $f_p$  by moving the puncture  $p$  in the domain  $D$ . Such a variation is called a puncture variation. Theorem 4.2. is the main result of this paper which gives explicitly the variation of  $f_p$ . As a corollary we have a puncture variation of the Poincaré metric. To obtain the formula we use quasiconformal mappings and apply a well-known representation theorem for quasiconformal maps with small dilatation.

## 2. Construction of $f_{p+\varepsilon}$ from $f_p$ .

For sufficiently small  $\rho \in \mathbb{R}$  let  $N = \{z \mid 0 < |z-p| < e^\rho\}$  be a punctured disc contained in  $f_p(\Delta)$  with  $0 \notin N$ . Let  $\Delta_0$  be a fixed component of  $f^{-1}(N)$ . Note that  $\Delta_0$  does not contain the origin. Let  $\Gamma$  be the covering group of  $f_p$ . The following Lemma is well known.

**LEMMA 2.1.** There exist a parabolic element  $\beta \in \Gamma$  and a Möbius transformation  $A$  with the following properties,

- (1)  $A$  maps the upper half-plane onto  $\Delta$ ,
- (2)  $A(\infty) \in \partial\Delta$  is the fixed point of  $\beta$  and  $A^{-1} \cdot \beta \cdot Az = z+1$ ,
- (3)  $\Delta_0$  is simply connected and contains a disc  $A(U_c)$  with  $U_c = \{z \in \mathbb{C} \mid \text{Im}z > c\}$  ( $c > 0$ ), and
- (4) two points  $z_1$  and  $z_2$  of  $\Delta_0$  are equivalent under  $\Gamma$  if and only if  $z_2 = \beta^n(z_1)$  for some integer  $n$ .

**PROOF.** See Kra [3, p.52] or Ahlfors [1, Lemma 1] where more general Kleinian case is considered. q.e.d

Let  $\Gamma_0$  be the cyclic subgroup of  $\Gamma$  generated by  $\beta \in \Gamma$ . Expressing  $f_p^{-1}(N)$  as a disjoint union of the components, we have

$$f_p^{-1}(N) = \bigcup_{\gamma \in \Gamma/\Gamma_0} \gamma \Delta_0 \quad (2.1)$$

where  $\Gamma/\Gamma_0$  denotes the set of left cosets. Let  $\Pi: L \rightarrow N$  be a universal covering given by

$$\Pi(z) = p + e^{z+\rho},$$

where  $L$  is the left half-plane  $\{z \mid \text{Re } z < 0\}$ . By the theory of covering surface we can find a conformal map  $\varphi: \Delta_0 \rightarrow L$  such that

$$\varphi \circ \beta = \varphi + 2\pi i \quad (2.2)$$

and

$$f_p = \Pi \circ \varphi \quad \text{on } \Delta_0. \quad (2.3)$$

**LEMMA 2.2.** For  $\varepsilon \in \mathbb{C}$  small, there exists a quasiconformal map  $\psi_\varepsilon: L \rightarrow \mathbb{C}$  such that

$$\psi_\varepsilon(z+2\pi i) = \psi_\varepsilon(z) + 2\pi i \quad \text{on } L \quad (2.4)$$

and

$$\varepsilon + \Pi \circ \psi_\varepsilon = \Pi \quad \text{on } \partial L \quad (2.5)$$

with complex dilatation

$$-\varepsilon e^{\bar{z}-\rho} + o(\varepsilon^2) \quad (2.6)$$

for  $z \in L$ . The estimate is uniform for  $z \in L$ .

**PROOF.** Taking a branch of the logarithm, we set

$$\psi_\varepsilon(z) = z + \ln(1 - \varepsilon e^{\bar{z}-\rho}).$$

It is easy to see that this is a desired quasiconformal map. q.e.d.

Define a map  $\tilde{f}: \Delta \rightarrow D_{p+\varepsilon}$  by

$$\tilde{f}(z) = \begin{cases} f_p(z) & , z \in f_p^{-1}(N) \\ \varepsilon + \Pi \circ \psi_\varepsilon \circ \varphi \circ \gamma^{-1}(z) & , z \in \gamma \Delta_0 \quad (\gamma \in \Gamma/\Gamma_0). \end{cases}$$

It is seen from (2.1)-(2.5) that  $\tilde{f}$  is a well-defined (topological) covering of  $D_{p+\varepsilon}$ . Let  $g_\mu$  denote the quasiconformal automorphism of  $\Delta$  with complex dilatation  $\mu$  which is holomorphic near the origin and satisfies  $g_\mu(0) = 0$ ,  $g'_\mu(0) > 0$ .

LEMMA 2.3. We have the identity

$$f_{p+\varepsilon} = \tilde{f} \cdot g_{\varepsilon\mu}^{-1} \quad \text{on } \Delta$$

where the complex dilatation  $\mu$  is given by

$$\mu(z) = \begin{cases} 0 & , z \in f_p^{-1}(N) \\ \varepsilon^{-1} (\varphi \circ \gamma^{-1})^* \mu_{\psi_\varepsilon}(z) & , z \in \gamma \Delta_0 \quad (\gamma \in \Gamma/\Gamma_0) . \end{cases} \quad (2.7)$$

Here,  $\varphi^* \mu$  denotes as usual the pull-back  $\mu \circ \varphi \cdot \frac{\overline{\varphi'}}{\varphi'}$  of the Beltrami coefficient  $\mu$ .

PROOF. Computing the complex dilatation we have

$\mu_{\tilde{f} \cdot g_{\varepsilon\mu}^{-1}} = 0$  a.e. on  $\Delta$ . Hence  $\tilde{f} \cdot g_{\varepsilon\mu}^{-1}$  is a holomorphic covering of

$D_{p+\varepsilon}$  such that

$$\tilde{f} \cdot g_{\varepsilon\mu}^{-1}(0) = 0, \quad (\tilde{f} \cdot g_{\varepsilon\mu}^{-1})'(0) > 0.$$

Since these conditions determine a holomorphic covering uniquely,

we conclude that  $f_{p+\varepsilon} = \tilde{f} \cdot g_{\varepsilon\mu}^{-1}$ . q.e.d.

### 3. Integral representation of the variation.

Let  $f_\mu$  be the quasiconformal automorphism of  $\Delta$  with complex dilatation  $\mu$  which leaves 0 and 1 fixed. The following perturbation formula is well known [2, p.105].

LEMMA 3.1. For  $\varepsilon \in \mathbb{C}$  small and  $\zeta \in \Delta$ ,  $f_{\varepsilon\mu}$  is given by

$$f_{\varepsilon\mu}(\zeta) = \zeta + \hat{f}(\zeta) + O(\varepsilon^2)$$

where

$$\hat{f}(\zeta) = -\frac{\varepsilon}{\pi} \iint_{\Delta} \mu(z) R(z, \zeta) dx dy + \frac{\overline{\varepsilon}}{\pi} \iint_{\Delta} \overline{\mu(z)} \zeta^2 \overline{R(z, 1/\zeta)} dx dy$$

and

$$R(z, \xi) = \frac{\xi(\xi-1)}{z(z-1)(z-\xi)}.$$

The estimate is uniform for compact subsets of  $\Delta$ .

It is convenient for our purpose to have a lemma with different normalization. The next lemma is a useful perturbation formula for  $g_\mu$ . Recall that  $\mu$  vanishes near the origin and that  $g_\mu(0)=0$  and  $g_\mu'(0)>0$ .

**LEMMA 3.2.** For  $\varepsilon \in \mathbb{C}$  small and  $\xi \in \Delta$ ,  $g_{\varepsilon\mu}$  is given by

$$g_{\varepsilon\mu}(\xi) = \xi + \dot{g}(\xi) + O(\varepsilon^2)$$

where

$$\dot{g}(\xi) = -\frac{\varepsilon\xi}{2\pi} \iint_{\Delta} \mu(z) Q(z, \xi) dx dy + \frac{\bar{\varepsilon}\bar{\xi}}{2\pi} \iint_{\Delta} \overline{\mu(z) Q(z, 1/\bar{\xi})} dx dy$$

and

$$Q(z, \xi) = \frac{z+\xi}{z^2(z-\xi)}.$$

The estimate is uniform for compact subsets of  $\Delta$ .

**PROOF.** Observe that

$$\dot{g}(\xi) = \dot{f}(\xi) + \frac{1}{2}\xi(\overline{\dot{f}'(0) - \dot{f}'(0)}).$$

**LEMMA 3.1.** yields

$$\dot{f}'(0) = \frac{\varepsilon}{\pi} \iint_{\Delta} \frac{\mu(z)}{z^2(z-1)} dx dy - \frac{\bar{\varepsilon}}{\pi} \iint_{\Delta} \frac{\overline{\mu(z)}}{\bar{z}(\bar{z}-1)} dx dy.$$

Combining these identities, we obtain the Lemma. q.e.d.

From Lemmas 2.3 and 3.2 we have, for  $\xi \in f_p^{-1}(N)$ ,

$$f_{p+\varepsilon} = f_p(\zeta) + \zeta f_p'(\zeta)(\varepsilon I(\zeta) + \bar{\varepsilon} J(\zeta)) + O(\varepsilon^2) \quad (3.1)$$

where

$$I(\zeta) = \frac{1}{2\pi} \iint_{F_p^{-1}(N)} \mu(z) Q(z, \zeta) dx dy$$

and

$$J(\zeta) = \overline{-I(1/\bar{\zeta})}. \quad (3.2)$$

Since (2.1) is a disjoint union,  $I(\zeta)$  is expressed as a series of the form

$$I(\zeta) = \sum_{\gamma \in \Gamma/\Gamma_0} I_\gamma(\zeta) \quad (3.3)$$

where

$$I_\gamma(\zeta) = \frac{1}{2\pi} \iint_{\gamma \Delta_0} \mu(z) Q(z, \zeta) dx dy.$$

#### 4. Evaluation of $I_\gamma(\zeta)$ .

By (2.6) and (2.7) we have

$$\begin{aligned} I_\gamma(\zeta) &= \frac{1}{2\pi} \iint_{\Delta_0} \gamma^* \mu(z) \gamma^* Q(z, \zeta) dx dy \\ &= \frac{1}{2\pi} \iint_{\Delta_0} \varepsilon^{-1} \varphi^* \mu_{\psi_\varepsilon}(z) \gamma^* Q(z, \zeta) dx dy \\ &= \frac{1}{2\pi} \iint_L \varepsilon^{-1} \mu_{\psi_\varepsilon}(z) (\gamma \circ \varphi^{-1})^* Q(z, \zeta) dx dy \\ &= -\frac{1}{2\pi} \iint_{x \leq 0} e^{\bar{z}-\rho} (\gamma \circ \varphi^{-1})^* Q(z, \zeta) dx dy + O(\varepsilon), \end{aligned}$$

where  $\gamma^* Q(z, \zeta) = Q(\gamma(z), \zeta) \left(\frac{d\gamma}{dz}\right)^2$  is the pull-back of  $Q$  considered

as a quadratic differential of  $z$ . Therefore,

$$I_\gamma(\zeta) = I + O(\varepsilon) \quad (4.1)$$

where

$$I = -\frac{1}{2\pi} \iint_{x \leq 0} e^{\bar{z} - \rho} (\gamma \circ \varphi^{-1})^* Q(z, \zeta) dx dy.$$

Our task is to evaluate the double integral  $I$  by using the calculus of residues. For convenience we introduce the functions  $u$  and  $w$  with the following properties,

$$(1) \varphi = u \circ w,$$

(2)  $w: \Delta \rightarrow L$  is a Möbius transformation onto  $L$  such that  $w \circ \beta = w + 2\pi i$ , and

(3)  $u: w(\Delta_0) \rightarrow L$  is a conformal surjection such that  $u(z + 2\pi i) = u(z) + 2\pi i$ .

Obviously, such  $u$  and  $w$  exist but not uniquely. We fix once and for all a choice of  $w$ .

**LEMMA 4.1.** For fixed  $\zeta \in f_p^{-1}(N)$ ,

$$(\gamma \circ \varphi^{-1})^* Q(z, \zeta) = O(z^{-4}) \quad \text{as } z \rightarrow \infty, z \in L.$$

**PROOF.** Setting  $\gamma_1 = \gamma \circ w^{-1}$  and  $u_1 = u^{-1}$ , we have

$$\gamma \circ \varphi^{-1} = \gamma_1 \circ u_1$$

where  $\gamma_1: L \rightarrow \Delta$  is a Möbius transformation and  $u_1: L \rightarrow L$  is holomorphic. Clearly,

$$\gamma_1'(z) = O(z^{-2}) \quad (z \rightarrow \infty). \quad (4.2)$$

On the other hand, by expanding the function  $u_1(z) - z$ , which is periodic with period  $2\pi i$ , in a Fourier series, it is not hard to see that

$$u_1(z) = z + o(1) \text{ and } u_1'(z) = o(1) \text{ (} z \rightarrow \infty \text{)}, \quad (4.3)$$

since  $u_1(z)$  is analytic on  $\partial L$  and  $u_1$  maps  $L$  into itself. (4.2) and (4.3) show that  $(\gamma \circ \varphi^{-1})'(z) = o(z^{-2})$  ( $z \rightarrow \infty$ ). This immediately gives the Lemma. q.e.d.

Cauchy's integral theorem and Lemma 4.1. imply that the integral

$$\int_{-\infty}^{\infty} e^{-z-\rho} (\gamma \circ \varphi^{-1})^* Q(z, \xi) dy$$

is independent of  $x = \operatorname{Re} z$ . Thus

$$\begin{aligned} I &= - \frac{1}{2\pi} \int_{-\infty}^0 e^{2x} dx \int_{-\infty}^{\infty} e^{-z-\rho} (\gamma \circ \varphi^{-1})^* Q dy \\ &= - \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-iy-\rho} (\gamma \circ \varphi^{-1})^* Q(iy, \xi) dy \\ &= - \frac{1}{4\pi i} \int_{\partial L} e^{-z-\rho} (\gamma \circ \varphi^{-1})^* Q(z, \xi) dz \\ &= - \frac{1}{4\pi i} \int_{\ell} \frac{(\gamma \circ w^{-1})^* Q(z, \xi)}{u'(z) e^{u(z)+\rho}} dz \end{aligned}$$

where  $\ell$  is a vertical line contained in  $w(\Delta_0)$ . Since the function  $f_p \circ w^{-1}(z)$  is periodic with period  $2\pi i$  on  $L$ , it is of the form  $f_p \circ w^{-1}(z) = F(e^z)$  where  $F(z)$  is regular in  $\Delta$  with  $F(0) = p$ . Differentiating both sides of the identity  $F(e^z) = p + e^{u(z)+\rho}$ , we have

$$u'(z) e^{u(z)+\rho} = F'(e^z) e^z.$$

Hence

$$I = - \frac{1}{4\pi i} \int_{\ell} \frac{(\gamma \circ w^{-1})^* Q}{F'(e^z) e^z} dz.$$

By noting the estimates

$$(\gamma \cdot w^{-1})^* Q = O(z^{-4}), \quad (z \rightarrow \infty)$$

and

$$\frac{1}{Ce^z} - \frac{1}{F'(e^z)e^z} = O(1), \quad (z \rightarrow \infty)$$

with  $C=F'(0)$ , a standard application of Cauchy's integral theorem yields

$$I = -\frac{1}{4\pi i C} \int_{\varrho} e^{-z} (\gamma \cdot w^{-1})^* Q(z, \xi) dz.$$

Although this integral can be evaluated by computing the residues in the right half-plane determined by  $\varrho$ , it is easier to evaluate the integral by changing the variable  $z$  to  $w \cdot \gamma^{-1}(z)$ . Thus

$$I = -\frac{1}{4\pi i C} \int_{\gamma(h)} \frac{e^{-w \cdot \gamma^{-1}(z)}}{(w \cdot \gamma^{-1})'(z)} Q(z, \xi) dz$$

where  $h$  is a circle in  $\Delta$  which is tangent to  $\partial\Delta$  at the fixed point of  $\beta$ . Denoting the residue of the integrand at  $z$  by  $\text{Res}(z)$ , we have

$$I = \frac{1}{2C} [\text{Res}(\xi) + \text{Res}(0) + \text{Res}(\infty)].$$

Observe that  $w \cdot \gamma^{-1}(z)$  is of the form

$$w \cdot \gamma^{-1}(z) = \frac{t\alpha + \bar{t}z}{\alpha - z}, \quad |\alpha|=1, \quad \text{Re } t < 0.$$

After elementary calculations, we obtain

$$I = \frac{1}{C\xi} \left[ \frac{e^{-iu_\gamma}}{(w \cdot \gamma^{-1})'(\xi)} \right] \left[ e^{-w \cdot \gamma^{-1}(\xi) + iu_\gamma} + (w \cdot \gamma^{-1}(\xi) - iu_\gamma) \frac{\sinh t_\gamma}{t_\gamma} \right]$$

$$-\cosh t_\gamma] - \zeta e^{-iu_\gamma} \sinh t_\gamma \} \quad (4.4)$$

with  $w \cdot \gamma^{-1}(0) = t_\gamma + iu_\gamma$  ( $t_\gamma, u_\gamma \in \mathbb{R}$ ). Since

$$w \cdot \gamma^{-1}(1/\bar{\zeta}) = -\overline{w \cdot \gamma^{-1}(\zeta)}, \quad \zeta \in \Delta,$$

identities (3.1)-(3.3), (4.1) and (4.4) give us the following final form of the variation of  $f_p$ .

**THEOREM 4.2.** For sufficiently small  $\epsilon \in \mathbb{C}$ , the universal covering  $f_{p+\epsilon}$  of  $D_{p+\epsilon}$  is given by

$$f_{p+\epsilon}(z) = f_p(z) + f_p'(z) \left[ \frac{\epsilon}{C} I_1 - \frac{\bar{\epsilon}}{C} I_2 \right] + O(\epsilon^2), \quad z \in \Delta$$

where

$$I_1 = \sum_{\gamma \in \Gamma/\Gamma_0} \left\{ \frac{e^{-iu_\gamma}}{(w \cdot \gamma^{-1})'(z)} \left[ e^{-w \cdot \gamma^{-1}(z) + iu_\gamma} + (w \cdot \gamma^{-1}(z) - iu_\gamma) \frac{\sinh t_\gamma}{t_\gamma} - \cosh t_\gamma \right] - z e^{-iu_\gamma} \sinh t_\gamma \right\},$$

and

$$I_2 = \sum_{\gamma \in \Gamma/\Gamma_0} \left\{ \frac{e^{iu_\gamma}}{(w \cdot \gamma^{-1})'(z)} \left[ e^{w \cdot \gamma^{-1}(z) - iu_\gamma} - (w \cdot \gamma^{-1}(z) - iu_\gamma) \frac{\sinh t_\gamma}{t_\gamma} - \cosh t_\gamma \right] - z e^{iu_\gamma} \sinh t_\gamma \right\}$$

with  $w \cdot \gamma^{-1}(0) = t_\gamma + iu_\gamma$  ( $t_\gamma, u_\gamma \in \mathbb{R}$ ). The constant  $C$  denotes the derivative  $F'(0)$  of the function  $F$  satisfying the identity

$f_p \circ w^{-1}(z) = F(e^z)$ . The estimate is uniform as long as  $z$  stays in compact subsets of  $\Delta$ .

Let  $\lambda_p(z)|dz|$  be the Poincaré metric of the domain  $D_p$ . By definition  $\lambda_p(z)$  satisfies

$$\lambda_p(f_p(z))|f_p'(z)| = \frac{1}{1-|z|^2}, \quad z \in \Delta.$$

In particular, we have  $\lambda_p(0) = 1/f_p'(0)$ . Theorem 4.2. easily gives the following

**COROLLARY.** For sufficiently small  $\varepsilon \in \mathbb{C}$ ,  $\lambda_{p+\varepsilon}(0)$  is given by

$$\ln \lambda_{p+\varepsilon}(0) = \ln \lambda_p(0) + 2\operatorname{Re} \left[ \frac{\varepsilon}{\mathbb{C}} \sum_{\gamma \in \Gamma/\Gamma_0} e^{-iu_\gamma} \left( \cosh t_\gamma - \frac{\sinh t_\gamma}{t_\gamma} \right) \right] + O(\varepsilon^2).$$

#### REFERENCES

1. L.V. Ahlfors, Finitely generated Kleinian groups, Amer. J. Math. 86 (1964), 413-429.
2. L.V. Ahlfors, Lectures on quasiconformal mappings, Van Nostrand, Princeton, New Jersey, 1966.
3. I. Kra, Automorphic forms and Kleinian groups, Benjamin, Reading, Massachusetts, 1972.